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# CONVOLUTIONS OF STABLE LAWS AS LIMIT DISTRIBUTIONS OF PARTIAL SUMS[1]

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**0. Summary.** Let $\{X_n\}$ be a sequence of independent random variables and let $Y_n = B_n^{-1}\sum_{i=1}^n X_i - A_n$ be a sequence of normed, centered sums such that, for appropriately chosen normalizing coefficients ($B_n \to \infty$) and centering constants, $\{Y_n\}$ converges in law to a nondegenerate limit distribution $G$. B. V. Gnedenko asked the following question: What characterizes the class of limit distributions $\{G\}$ when there are $r$ different distribution functions among those of the random variables $\{X_n\}$?

Let $\mathscr{P}_r$ denote this class of distribution functions. As is well-known, $\mathscr{P}_1$ is the class of stable distributions. V. M. Zolotarev and V. S. Korolyuk [8] have shown that $\mathscr{P}_2$ consists solely of stable distributions and convolutions of two stable distributions. It was thought that O. K. Lebedintseva [4] had shown that this was true for $r > 2$ (with *two* replaced by *less than or equal to* $r$) with the added hypothesis that one of the $r$ possible distribution functions of the summands $X_n$ belongs either to the domain of attraction of a stable distribution, or to a domain of partial attraction of only one type. However, V. M. Zolotarev and V. S. Korolyuk [8] gave an example that showed that O. K. Lebedintseva's theorem did not completely settle the matter. However, A. A. Zinger [7] gave a necessary and sufficient condition on the Lévy spectral function of $G$ in order that $G$ be in $\mathscr{P}_r$. His theorem shows that Lebedintseva's result is incorrect.

In this same paper, A. A. Zinger proved a theorem that gives a necessary condition on the distribution functions of the summands $X_n$ in order that $G$ be a convolution of $r$ distinct stable distributions. Here we expand Zinger's theorem to obtain a necessary and sufficient condition that $G$ be a convolution of $r$ distinct stable distributions. Some related results are also obtained.

**1. Main results.** In this section, we state results connected with B. V. Gnedenko's conjecture on limit distribution functions of normed, centered sums of sequences of independent random variables when there are $r$ different distribution functions among those of the sequence. The proofs are given in Section 3.

A distribution function $F$ being in the domain of attraction of a stable distribution of characteristic exponent $\alpha$, denoted by $F \in \mathscr{D}(\alpha)$, means that if $\{X_n\}$ is a sequence of independent identically distributed random variables with common

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distribution function $F$, then there exist a sequence of positive constants $\{B_n\}$, called *normalizing coefficients*, with $B_n \to \infty$, and a sequence of real numbers $\{A_n\}$, called *centering constants*, such that the limiting distribution of $B_n^{-1}(X_1 + \cdots + X_n) - A_n$ is a stable distribution with characteristic exponent $\alpha$. Lamperti [3] essentially proved the "if" part and Tucker [5] proved the "converse" part of the following useful lemma.

LEMMA 1. *If $F \in \mathscr{D}(\alpha)$, $0 < \alpha \leqq 2$, and if $\{B_n\}$ is a sequence of normalizing coefficients for $F$, then there is a measurable slowly varying function $L$ defined over $(0, \infty)$, which must be asymptotic to a non-decreasing function when $\alpha = 2$, such that $B_n \sim n^{\alpha^{-1}}L(n)$. Conversely, if $L$ is a measurable slowly varying function over $(0, \infty)$, and if $0 < \alpha < 2$, or if $\alpha = 2$ and $L$ is a measurable slowly varying function asymptotic to a non-decreasing function, then there is an $F \in \mathscr{D}(\alpha)$ such that $\{n^{\alpha^{-1}}L(n)\}$ is a sequence of normalizing coefficients for $F$.*

For the rest of this section, we will let $\{X_n\}$ be a sequence of independent random variables such that the distribution function of $X_n$ is one of $F_1, \cdots, F_r$, with $r \geqq 2$, for all $n$. We also assume that there exist normalizing coefficients $\{B_n\}$ and centering constants $\{A_n\}$ such that $B_n^{-1}(X_1 + \cdots + X_n) - A_n$ converges in law to a non-degenerate distribution function $G$.

For each positive integer, let $n_i(n)$ denote the number of random variables among $X_1, \cdots, X_n$ which have $F_i$ as their distribution function, $1 \leqq i \leqq r$. We assume that $n_i(n) \to \infty$ as $n \to \infty$ for each $i$.

THEOREM 1. *If $F_i \in \mathscr{D}(\lambda_i)$, $1 \leqq i \leqq r$, with $0 < \lambda_1 < \cdots < \lambda_r \leqq 2$, if there is a $k < r$ such that* $\limsup_{n\to\infty}(n_k(n)/n) = s_k > 0$ *and* $\lim_n(n_i(n)/n) = 0$ *for $i < k$, then $G$ is the convolution of not more than $k$ stable distributions with characteristic exponents in the set $\{\lambda_1, \cdots, \lambda_k\}$.*

Zinger ([7], Theorem 3) has proved the following theorem.

THEOREM (Zinger). *If $G$ is a convolution of $r$ stable distributions with characteristic exponents $\lambda_i$ such that $0 < \lambda_1 < \cdots < \lambda_r \leqq 2$, then under some permutation of the indices of the $F$'s, we have $F_i \in \mathscr{D}(\lambda_i)$, $1 \leqq i \leqq r$.*

Now, in the statement of our principal theorem, two other conclusions are added to the conclusion of Zinger's theorem, which together are not only necessary but also sufficient.

THEOREM 2. *Let $\{X_n\}$ be independent random variables with distribution functions among $\{F_1, \cdots, F_r\}$ such that $\lim_{n\to\infty} n_i(n) = \infty$ for all $i$. Let $\{B_n\}$ and $\{A_n\}$ be such that $0 < B_n \to \infty$ and $B_n^{-1}(X_1 + \cdots + X_n) - A_n$ converges in law to $G$, where $G$ is a non-degenerate distribution function. Then, $G$ is a convolution of $r$ stable distributions with characteristic exponents $\lambda_1, \cdots, \lambda_r$, $(0 < \lambda_1 < \cdots < \lambda_r \leqq 2)$, if and only if*

(i) *$F_i \in \mathscr{D}(\lambda_i)$ for $1 \leqq i \leqq r$, (possibly requiring a permutation of the indices of the $F$'s);*

(ii) $\lim_{n\to\infty} n_r(n)/n = 1$;

(iii) $\lim_{n\to\infty} B(i, n_i(n))/B(j, n_j(n)) = P_{ij}$, *with* $0 < P_{ij} < \infty$, *for all* $i, j \in \{1, \cdots, r\}$, *where $\{B(i,n)\}$ are normalizing coefficients for $F_i$, $1 \leqq i \leqq r$. Further, in this case, the normalizing coefficients $\{B_n\}$ can be chosen to be normalizing coefficients for $F_r$.*

**2. Further results and some examples.** One might question whether or not the existence of a non-degenerate distribution function forces $n_i(n)/n$ to converge. The following example shows that this need not occur.

Let $0 < \lambda_1 < \lambda_2 \leqq 2$. Let $F_1$ and $F_2$ be *symmetric* distribution functions defined by, for $i = 1, 2$,

$$\begin{aligned} F_i(x) &= 1 - x^{-\lambda_i}, && x \geqq 2^{\lambda_i^{-1}}, \\ &= \tfrac{1}{2}, && 0 \leqq x \leqq 2^{\lambda_i^{-1}}. \end{aligned}$$

Let $\#\{\ \}$ denote the number of elements in the set defined within the braces. Define a function $e(\cdot)$ on the positive integers as follows:

$$e(1) = 1, \qquad e(2) = 2, \quad \text{and for} \quad n > 2$$

$$\begin{aligned} e(n) &= 1 \quad \text{if} \quad e(n-1) = 2 \quad \text{and} \quad \#\{i: 1 \leqq i \leqq n-1, e(i) = 1\}/(n-1) = \tfrac{1}{2}, \\ &= 1 \quad \text{if} \quad e(n-1) = 1 \quad \text{and} \quad \#\{i: 1 \leqq i \leqq n-1, e(i) = 1\}/(n-1) < \tfrac{3}{4}, \\ &= 2 \quad \text{if} \quad e(n-1) = 2 \quad \text{and} \quad \#\{i: 1 \leqq i \leqq n-1, e(i) = 1\}/(n-1) > \tfrac{1}{2}, \\ &= 2 \quad \text{if} \quad e(n-1) = 1 \quad \text{and} \quad \#\{i: 1 \leqq i \leqq n-1, e(i) = 1\}/(n-1) = \tfrac{3}{4}. \end{aligned}$$

In other words, $e(n)$ is 1 until the proportion of ones assigned reaches $\frac{3}{4}$, then $e(n)$ is 2 until the proportion of ones reaches $\frac{1}{2}$ at which time $e(n)$ becomes 1 again and we repeat the process.

Let $\{X_n\}$ be a sequence of independent random variables such that $F_{X_n} = F_{e(n)}$ for all $n$. Then $\limsup_{n\to\infty} n_1(n)/n = \frac{3}{4}$ and $\liminf_{n\to\infty} n_1(n)/n = \frac{1}{2}$.

Let $B_n = (n_1(n))^{\lambda_1^{-1}}$ for all $n$.

According to Gnedenko and Kolmogorov ([2], 124, Theorem 4), in order to show that there exists $\{A_n\}$ such that $B_n^{-1}(X_1 + \cdots + X_n) - A_n$ converges in law to a stable distribution with characteristic exponent $\lambda_1$, it is sufficient to show that, for any $x > 0$,

(i) $n_1(n)F_1(-B_n x) + n_2(n)F_2(-B_n x) \to x^{-\lambda_1}$,

(ii) $n_1(n)(1 - F_1(B_n x)) + n_2(n)(1 - F_2(B_n x)) \to x^{-\lambda_1}$,

(iii) $\lim_{\varepsilon\to 0+} \limsup_{n\to\infty} D_n(\varepsilon) = \lim_{\varepsilon\to 0+} \liminf_{n\to\infty} D_n(\varepsilon) = 0$,

where

$$D_n(\varepsilon) = \textstyle\sum_{i=1}^{2} n_i(n)\{\int_{|x|<\varepsilon} x^2\, dF_i(B_n x) - (\int_{|x|<\varepsilon} x\, dF_i(B_n x))^2\}.$$

By the definition of $B_n$, we have that $n_1(n)/B_n^{\lambda_1} \to 1$ and $n_2(n)/B_n^{\lambda_2} \leqq (n/2)^{1-\lambda_2/\lambda_1} \to 0$.

Hence, for $x > 0$, $n_1(n)F_1(-B_n x) + n_2(n)F_2(-B_n x) \to x^{-\lambda_1}$ as $n \to \infty$. Therefore, (i) holds, and by the symmetry involved, (ii) also holds.

To show that (iii) holds, we note that $\int_{|x|<\varepsilon} x\, dF_i(B_n x) = 0$, for $i = 1, 2$. An easy calculation shows that $\lim_{n\to\infty} D_n(\varepsilon) = 2\lambda_1 \varepsilon^{2-\lambda_1}(2-\lambda_1)^{-1}$.

Another question we ask is: given $0 < \lambda_1 < \cdots < \lambda_r < 2$, does there exist a sequence $\{X_n\}$ of independent random variables containing $r$ distributions such that, for some sequences $\{B_n\}$ and $\{A_n\}$, $B_n^{-1}(X_1 + \cdots + X_n) - A_n$ converges in law, as n $\to \infty$, to a convolution of stable distributions with exponents $\lambda_1, \cdots, \lambda_r$? Zinger ([7], 620–621) has suggested a method to generate more general examples. Following his suggestion we define $r$ distribution functions by

$$\begin{aligned} F_i(x) &= |x|^{-\lambda_i} \quad \text{for} \quad x \leqq -2^{-\lambda_i^{-1}}, \\ &= \tfrac{1}{2} \quad \text{for} \quad |x| \leqq 2^{-\lambda_i^{-1}}, \\ &= 1 - x^{-\lambda_i} \quad \text{for} \quad x \geqq 2^{-\lambda_i^{-1}} \end{aligned}$$

For $n$ a positive integer, define $B_n$ to be the smallest positive solution of $y^{\lambda_1} + \cdots + y^{\lambda_r} = n$.

For $1 \leqq i \leqq r-1$, let $n_i(n) = [B_n^{\lambda_i}]$, and let $n_r(n) = n - \sum_{i=1}^{r-1} n_i(n)$, where $[a]$ denotes the integral part of the real number $a$. Since $n_i(n)$ is integer-valued, nondecreasing, and $\sum_{i=1}^{r} n_i(n+1) - \sum_{i=1}^{r} n_i(n) = n+1-n = 1$, we have that exactly one of the $n_i(n)$s increases by one when the argument increases from $n$ to $n+1$. Let $e(n)$ be the index $i$ such that $n_i(n)$ increases when the argument goes from $n-1$ to $n$.

Let $\{X_n\}$ be a sequence of independent random variables such that $F_{X_n}$ is $F_{e(n)}$.

In order to show that, for appropriately chosen constants $\{A_n\}$, $B_n^{-1}(X_1 + \cdots + X_n) - A_n$ converges in law to a convolution of $r$ stable distribution with characteristic exponents $\lambda_1, \cdots, \lambda_r$, it is sufficient to show that, for $x > 0$,

(i) $\sum_{i=1}^{r} n_i(n) F_i(-B_n x) \to \sum_{i=1}^{r} x^{-\lambda_i}$,

(ii) $\sum_{i=1}^{r} n_i(n)(1 - F_i(B_n x)) \to \sum_{i=1}^{r} x^{-\lambda_i}$,

(iii) $\lim_{\varepsilon \to 0+} \limsup_{n \to \infty} D_n(\varepsilon) = \lim_{\varepsilon \to 0+} \liminf_{n \to \infty} D_n(\varepsilon) = 0$

where

$$D_n(\varepsilon) = \sum_{i=1}^{r} n_i(n) \{ \textstyle\int_{|x| < \varepsilon} x^2 \, dF_i(B_n x) - (\int_{|x| < \varepsilon} x \, dF_i(B_n x))^2 \}.$$

For $1 \leqq i \leqq r-1$, $n_i(n)/B_n^{\lambda_i} \to 1$ as $n \to \infty$. Letting $\{a\}$ denote the fractional part of the real number $a$, we have that

$$n_r(n)/B_n^{\lambda_r} = (\textstyle\sum_{i=1}^{r} \{B_n^{\lambda_i}\} + [B_n^{\lambda_r}])/B_n^{\lambda_r}.$$

The right-hand side is bounded above by $(r + [B_n^{\lambda_r}])B_n^{-\lambda_r}$ and is bounded below by $[B_n^{\lambda_r}]B_n^{-\lambda_r}$; both of which converge to unity as $n \to \infty$. This establishes (i) and (ii).

It can be shown that $\lim_{n \to \infty} D_n(\varepsilon) = \sum_{i=1}^{r} 2\lambda_i \varepsilon^{2-\lambda_i}(2-\lambda_i)^{-1}$. Hence, (iii) holds and the posed question is answered in the affirmative.

V. M. Zolotarev and V. S. Korolyuk [8] gave an example that shows that the limit distribution being stable does not imply that at least one of the distribution functions $F_1$ and $F_2$ of the summands belongs to the domain of attraction, or even the domain of partial attraction, of any distribution. However, there are some errors in their example: $c_1$ should be $c_2$ and $c_2$ should be $c_1$; $n_1$ should be $[n/3]$ and $n_2$ should be $2[n/3] + 3\{n/3\}$.

Zinger [6] gave an example when $r = 3$ and the limit distribution is neither a stable distribution nor a convolution of any number of stable distributions.

We will conclude this section with three theorems and their corollaries. For the rest of this section we assume that $\{X_n\}$ is a sequence of independent random variables such that the distribution function of $X_n$ is one of $F_1, \cdots, F_r$, and $F_i \in \mathscr{D}(\lambda_i)$ for $1 \leqq i \leqq r$, with $0 < \lambda_1 < \lambda_2 < \lambda_3 \leqq \cdots \leqq \lambda_r \leqq 2$. We also assume that there exist sequences $\{B_n\}$ and $\{A_n\}$ such that $B_n^{-1}(X_1 + \cdots + X_n) - A_n$ converges in law to a non-degenerate distribution function $G$.

THEOREM 3. *If, for some* $\eta \in (0, 1-\lambda_1/\lambda_2)$, $\limsup_{n\to\infty}(n_1(n)/n^{1-\eta}) = s_1$, *with* $0 < s_1 < \infty$, *then $G$ is a stable distribution with characteristic exponent* $\lambda_1$.

It can occur, even though each of the $r$ distribution functions $F_1, \cdots, F_r$ belongs to the domain of attraction of a stable distribution with characteristic exponent from the set $\{\lambda_1, \cdots, \lambda_r\}$, that the normalizing coefficients are not normalizing coefficients for any distribution function in the domain of attraction of a stable distribution with characteristic exponent from the set $\{\lambda_1, \cdots, \lambda_r\}$. This follows from the following Corollary and Lemma 1.

COROLLARY TO THEOREM 3. *If, for some* $\eta \in (0, 1-\lambda_1/\lambda_2)$, $\lim_{n\to\infty}(n_1(n)/n^{1-\eta}) = s_1$, *with* $0 < s_1 < \infty$, *if, for* $i > 1$, $\lim_{n\to\infty}(n_i(n)/n) = s_i > 0$, *then $G$ is a stable distribution with characteristic exponent $\lambda_1$ and* $B_n \sim n^{\lambda^{-1}}L(n)$, *where* $\lambda = \lambda_1/(1-\eta)$ *and $L$ is a measurable slowly varying function defined over* $(0, \infty)$.

THEOREM 4. *If there exists* $\eta > 1-\lambda_1/\lambda_2$ *such that* $\lim_{n\to\infty}(n_1(n)/n^{1-\eta}) = s_1$, *with* $0 < s_1 < \infty$, *and if* $\limsup_{n\to\infty}(n_2(n)/n) = s_2 > 0$, *then $G$ is a stable distribution with characteristic exponent* $\lambda_2$.

COROLLARY TO THEOREM 4. *If there exists* $\eta > 1-\lambda_1/\lambda_2$ *such that* $\lim_{n\to\infty}(n_1(n)/n^{1-\eta}) = s_1$, *with* $0 < s_1 < \infty$, *and if* $\lim_{n\to\infty}(n_i(n)/n) = s_i > 0$ *for all* $i > 1$, *then $G$ is a stable distribution with characteristic exponent $\lambda_2$ and normalizing coefficients of $F_2$ can be used for the sequence* $\{B_n\}$.

THEOREM 5. *If* $n_1(n)/n^{\lambda_1/\lambda_2} \to s_1$, *with* $0 < s_1 < \infty$, *if* $n_2(n)/n \to s_2 > 0$, *then*

(i) *$G$ is a stable distribution with characteristic exponent $\lambda_1$ if and only if* $L_2(n)/L_1(n^{\lambda_1/\lambda_2}) \to 0$ *as* $n \to \infty$;

(ii) *$G$ is a stable distribution with characteristic exponent $\lambda_2$ if and only if* $L_2(n)/L_1(n^{\lambda_1/\lambda_2}) \to \infty$ *as* $n \to \infty$;

(iii) *$G$ is a convolution of two stable distributions with characteristic exponents $\lambda_1$ and $\lambda_2$ if and only if $L_2(n)/L_1(n^{\lambda_1/\lambda_2})$ is bounded away from zero and infinity.*

We should note that there are many assumptions made before (i), (ii) and (iii) of Theorem 5. In particular, not all distribution functions in $\mathscr{D}(\lambda_1)$ and $\mathscr{D}(\lambda_2)$ can be paired and satisfy the hypothesis of Theorem 5. For example, Tucker [5] gives an example of two non-decreasing slowly varying functions $L_1$ and $L_2$ such that $\limsup_{x\to\infty} L_1(x)/L_2(x) = \infty$ and $\liminf_{x\to\infty} L_1(x)/L_2(x) = 0$. Let $L_1'(x) = L_1(x^{\lambda_2/\lambda_1})$. Then $L_1'$ is a slowly varying function. Also, $\limsup_{x\to\infty} L_1'(x^{\lambda_1/\lambda_2})/L_2(x) = \infty$ and $\liminf_{x\to\infty} L_1'(x^{\lambda_1/\lambda_2})/L_2(x) = 0$. By the converse of Lemma 1, there are $F_1 \in \mathscr{D}(\lambda_1)$

and $F_2 \in \mathscr{D}(\lambda_2)$ such that $\{n^{\lambda_1^{-1}}L_1'(n)\}$ and $\{n^{\lambda_2^{-1}}L_2(n)\}$ are normalizing coefficients for $F_1$ and $F_2$, respectively. If a sequence $\{X_n\}$ of independent random variables exists such that $F_{X_n}$ is either $F_1$ or $F_2$, and if there are normalizing coefficients and centering constants so that the normed, centered sums of the $X_n$s converge in law, then we must have either $n_1(n)/n^{\lambda_1/\lambda_2} \to 0$ or $\infty$ as $n \to \infty$. Otherwise, Theorem 5 would apply, which obviously would be a contradiction to our choice of $L_1'$ and $L_2$.

**3. Proofs.**

PROOF OF THEOREM 1. Let $X(i,m)$ be the $m$th random variable in the sequence $\{X_n\}$ whose distribution function is $F_i$, for $1 \leqq i \leqq r$. Then, for each $i$, there are constants $\{B(i,n)\}$ and $\{A(i,n)\}$ such that

$$[B(i,n)]^{-1}[X(i,1)+\cdots+X(i,n)]-A(i,n)$$

converges in law to a stable distribution with characteristic exponent $\lambda_i$.

By Lemma 1, for each $i$, there exists a measurable slowly varying function $L_i$ defined over $(0,\infty)$ such that $B(i,n) \sim n^{\lambda_i^{-1}}L_i(n)$.

Let $\{m(j)\}$ be an increasing sequence of positive integers such that $\lim_{j\to\infty}(n_i(m(j))/m(j))$ exists and is equal to $s_i$, with $s_k > 0$, for $1 \leqq i \leqq r$.

Let

$$f_i(j) = \left(\frac{n_i(m(j))}{(m(j))^{\lambda_i/\lambda_k}}\right)^{1/\lambda_i}\frac{L_i(n_i(m(j)))}{L_k(m(j))}, \qquad \text{for} \quad 1 \leqq i \leqq k-1.$$

Now, select an increasing sequence $\{j(p)\}$ of positive integers such that $\lim_{p\to\infty} f_i(j(p))$ exists and is equal to $\alpha_i$, for $1 \leqq i \leqq k-1$. It is possible that some $\alpha_i$ may be $\infty$. Let $a(p) = m(j(p))$ for all $p$. Then, for $i > k$, $B(i,n_i(a(p)))/B(k,n_k(a(p))) \to 0$ as $p \to \infty$.

Let

$$f(i,p) = \left(\frac{n_i(a(p))}{(a(p))^{\lambda_i/\lambda_k}}\right)^{1/\lambda_i}\frac{L_i(n_i(a(p)))}{L_k(a(p))}, \qquad \text{for} \quad 1 \leqq i \leqq k-1.$$

Then $\lim_{p\to\infty} f(i,p) = \alpha_i$, $1 \leqq i \leqq k-1$.

The proof will be concluded by considering three exhaustive cases.

CASE 1. $\alpha_i < \infty$ for all $i$. Then for $1 \leqq i \leqq k-1$, we have

$$\frac{B(i,n_i(a(p)))}{B(k,n_k(a(p)))} \sim s_k^{-1/\lambda_k} f(i,p) \to s_k^{-1/\lambda_k}\alpha_i, \qquad \text{as} \quad p \to \infty.$$

We observe that

$$\begin{aligned}\frac{X_1+\cdots+X_{a(p)}}{B(k,n_k(a(p)))} &- \frac{1}{B(k,n_k(a(p)))}\sum_{i=1}^r B(i,n_i(a(p)))\cdot A(i,n_i(a(p))) \\ &= \frac{X(k,1)+\cdots+X(k,n_k(a(p)))}{B(k,n_k(a(p)))} - A(k,n_k(a(p))) \\ &+ \sum_{i=1,i\neq k}^r \frac{B(i,n_i(a(p)))}{B(k,n_k(a(p)))}\left\{\frac{X(i,1)+\cdots+X(i,n_i(a(p)))}{B(i,n_i(a(p)))} - A(i,n_i(a(p)))\right\}.\end{aligned}$$

We see that the right-hand side of the above equality converges in law to a convolution of not more than $k$ stable laws with characteristic exponents in the set $\{\lambda_1, \cdots, \lambda_k\}$. Hence,

$$\frac{X_1 + \cdots + X_{a(p)}}{B(k, n_k(a(p)))} - \frac{1}{B(k, n_k(a(p)))} \sum_{i=1}^{r} B(i, n_i(a(p)))A(i, n_i(a(p)))$$

converges in law, as $p \to \infty$, to a convolution of not more than $k$ stable laws with characteristic exponents in the set $\{\lambda_1, \cdots, \lambda_k\}$, including $\lambda_k$.

Since $B_{a(p)}^{-1}[X_1 + \cdots + X_{a(p)}] - A_{a(p)}$ converges in law, as $p \to \infty$, to $G$, we have that $G$ is of this same type, which establishes the theorem for Case 1.

CASE 2. For exactly one $j \in \{1, \cdots, k-1\}$, $\alpha_j = \infty$ and $\alpha_i < \infty$ for $i \in \{1, \cdots, k-1\} \backslash \{j\}$.

Then, for $i \in \{1, \cdots, k-1\} \backslash \{j\}$,

$$B(i, n_i(a(p)))/B(j, n_j(a(p))) \sim f(i, p)/f(j, p) \to 0, \quad \text{as} \quad p \to \infty.$$

Also,

$$B(k, n_k(a(p)))/B(j, n_j(a(p))) \sim s_k^{\lambda_k^{-1}}/f(j, p) \to 0, \quad \text{as} \quad p \to \infty.$$

Continuing as in Case 1, but dividing by $B(j, n_j(a(p)))$ instead of $B(k, n_k(a(p)))$, and noting that for $i > k$,

$$\frac{B(i, n_i(a(p)))}{B(j, n_j(a(p)))} = \frac{B(i, n_i(a(p)))/B(k, n_k(a(p)))}{B(j, n_j(a(p)))/B(k, n_k(a(p)))} \to 0,$$

as $p \to \infty$, we can conclude that $G$ is a stable distribution with characteristic exponent $\lambda_j \in \{\lambda_1, \cdots, \lambda_k\}$. This establishes the theorem for Case 2.

CASE 3. $\alpha_{i_s} = \infty$ for $s \in \{1, \cdots, u\}$ with $u \geqq 2$ and $\alpha_i < \infty$ for $i \in \{1, \cdots, k-1\} \backslash \{i_i, \cdots, i_u\}$.

Assume $i_1 < \cdots < i_u$. Let $\{p(j)\}$ be an increasing sequence of positive integers such that $\lim_{j \to \infty} f(i_s, p(j))/f(i_u, p(j))$ exists and is equal to $\beta_s$, for $1 \leqq s \leqq u-1$.

(i) If $\beta_s < \infty$ for all $s$, then, for $i \in \{i_1, \cdots, i_{u-1}\}$,

$$\frac{B(i, n_i(a(p(j))))}{B(i_u, n_{i_u}(a(p(j))))} \sim \frac{f(i, p(j))}{f(i_u, p(j))} \to \beta_i < \infty.$$

For $i \in \{1, \cdots, k-1\} \backslash \{i_1, \cdots, i_u\}$,

$$\frac{B(i, n_i(a(p(j))))}{B(i_u, n_{i_u}(a(p(j))))} \sim \frac{f(i, p(j))}{f(i_u, p(j))} \to 0,$$

since $\alpha_i < \infty$ and $\alpha_{i_u} = \infty$. Also

$$\frac{B(k, n_k(a(p(j))))}{B(i_u, n_{i_u}(a(p(j))))} \sim \frac{s_k^{\lambda_k^{-1}}}{f(i_u, p(j))} \to 0.$$

As at the end of Case 2, for $i > k$,

$$B(i, n_i(a(p(j))))/B(i_u, n_{i_u}(a(p(j)))) \to 0, \quad \text{as} \quad j \to \infty.$$

Continuing as in Case 1, but dividing by $B(i_u, n_{i_u}(a(p(j))))$ instead of $B(k, n_k(a(p)))$ and summing the $X$s to $a(p(j))$ instead of $a(p)$, we can conclude that $G$ is a convolution of not more than $u < k$ stable distributions with characteristic exponents in the set $\{\lambda_{i_1}, \cdots, \lambda_{i_u}\} \subset \{\lambda_1, \cdots, \lambda_k\}$.

(ii) If exactly one of the $\beta$s, say $\beta_s$, is infinite, then, for $i \in \{i_1, \cdots, i_u\} \backslash \{i_s\}$,

$$\frac{B(i, n_i(a(p(j))))}{B(i_s, n_{i_s}(a(p(j))))} \sim \frac{f(i, p(j))/f(i_u, p(j))}{f(i_s, p(j))/f(i_u, p(j))} \to 0.$$

For $i \in \{1, \cdots, k-1\} \backslash \{i_1, \cdots, i_u\}$,

$$B(i, n_i(a(p(j))))/B(i_s, n_{i_s}(a(p(j)))) \sim f(i, p(j))/f(i_s, p(j)) \to 0,$$

since $\alpha_i < \infty$ and $\alpha_{i_s} = \infty$.

Again $B(k, n_k(a(p(j))))/B(i_s, n_{i_s}(a(p(j)))) \sim s_k^{\lambda_k^{-1}}/f(i_s, p(j)) \to 0$.
As in (i), for $i > k$,

$$B(i, n_i(a(p(j))))/B(i_s, n_{i_s}(a(p(j)))) \to 0, \quad \text{as} \quad j \to \infty.$$

Now continuing as in (i), except dividing by $B(i_s, n_{i_s}(a(p(j))))$, we can conclude that $G$ is a stable distribution with characteristic exponent $\lambda_{i_s} \in \{\lambda_1, \cdots, \lambda_k\}$.

(iii) If more than one of the $\beta$s, say $\beta_{t_1}, \cdots, \beta_{t_w}$, is infinite, then repeat the above process by forming the ratios $f(i, a(p(j)))/f(i_{t_w}, a(p(j)))$ for $i \in \{i_{t_1}, \cdots, i_{t_{w-1}}\}$. Continuing this process as long as necessary, we will eventually arrive at one of two possible stages.

*Stage* 1. We have a subset $\{f(v_1, \cdot), \cdots, f(v_t, \cdot)\}$ of $\{f(1, \cdot), \cdots, f(k-1, \cdot)\}$ with $t \geqq 2$ and a subsequence $\{p''(j)\}$ of $\{p(j)\}$ such that, for $i \in \{v_1, \cdots, v_{t-1}\}$,

$$\lim_{j \to \infty} f(i, a(p''(j)))/f(v_t, a(p''(j)))$$

exists and is finite.

Then, for $i \in \{1, \cdots, k-1\} \backslash \{v_1, \cdots, v_t\}$, $f(i, \cdot) \notin \{f(v_1, \cdot), \cdots, f(v_t, \cdot)\}$ for one of the three following reasons:

(i) $\alpha_i < \infty$;

(ii) there exists an $f(z, \cdot) \in \{f(1, \cdot), \cdots, f(k-1, \cdot)\}$ and there exists a subsequence $\{p'(j)\}$ of $\{p(j)\}$ containing $\{p''(j)\}$ such that

$$\lim_{j \to \infty} f(i, a(p'(j)))/f(z, a(p'(j))) \quad \text{is finite and}$$

$$\lim_{j \to \infty} f(v_t, a(p'(j)))/f(z, a(p'(j))) = \infty;$$

(iii) $\lim_{j \to \infty} f(v_t, a(p'(j)))/f(i, a(p'(j))) = \infty$, for some subsequence $\{p'(j)\}$ of $\{p(j)\}$ containing $\{p''(j)\}$.

In each of these three cases, we have

$$\lim_{j \to \infty} f(i, a(p''(j)))/f(v_t, a(p''(j))) = 0.$$

Hence, for $i \in \{1, \cdots, k-1\} \backslash \{v_1, \cdots, v_t\}$,

$$B(i, n_i(a(p''(j))))/B(v_t, n_{v_t}(a(p''(j)))) \to 0 \quad \text{as} \quad j \to \infty.$$

Clearly, for $i \geqq k$, we also have convergence to zero.

Continuing as always, we can conclude that $G$ is a convolution of not more than $t < k$ stable distributions with characteristic exponents in the set $\{\lambda_{v_1}, \cdots, \lambda_{v_t}\} \subset \{\lambda_1, \cdots, \lambda_k\}$.

*Stage* 2. We have a subset $\{f(v_1, \cdot), \cdots, f(v_t, \cdot)\}$ of $\{f(1, \cdot), \cdots, f(k-1, \cdot)\}$ with $t \geqq 2$ and we have a sequence $\{p'(j)\}$ of $\{p(j)\}$ such that for some $i_* \in \{v_1, \cdots, v_{t-1}\}$

$$\lim_{j \to \infty} f(i_*, a(p'(j)))/f(v_t, a(p'(j))) = \infty$$

and for $i \in \{v_1, \cdots, v_{t-1}\} \backslash \{i_*\}$

$$\lim_{j \to \infty} f(i, a(p'(j)))/f(v_t, a(p'(j))) \qquad \text{exists and is finite.}$$

As in Stage 1, we have for $i \in \{1, \cdots, k-1\} \backslash \{v_1, \cdots, v_t\}$, $\lim_{j \to \infty} f(i, a(p'(j)))/f(i_*, a(p'(j))) = 0$. Hence, for $i \in \{1, \cdots, k-1\} \backslash \{v_1, \cdots, v_t\}$,

$$B(i, n_i(a(p'(j))))/B(i_*, n_{i_*}(a(p'(j)))) \to 0 \quad \text{as} \quad j \to \infty.$$

Clearly, for $i \geqq k$, the above convergence holds.

Since, for $i \in \{v_1, \cdots, v_t\} \backslash \{i_*\}$

$$\lim_{j \to \infty} f(i, a(p'(j)))/f(i_*, a(p'(j))) = 0,$$

we have

$$B(i, n_i(a(p'(j))))/B(i_*, n_{i_*}(a(p'(j)))) \to 0$$

as $j \to \infty$.

Therefore, we can conclude that $G$ is a stable distribution with characteristic exponent $\lambda_{i_*} \in \{\lambda_1, \cdots, \lambda_k\}$.

We have finally exhausted all cases and established the theorem in each of these. $\square$

For the proof of Theorem 2, we need the following lemma.

LEMMA 2. *If $Y_1, \cdots, Y_r$ are independent stable random variables with the characteristic exponent of $Y_i$ being $\alpha_i$ and $0 < \alpha_1 < \cdots < \alpha_r \leqq 2$, if $a_1, \cdots, a_r, b_1, \cdots, b_r$ are constants such that $a_1 Y_1 + \cdots + a_r Y_r$ and $b_1 Y_1 + \cdots + b_r Y_r$ have the same distribution function, then $a_i = b_i$ for all $i$.*

PROOF. Let $Z_1 = a_1 Y_1 + \cdots + a_r Y_r$ and $Z_2 = b_1 Y_1 + \cdots + b_r Y_r$. Also, let $Z_i'$, $i = 1, 2$, be random variables such that $F_{Z_i} = F_{Z_i'}$ and $(Z_1, Z_2)$, $(Z_1', Z_2')$ are independent.

Let $\psi$ be the logarithm of the characteristic function of $Z_1 - Z_1'$. Using the canonical representation of the logarithm of a stable distribution ([2], 164), we have $\psi(t) = a_1|t|^{\alpha_1} + \cdots + a_r|t|^{\alpha_r}$ and $\psi(t) = b_1|t|^{\alpha_1} + \cdots + b_r|t|^{\alpha_r}$, for all $t$.

Since $\alpha_1, \cdots, \alpha_r$ are distinct, $|t|^{\alpha_1}, \cdots, |t|^{\alpha_r}$ are linearly independent, hence, $a_i = b_i$ for all $i$. $\square$

PROOF OF SUFFICIENCY PART OF THEOREM 2. Let $X(i,m)$ be the $m$th random variable in the sequence $\{X_n\}$ whose distribution function is $F_i$, for $1 \leqq i \leqq r$. Then for each $i$, there are constants $\{B(i,n)\}$ and $\{A(i,n)\}$, with $B(i,n) \to \infty$ as $n \to \infty$ for all $i$, and there is a random variable $Y_i$ such that the joint distribution function of

$$\left\{\frac{X(i,1)+\cdots+X(i,n)}{B(i,n)} - A(i,n), \qquad 1 \leqq i \leqq r\right\}$$

converges in law to $\{Y_1, \cdots, Y_r\}$, where $Y_i$ has a stable distribution with characteristic exponent $\lambda_i$, and where $Y_1, \cdots, Y_r$ are independent.

We observe that

$$\begin{aligned}\frac{X_1+\cdots+X_n}{B(r,n_r(n))} &- \frac{1}{B(r,n_r(n))}\sum_{i=1} B(i,n_i(n))A(i,n_i(n)) \\ &= \frac{X(r,1)+\cdots+X(r,n_r(n))}{B(r,n_r(n))} - A(r,n_r(n)) + \sum_{i=1}^{r-1} \frac{B(i,n_i(n))}{B(r,n_r(n))} \\ &\qquad \cdot \left\{\frac{X(i,1)+\cdots+X(i,n_i(n))}{B(i,n_i(n))} - A(i,n_i(n))\right\}.\end{aligned}$$

Since normalizing coefficients for a distribution function are unique up to a positive constant factor (see, e.g. Feller [1], Lemma 1, 246), Theorem 2 (iii) implies that, for any choice of normalizing coefficients for the $F_i$s, $B(i,n_i(n))/B(r,n_r(n))$ converges to a positive, finite constant, for $1 \leqq i \leqq r-1$. Hence, by the argument used at the end of Case 1 in the proof of Theorem 1, we have that $G$ is the distribution function of a linear combination of $Y_1, \cdots, Y_r$, with each $Y_i$ needed. □

PROOF OF NECESSITY PART OF THEOREM 2. We now assume that $G$ is a convolution of $r$ stable distributions with characteristic exponents $0 < \lambda_1 < \cdots < \lambda_r \leqq 2$.

By Zinger's Theorem, Theorem 2 (i) holds.

By Theorem 1, $n_i(n)/n \to 0$ as $n \to \infty$ for $1 \leqq i \leqq r-1$, hence, $n_r(n)/n \to 1$ as $n \to \infty$, i.e. Theorem 2 (ii) holds.

Now we will show that Theorem 2 (iii) holds also. Let $X(i,n)$, $B(i,n)$, $A(i,n)$ and $Y_i$ be as in the proof of the sufficiency. Then by Lemma 1, for each $i$, there exists a measurable slowly varying function $L_i$ such that $B_n \sim n^{\lambda_i^{-1}}L_i(n)$.

Hence, for $1 \leqq i \leqq r-1$,

$$\frac{B(i,n_i(n))}{B(r,n_r(n))} \sim \left(\frac{n_i(n)}{n^{\lambda_i/\lambda_r}}\right)^{\lambda_i^{-1}} \frac{L_i(n_i(n))}{L_r(n)}.$$

For $1 \leqq i \leqq r-1$, we define

$$g(i,n) = \left(\frac{n_i(n)}{n^{\lambda_i/\lambda_r}}\right)^{\lambda_i^{-1}} \frac{L_i(n_i(n))}{L_r(n)}.$$

In order to conclude the proof of the necessity, we need to show that $\lim_{n\to\infty} g(i,n)$ exists and is positive and finite, for $1 \leqq i \leqq r-1$.

Let $a_i = \liminf_{n\to\infty} g(i,n)$ and $b_i = \limsup_{n\to\infty} g(i,n)$ for $1 \leqq i \leqq r-1$. We will first show that $b_i < \infty$ for all $i$.

Suppose not. Then there is a $k$ such that $b_k = \infty$. Select an increasing sequence $\{m(j)\}$ of positive integers such that $\lim_{j\to\infty} g(i, m(j)) = b_i'$, for $1 \leqq i \leqq r-1$, with $b_k' = \infty$.

If $b_k'$ is the only infinite one among $b_1', \cdots, b_{r-1}'$, then, by the argument in Case 2 of the proof of Theorem 1, we can conclude that $G$ is a stable distribution with characteristic exponent $\lambda_k$. This contradicts $G$ being a convolution of $r$ distinct stable distributions.

If several of the $b$'s, say $b_{i_1}', \cdots, b_{i_u}'$, are infinite, then, by an argument analogous to the one used in Case 3 of the proof of Theorem 1, we can conclude that $G$ is a convolution of not more than $u < r$ stable distributions. Again, we have a contradiction.

Therefore, $b_i < \infty$ for all $i$.

Next, we will show that $a_i > 0$, for all $i$.

Suppose not. Let $k$ be such that $a_k = 0$. Select an increasing sequence $\{m(j)\}$ of positive integers such that $\lim_{j\to\infty} g(i, m(j)) = a_i'$ for $1 \leqq i \leqq r-1$, with $a_k' = 0$ and $a_i' < \infty$ for all $i$. Then

$$B(i, n_i(m(j)))/B(r, n_r(m(j))) \sim g(i, m(j)) \to a_i' \quad \text{as} \quad j \to \infty,$$

for $1 \leqq i \leqq r-1$. As in the proof of the sufficiency, this implies that $G$ is the distribution function of $c\{a_1'Y_1 + \cdots + a_k'Y_k + \cdots + a_{r-1}'Y_{r-1} + Y_r\}$ for some $c > 0$. Since $a_k' = 0$, $G$ is a convolution of not more than $r-1$ stable distributions, which contradicts our hypothesis of $r$ distinct stable components. Therefore, $a_i > 0$ for all $i$.

Finally, we will show that $a_i = b_i$ for all $i$.

Suppose not. Let $k$ be such that $a_k < b_k$. Select increasing sequences $\{m(j)\}$ and $\{m'(j)\}$ of positive integers such that $\lim_{j\to\infty} g(i, m(j)) = b_i'$ and $\lim_{j\to\infty} g(i, m'(j)) = a_i'$ for $1 \leqq i \leqq r-1$, with $b_k' = b_k$ and $a_k' = a_k$. Then, for $1 \leqq i \leqq r-1$,

$$B(i, n_i(m(j)))/B(r, n_r(m(j))) \sim g(i, m(j)) \to b_i' \quad \text{as} \quad j \to \infty,$$

and

$$B(i, n_i(m'(j)))/B(r, n_r(m'(j))) \sim g(i, m'(j)) \to a_i' \quad \text{as} \quad j \to \infty.$$

Hence, there exist constants $d_1 > 0$ and $d_2 > 0$ such that $G$ is the distribution function of both $d_1\{b_1'Y_1 + \cdots + b_{r-1}'Y_{r-1} + Y_r\}$ and $d_2\{a_1'Y_1 + \cdots + a_{r-1}'Y_{r-1} + Y_r\}$. By Lemma 2, $d_1 b_i' = d_2 a_i'$, and $1 \leqq i \leqq r-1$, and $d_1 = d_2$. Hence, $a_k' = b_k'$, which implies $a_k = b_k$. We have the desired contradiction which shows that $\lim_{n\to\infty} g(i,n)$ exists and is positive and finite for $1 \leqq i \leqq r-1$. □

To complete the proof of Theorem 2, we need to show that $\{B_n\}$ can be chosen as normalizing coefficients for $F_r$. We have shown that there exists a constant $d > 0$ such that $B_n \sim dB(r, n_r(n))$. Hence,

$$B_n \sim dn^{\lambda_r^{-1}}\left(\frac{n_r(n)}{n}\right)^{\lambda_r^{-1}} L_r(n)\frac{L_r(n(n_r(n)/n))}{L_r(n)}$$
$$\sim dn^{\lambda_r^{-1}} L_r(n).$$

Therefore, $B_n \sim dn^{\lambda_r^{-1}} L_r(n)$. □

PROOF OF THEOREM 3. Let $\{X(i,n)\}$, $\{B(i,n)\}$ and $\{A(i,n)\}$ be as in the proof of Theorem 1. Then, by Lemma 1, for each $i$ there exists a measurable slowly varying function $L_i$ defined over $(0,\infty)$ such that $B(i,n) \sim n^{\lambda_i^{-1}}L_i(n)$.

Also, as in the proof of Theorem 1, select an increasing sequence $\{m(j)\}$ of positive integers such that $\lim_{j\to\infty}(n_1(m(j))/m(j)^{1-\eta}) = s_1$ and, for all $i > 1$, $\lim_{j\to\infty}(n_i(m(j))/m(j)) = s_i$ say.

Since $\eta \in (0, 1-\lambda_1/\lambda_2)$, we see that for all $i > 1$, $B(i,n_i(m(j)))/B(1,n_1(m(j))) \to 0$ as $j \to \infty$. Hence, the distribution function of

$$\frac{X(1,1)+\cdots+X(1,n_1(m(j)))}{B(1,n_1(m(j)))} - A(1,n_1(m(j))) + \sum_{i=2}^{r} \frac{B(i,n_i(m(j)))}{B(1,n_1(m(j)))} \cdot\left\{\frac{X(i,1)+\cdots+X(i,n_i(m(j)))}{B(i,n_i(m(j)))} - A(i,n_i(m(j)))\right\}$$

converges to a stable distribution with characteristic exponent $\lambda_1$. Therefore, $G$ is a stable distribution with characteristic exponent $\lambda_1$. □

PROOF OF THEOREM 4. Notation is as usual. Then, there exists a sequence $\{m(j)\}$ of positive integers such that $\lim_{j\to\infty}(n_1(m(j))/m(j)^{1-\eta}) = s_1$, $\lim_{j\to\infty}(n_2(m(j))/m(j)) = s_2$ and, for all $i > 2$, $\lim_{j\to\infty}(n_i(m(j))/m(j)) = s_i$ say. Then

$$\frac{B(1,n_1(m(j)))}{B(2,n_2(m(j)))} \sim \frac{s_1^{\lambda_1^{-1}} m(j)^{(1-\eta)/\lambda_1} L_1(m(j)^{(1-\eta)/\lambda_1})}{s_2^{\lambda_2^{-1}} m(j)^{\lambda_2^{-1}} L_2(m(j))} \to 0$$

as $j \to \infty$, since $\eta > 1-\lambda_1/\lambda_2$ implies that $(1-\eta)/\lambda_1 - 1/\lambda_2 < 0$.

Since $\lambda_i^{-1} - \lambda_2^{-1} < 0$ for all $i > 2$, we also have that, for all $i > 2$, $B(i,n_i(m(j)))/B(2,n_2(m(j))) \to 0$ as $j \to \infty$.

Therefore we can conclude that $G$ is a stable distribution with characteristic exponent $\lambda_2$. □

The proofs of the Corollaries to Theorems 3 and 4 are obvious.

PROOF OF THEOREM 5. Note that

$$B(2,n_2(n))/B(1,n_1(n)) \sim s_2^{\lambda_2^{-1}} s_1^{-\lambda_1^{-1}} L_2(n)/L_1(n^{\lambda_1/\lambda_2}).$$

(i) $L_2(n)/L_1(n^{\lambda_1/\lambda_2}) \to 0$ as $n \to \infty$ implies that $B(2,n_2(n))/B(1,n_1(n)) \to 0$ as $n \to \infty$. Since $\lambda_i^{-1} - \lambda_2^{-1} < 0$, for all $i > 2$, we have that there exists a sequence $\{m(j)\}$ of positive integers such that, for all $i > 2$, $B(i,n_i(m(j)))/B(1,n_1(m(j))) \to 0$ as $j \to \infty$. This concludes (i).

(ii) $L_2(n)/L_1(n^{\lambda_1/\lambda_2}) \to \infty$ as $n \to \infty$ implies that $B(1,n_1(n))/B(2,n_2(n)) \to 0$ as $n \to \infty$. For all $i > 2$, there exists a sequence $\{m(j)\}$ of positive integers such that $B(i,n_i(m(j)))/B(2,n_2(m(j))) \to 0$ as $j \to \infty$. This concludes (ii).

(iii) Let $a = \liminf_{n\to\infty} L_2(n)/L_1(n^{\lambda_1/\lambda_2})$ and $b = \limsup_{n\to\infty} L_2(n)/L_1(n^{\lambda_1/\lambda_2})$. We first wish to prove that $b < \infty$. Suppose to the contrary that $b = \infty$. We have $a < \infty$. Then select a sequence $\{m_1(j)\}$ of positive integers such that

$$L_2(m_1(j))/L_1(m_1(j)^{\lambda_1/\lambda_2}) \to \infty$$

as $j \to \infty$. Hence, $B(1, n_1(m_1(j)))/B(2, n_2(m_1(j))) \to 0$ as $j \to \infty$. For all $i > 2$, $B(i, n_i(m_1(j)))/B(2, n_2(m_1(j))) \to 0$ as $j \to \infty$. Therefore, $G$ is a stable distribution with characteristic exponent $\lambda_2$. However, since $\{L_2(n)/L_1(n^{\lambda_1/\lambda_2})\}$ does not converge to $\infty$, we can also select a sequence $\{m_2(j)\}$ of positive integers such that

$$L_2(m_2(j))/L_1(m_2(j)^{\lambda_1/\lambda_2}) \to a < \infty$$

as $j \to \infty$. Hence, $B(2, n_2(m_2(j)))/B(1, n_1(m_2(j))) \to s_2^{\lambda_2^{-1}} s_1^{-\lambda_1^{-1}} a$ as $j \to \infty$. For all $i > 2$, $B(i, n_i(m_2(j)))/B(1, n_1(m_2(j))) \to 0$ as $j \to \infty$. If $a = 0$, then $G$ is a stable distribution with characteristic exponent $\lambda_1$, and if $a > 0$, then $G$ is a convolution of two stable distributions with characteristic exponents $\lambda_1$ and $\lambda_2$. If $a \neq b = \infty$, there are two possible combinations: (i) $b = \infty$ and $a = 0$, (ii) $b = \infty$ and $0 < a < \infty$. Under (i), we have a contradiction since $b = \infty$ implies that $G$ is a stable distribution with characteristic exponent $\lambda_2$, while $a = 0$ implies that $G$ is a stable distribution with characteristic exponent $\lambda_1$. Also, we have a contradiction under (ii) since $0 < a < \infty$ implies that $G$ is a convolution of two stable distributions with characteristic exponents $\lambda_1 \neq \lambda_2$.

Therefore $b < \infty$. We next show that $a > 0$.

Suppose to the contrary that $a = 0$. We have $0 < b < \infty$. Select a sequence $\{m_1(j)\}$ of positive integers such that $L_2(m_1(j))/L_1(m_1(j)^{\lambda_1/\lambda_2}) \to 0$ as $j \to \infty$. Hence, $B(2, n_2(m_1(j)))/B(1, n_1(m_1(j))) \to 0$ as $j \to \infty$. For all $i > 2$,

$$B(i, n_i(m_1(j)))/B(1, n_1(m_1(j))) \to 0$$

as $j \to \infty$. Therefore, $G$ is a stable distribution with characteristic exponent $\lambda_1$. However, we can also select a sequence $\{m_2(j)\}$ of positive integers such that $L_2(m_2(j))/L_1(m_2(j)^{\lambda_1/\lambda_2}) \to b$ as $j \to \infty$. Hence, $B(2, n_2(m_2(j)))/B(1, n_1(m_2(j))) \to s_2^{\lambda_2^{-1}} s_1^{-\lambda_1^{-1}} b$, as $j \to \infty$, which is positive and finite. For all $i > 2$,

$$B(i, n_i(m_2(j)))/B(1, n_1(m_2(j))) \to 0$$

as $j \to \infty$. Therefore, $G$ is a convolution of two stable distributions with characteristic exponents $\lambda_1$ and $\lambda_2$. Again, we have a contradiction since $\lambda_1 \neq \lambda_2$.

Therefore, $0 < a \leqq b < \infty$. Hence, we can select a sequence $\{m(j)\}$ of positive integers such that $L_2(m(j))/L_1(m(j)^{\lambda_1/\lambda_2}) \to a$, as $j \to \infty$, which is positive and finite. As above, this implies that $G$ is a convolution of two stable distributions with characteristic exponents $\lambda_1$ and $\lambda_2$. This concludes (iii). □

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