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ON THE ABSOLUTE CONTINUITY OF MEASURES[1]

BY TYRONE E. DUNCAN

The University of Michigan

1. Introduction. The problem of determining absolute continuity of measures on function spaces has been investigated for some time. Much effort has been devoted to the problem of obtaining criteria for the absolute continuity of Gaussian measures for example [13, 16, 19, 21, 23]. A well-known dichotomy for absolute continuity of Gaussian measures exists [5, 7], and some useful complete results exist for certain Gaussian measures. More recently there has been interest in obtaining conditions for absolute continuity of measures which correspond to solutions of stochastic differential equations.

We shall consider the problem of absolute continuity for processes with a continuity property on certain sub-σ-fields of the processes and indicate some simple structure on the Radon-Nikodym derivative by using the Doob-Meyer results for decomposition of supermartingales [15].

Our results will simplify and clarify some results for absolute continuity for measures corresponding to solutions of stochastic differential equations and for Gaussian measures equivalent to Wiener measure. For Gaussian measures equivalent to Wiener measure we shall relate the Gaussian process to Brownian motion via a stochastic differential equation. In this manner we obtain a "nice" transformation of Brownian motion to the other Gaussian process. The Radon-Nikodym derivative is also conveniently expressed.

2. Some general comments. When determining absolute continuity of measures we typically start from either discrete time or from some continuous time results where we know we have absolute continuity, and then try to take an appropriate limit. We have then either a martingale sequence or a martingale net, and in both cases we have necessary and sufficient conditions for absolute continuity in terms of uniform integrability (Doob [2], Helms [8]); in other words, conditions that the limit be a martingale. However, in the case of a martingale net we cannot immediately assert pointwise convergence (Dieudonné [1], Helms [8]). The importance of uniform integrability is also seen in the supermartingale work of Meyer [15].

Before discussing some results on absolute continuity, a few preliminaries will be useful. We first give a characterization of uniform integrability due to LaVallee Poussin (cf. Meyer [15]).

THEOREM 1. *Let H be a subset of L^1 $(\Omega, \mathscr{F}, P)$. The following properties are equivalent.*

(1) *H is uniformly integrable.*

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(2) *There exists a function* $G(t)$ *defined on* $R^+ = [0, \infty)$ *which is positive and increasing such that*

(1) $$\lim_{t\to+\infty} G(t)/t = +\infty$$ *and*

(2) $$\sup_{f\in H} E[G\circ|f|] < \infty.$$

For (1) $\Rightarrow$ (2) we can also assert that G is convex.

We shall usually obtain a family of sub-σ-fields $(\mathscr{F}_t)$ of the σ-field $\mathscr{F}$ of the probability space $(\Omega, \mathscr{F}, P)$ from the process $X = (X_t)$ by letting $\mathscr{F}_t = \mathscr{B}(X_u, u \leqq t)$, which is the sub-$\sigma$-field generated by the process X to time t. We shall assume that all sub-σ-fields are augmented, i.e., contain all the null sets of the measure P. We call the family of sub-σ-fields $(\mathscr{F}_t)$ right continuous if $\mathscr{F}_t = \mathscr{F}_{t+}$ where

(3) $$\mathscr{F}_{t+} = \bigcap_{s>t} \mathscr{F}_s.$$

Continuity of the sub-σ-fields is defined in the obvious manner.

We shall also state a result for the decomposition of right-continuous supermartingales due to Meyer [15]. The definitions of the various terms in the theorem can be found in Meyer's book [15].

THEOREM 2. *A right-continuous supermartingale* (X_t) *has a Doob decomposition*

(4) $$X_t = M_t - A_t$$

where (M_t) *denotes a right-continuous martingale and* (A_t) *an increasing process, if and only if* (X_t) *belongs to class (DL). There then exists a decomposition* (4) *for which the process* (A_t) *is natural, and this decomposition is unique.*

3. A characterization of the Radon-Nikodym derivative. A simple characterization will now be given for the Radon-Nikodym derivative $d\mu_1/d\mu_0$ where μ_0 and μ_1, are defined on Ω with σ-field $\mathscr{F}$. We shall also establish certain integrability properties of the two terms in the Radon-Nikodym derivative along with a relation between the two terms.

THEOREM 3. *Let* $\mu_1 \ll \mu_0$ *and* $(\Omega, \mathscr{F})$ *be the measurable space. Let* $(\mathscr{F}_t)$ $t \in [0, 1]$ *be an increasing family of sub-σ-fields of* $\mathscr{F}$ *which are continuous and augmented* w.r.t. μ_0 *and* $\mathscr{F} = \mathscr{F}_1$ *and* $\mathscr{F}_0$ *is the trivial σ-field. Assume that the only martingales on* $(\Omega, \mathscr{F}, \mu_0)$ *are continuous. The Radon-Nikodym derivative,* $M = d\mu_1/d\mu_0$, *has the following characterization on the set where* M *is strictly positive*

(5) $$M = \exp[X - A],$$

where (X_t) *and* (A_t) *are a martingale with continuous sample paths and a natural increasing process respectively, and* $t \in [0, 1]$.

M is strictly positive a.s. *only if* $\mu_0(A < \infty) = 1$.

PROOF. Define

(6) $$M_t = E[M \mid \mathscr{F}_t].$$

By the properties of the Radon-Nikodym derivative and conditional expectation it is clear that $(M_t, \mathscr{F}_t, \mu_0)$ is a martingale. By assumption the martingale admits a continuous modification.

We shall define a sequence of stopping times T_n as follows:

$$T_n = \inf\{t: M_t < 1/n \quad \text{or} \quad M_t > n\}$$
$$= 1 \quad \text{if the above set is empty.}$$

Recall $t \in [0, 1]$. Note $0 \leqq M < \infty$ a.s. μ_0 by properties of the Radon-Nikodym derivative. $(M_{t \wedge T_n}, \mathscr{F}_{t \wedge T_n}, \mu_0)$ is a martingale by the Optional Sampling Theorem (Meyer [15], page 98). Since $M_{t \wedge T_n}$ is bounded above and away from zero for each n, $\ln M_{t \wedge T_n}$ is bounded for each n. Let

$$X_t^{(n)} = \ln M_{t \wedge T_n}. \tag{7}$$

$X_t^{(n)}$ is a supermartingale by Jensen's inequality and in class (D). Using Meyer's results we have

$$X_t^{(n)} = Y_t^{(n)} - A_t^{(n)}. \tag{8}$$

Since $\ln M_{t \wedge T_n}$ is a continuous function of t, $Y_t^{(n)}$ and $A_t^{(n)}$ are continuous.

Now let $m > n$

$$X_t^{(n)} = X_t^{(m)}$$

for $t \leqq T_n$. By the uniqueness of the decomposition we have for $t \leqq T_n$

$$Y_t^{(n)} = Y_t^{(m)}, \qquad A_t^{(n)} = A_t^{(m)}.$$

Let $\Gamma = \{0 < M < \infty\}$. The following limits are well defined for $t \in [0, 1]$.

$$Y_t 1_\Gamma = \lim_n Y_t^{(n)} 1_\Gamma, \tag{9}$$

$$A_t 1_\Gamma = \lim_n A_t^{(n)} 1_\Gamma. \tag{10}$$

If $M > 0$ a.s. then $\mu_0(\Gamma) = 1$ and the characterization given above is valid a.s. By the above decomposition we then have $\mu_0(A_1 < \infty) = 1$. If we have the increasing process A_t defined a.s. then $\mu_0 (A_1 < \infty) = 1$ implies $M > 0$ a.s. □

K. Itô and S. Watanabe [10] obtain a decomposition of a positive supermartingale. The techniques used above are similar to their approach, and developed from discussions with S. Watanabe for characterizing Radon-Nikodym derivatives for solutions of stochastic differential equations (cf. Duncan [3]).

In the case where M is positive (or more generally on this set) we can relate the continuous martingale X and the increasing process A by the following result.

PROPOSITION 1. *Let M be given by* (5) *and assume* $M > 0$ a.s. *Then*

$$A_t = \tfrac{1}{2}\int_0^t d\langle X\rangle_s \tag{11}$$

where $\langle X\rangle$ is the unique increasing process such that the process $X_t^2 - \langle X\rangle_t$ is locally a martingale.

PROOF. Using the formula for stochastic differentials [11], extended by Kunita and S. Watanabe [14] to continuous locally square integrable martingales, we have that

$$M_t = \exp[X_t - A_t] = 1 + \int_0^t M_s\,dX_s - \int_0^t M_s\,dA_s + \tfrac{1}{2}\int_0^t M_s\,d\langle X\rangle_s \tag{12}$$

$$M_t - 1 - \int_0^t M_s\,dX_s = -\int_0^t M_s\,dA_s + \tfrac{1}{2}\int_0^t M_s\,d\langle X\rangle_s.$$

Since the left-hand side of the equation is a martingale and the right-hand side is of bounded variation by Meyer's uniqueness result (cf. Kunita-Watanabe [14]) we have that

$$M_t = 1 + \int_0^t M_s\,dX_s \quad \text{and} \tag{13}$$

$$\int M_s\,dA_s = \tfrac{1}{2}\int M_s\,d\langle X\rangle_s; \quad \text{so} \tag{14}$$

$$A_t = \tfrac{1}{2}\langle X\rangle_t \quad \text{a.s.} \tag{15}$$

since $M > 0$ a.s. and a continuous function of t. □

4. Comments on absolute continuity for some non-Gaussian measures. We shall now review two usual conditions for absolute continuity and see how they arise from uniform integrability. The divergence, J, is defined as

$$J = \int (X_0 - X_1)\log(X_0/X_1)\,d\mu \tag{16}$$

where μ is a measure that is absolutely continuous with respect to both μ_1 and μ_0 (e.g., $\mu = \mu_0 + \mu_1$) and $X_i = d\mu_i/d\mu$, $i = 0, 1$. The entropy of μ_1 with respect to μ_0 is

$$H_{\mu_1}(\mu_0) = \int M \ln M\,d\mu_0 \tag{17}$$

where $M = d\mu_1/d\mu_0$. For the entropy we usually have some discrete time approximations so that the density is well defined, and then we take a limit or a supremum. Note that finiteness of the divergence J is equivalent to finiteness of both the entropies $H_{\mu_1}(\mu_0)$ and $H_{\mu_0}(\mu_1)$. Using our characterization of uniform integrability in Theorem 1 and letting

$$G(t) = t \ln t \tag{18}$$

we see immediately why finiteness of the divergence is a sufficient condition for mutual absolute continuity, and why finiteness of the entropies implies the corresponding absolute continuities.

We shall show how some known results on absolute continuity can be cast in terms of our results to simplify interpretations. Girsanov [6] considers the problem of transforming solutions of stochastic differential equations by obtaining an appropriate absolute continuity relation. He assumes in the general stochastic differential equation case that the function must be bounded. Under this condition he shows that higher moments (than one) are finite, which is a sufficient condition for uniform integrability. Using uniform integrability we can easily state necessary and sufficient conditions for his transformation to work. By specializing the problem Girsanov obtains more general results, and these too seem more transparent from a uniform integrability interpretation.

Shepp [22] considers the problem of absolute continuity for discrete Itô processes, i.e., stochastic difference equations

$$y_n = \phi_{n-1}(y_1, y_2, \cdots, y_{n-1}) + \eta_n \tag{19}$$

where $\{\eta_n\}$ is a sequence of independent standard normal random variables. He notes that a known condition for absolute continuity is that the limit of the likelihood ratios be strictly positive a.s. For this problem it is not difficult to show that the increasing process is $\frac{1}{2}\sum_n \phi_{n-1}^2$. While the σ-fields are not continuous in this case we can still apply the decomposition result of Theorem 3, which in this case is due to Doob [2]. The increasing process is defined a.s. and we obtain Shepp's result that finiteness of the increasing process is necessary and sufficient for absolute continuity.

5. Comments on absolute continuity for some Gaussian measures. We shall now consider the problem of absolute continuity for Gaussian measures and show how our representation for the Radon-Nikodym derivative can be used in these problems. To simplify discussion of the absolute continuity problem we shall consider a Wiener measure as one of the Gaussian measures, though this restriction is not necessary. Necessary and sufficient conditions for equivalence of a Gaussian measure to a Wiener measure have been obtained by Shepp [21] in terms of the mean and the covariance for the Gaussian process.

We shall provide an interpretation of Shepp's theorem using our characterization for the Radon-Nikodym derivative and obtain a stochastic differential equation which relates the Gaussian process to Brownian motion. This will also provide another characterization for a Gaussian measure to be equivalent to a Wiener measure. First though, we give a specific characterization of the Radon-Nikodym derivative. This will specialize the result in Theorem 3.

LEMMA 1. *Let $\mu_Y \sim \mu_B$ where μ_B is the measure for Brownian motion and μ_Y is the measure for the (Gaussian) process Y. Then*

$$d\mu_Y/d\mu_B = \exp\left[\int \phi_s \, dB_s - \tfrac{1}{2}\int \phi_s^2 \, ds\right] \tag{20}$$

where $$\int \phi_s^2 \, ds < \infty \qquad \text{a.s. } \mu_B.$$

PROOF. From Theorem 3 we know that the martingale term in the Radon-Nikodym derivative is locally a square integrable martingale. Since this martingale is defined on the Brownian motion probability space, it can be represented by a stochastic integral. This fact is known (Wentzel [24], Kunita-Watanabe [14], Duncan [3]) and can be proved by using K. Itô's representation of L^2 functionals of Brownian motion in terms of multiple Wiener integrals [12], and then summing the integrals. □

Wentzel considers the homogeneous Markov case and obtains a similar representation for multiplicative functionals with expectation one.

If the process Y in Lemma 1 is a Gaussian process we know that the martingale is square integrable from the finiteness of the divergence.

We shall now proceed to characterize Gaussian measures equivalent to Wiener measures in terms of stochastic differential equations.

THEOREM 4. *Let μ_Y and μ_B be measures for the zero mean Gaussian process Y and for the Brownian motion process B respectively, defined for $t \in [0, 1]$. $\mu_Y \sim \mu_B$ if and only if a process Y with measure μ_Y can be obtained as the solution of*

$$dY_t = \phi_t dt + dB_t \tag{21}$$

where $t \in [0, 1]$, $Y_0 \equiv 0$

$$\phi_t = \int_0^t \alpha(t, s)\, dY_s \qquad \textit{and} \tag{22}$$

$$\int_0^1 \int_0^t \alpha^2(t, s)\, dt\, ds < \infty. \tag{23}$$

PROOF. ($\Rightarrow$) Let $\mu_Y \sim \mu_B$. Then by Lemma 1 we know

$$d\mu_Y / d\mu_B = \exp\left[\int \phi_s dB_s - \tfrac{1}{2} \int \phi_s^{\,2}\, ds\right] \qquad \text{and} \tag{24}$$

$$\int \phi_s^2\, ds < \infty \qquad \text{a.s. } \mu_B. \tag{25}$$

We can then approximate the integrals in the Radon-Nikodym derivative by a sequence of uniformly stepwise processes $\{\varphi_t^{(n)}\}$.

A process $\{C_t\}$ is a uniformly stepwise process for $t \in [0, 1]$ if there exists a subdivision of the interval $[0, 1]$, $0 = t_0 < t_1 < t_1 < \cdots < t_n = 1$, such that

$$C_t(\omega) = C_{t_i}(\omega), \qquad t_i \leqq t < t_{i+1} \qquad i = 0, 1, \cdots, n-1.$$

We shall simplify our computations by approximating the process ϕ by a sequence of uniformly stepwise processes. Using some straightforward bounds on the integrals we can show that the sequence of Radon-Nikodym derivatives will converge both pointwise and in $L^1(d\mu_B)$ and thus the sequence of measures will converge to μ_Y.

When ϕ is a uniformly stepwise process the processes B and Y can be related by the stochastic differential equation

$$dY_t = \phi_t dt + dB_t. \tag{26}$$

Let $\{t_0, t_1, \cdots, t_n\}$ be the partition corresponding to the stepwise process ϕ, i.e.,

$$\phi_t = C_{t_i}, \qquad t_i \leqq t < t_{i+1} \qquad i = 0, \cdots, n-1. \tag{27}$$

Each C_{t_i} can be expressed in terms of multiple Wiener integrals [12], because each C_{t_i} expressed in the Radon-Nikodym derivative is a functional of the past Brownian motion. We note that since Y and B are Gaussian, C_{t_i} will be a Gaussian random variable for each i with zero mean (use stochastic differential equation description). By summing the multiple Wiener integrals that represent C_{t_i} we obtain a stochastic integral. Since each C_{t_i} is Gaussian we know that the increasing process associated with the stochastic integral is not random (e.g., use a random measure argument) and therefore in the expansion we have only the Wiener integral term, since we have assumed that Y has zero mean. Therefore

(28) $$C_{t_i} = \int_0^{t_i} \alpha_{t_i}(s)\, dB_s.$$

By the Gaussian property of Y and B and the fact that $\mu_Y \sim \mu_B$ we have that

(29) $$E_{\mu_B} \int \phi_t{}^2\, dt < \infty.$$

Using the representation (27) for the function ϕ (29) gives

(30) $$\int_0^1 \sum_{i=0}^{n-1} \int^{t_i} \alpha_{t_i}^2(s)\, 1_{[t_i, t_{i+1}]}(t)\, ds\, dt < \infty$$

which in the limit of the sequence of uniformly stepwise processes $\{\phi_t{}^{(n)}\}$ gives a function $\alpha(t,s)$ with

(31) $$\int_0^1 \int_0^t \alpha^2(t,s)\, dt\, ds < \infty.$$

Therefore, for the general case, our stochastic differential equation is

(32) $$dY_t = \phi_t\, dt + dB_t$$

where

(33) $$\phi_t = \int_0^t \alpha(t,s)\, dY_s.$$

Since the function α is Volterra and in $L^2(dt\, ds)$, it is not difficult to verify that the stochastic differential equation has one and only one solution.
($\Leftarrow$) By the integrability condition on α we know that the process Y is a Gaussian process which is a functional of the "past" Brownian motion. Using the dichotomy result for Gaussian measures [5, 7] and the fact that the increasing process is finite a.s. μ_B the desired absolute continuity is established. Alternatively we could show the finiteness of the appropriate entropy or bound α and show that the sequence is uniformly integrable to establish absolute continuity. $\square$

We note that from our representation for the process Y given by (21) we can obtain Shepp's two conditions on the covariance of the process Y, i.e., (i) the integrability condition is satisfied since $\alpha \in L^2(dt\, ds)$ and (ii) the spectrum condition is verified since α is a Volterra kernel [18]. From our result we have the linear transformation of the Brownian motion in a form in which the process Y is a functional only of the "past" Brownian motion. Shepp was unable to obtain this representation. Furthermore, the representation we have obtained is unique, whereas without the restriction that the process Y be a functional only of the "past" Brownian motion, many representations for Y exist, as Shepp has noted.

In Theorem 4 we have assumed that the process Y has zero mean. We can also prove the result due to Segal [20] that the measure of a Gaussian process which is a translate of Brownian motion is equivalent to Wiener measure if and only if the mean value function has a derivative (with respect to Lebesgue measure) which is in $L^2(dt)$. In this case we get only the first term in the Itô expansion [12], which is a constant, and we can proceed as we did in the proof of Theorem 4 to establish that it is necessary and sufficient as Segal has shown. Rao and Varadarajan [17] have shown that if a measure of a nonzero mean Gaussian process is equivalent to another Gaussian measure then the measure of the former Gaussian process with zero mean is also equivalent to the latter Gaussian measure. We can establish this result for a Gaussian measure equivalent to Wiener measure because in the proof of Theorem 4 we would have two terms from Itô's expansion (i) the mean value (ii) the random drift, and obviously each term would have to be a square integrable functional of Brownian motion. It is worth noting that our expression for the Radon-Nikodym derivative (24) is not complicated when a term in the covariance is not of trace class.

REMARK. Since submission of this paper the result in Theorem 4 has appeared in a paper by M. Hitsuda [9].

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