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# CONVEX CONES AND FINITE OPTIMALS

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**1. Introduction.** Among the continuous payoff kernels $K(x, y)$ on the unit square for an infinite zero-sum two person game, the separable kernels, the generalized convex kernels and certain analytic kernels are known to possess optimal mixed strategies for the two players with finite spectrum [5]. It is known that even among $C^{\infty}$ kernels we can have very pathological optimal mixed strategies as their unique optimals [4]. Thus the problem of classifying kernels with finite optimals is unresolved. Here an attempt is made to look at this problem from the topological viewpoint. The binding geometric object between kernels and strategies could be chosen as the cone generated by functions $h_{\alpha}(x) = K(x, \alpha)$ where we fix $\alpha$ and view $K(x, \alpha)$ as a function of $x$. Some of the properties of the cones are reflected in the finiteness of the spectrum for an optimal strategy for a player. Similar versions could be stated for the other player, by considering a related kernel where now the second player becomes the maximizer. Further, these cones in certain other topologies also characterize extreme optimals for a class of games.

*Preliminaries.* Let $X$, $Y$ be compact metric spaces and $K(x, y) > 0$ be a continuous payoff on $X \times Y$. Let $E_X$, $E_Y$ be the Banach space of continuous functions on $X$ and $Y$ with their supremum norm ($\|\cdot\|$). Let $C$ be the closure of the convex cone in $E_X$ generated by functions $h_{\alpha}(x)$, where $h_{\alpha}(x) = K(x, \alpha)$, $\alpha \in Y$. Let $K$ be the cone of nonnegative functions in $E_X$. Let $E_0$ be the linear manifold $C$–$C$ and $\bar{E}_0$ its closure. By a positive operator we mean a linear operator $A$ from $\bar{E}_0 \to E_X$ which maps the cone $C$ into $K$. We would call the image of the cone $C$ under $A$ the range cone. The following is the main theorem.

THEOREM 1. *Let $A$ be a positive operator from $\bar{E}_0 \to E_X$ continuous on the cone $C$. If $A$ is isometric on $C$ and if the range cone has a relative interior point in the closed linear manifold spanned by this cone, then player II has always an optimal mixed strategy whose spectrum is finite. If the cone $C$ itself possesses an interior point relative to $\bar{E}_0$ then the conditions are trivially satisfied for the identity map and in this case the kernel is separable and both players have optimal mixed strategies with finite spectrum.*

PROOF. Let $P_Y$ denote the set of all probability measures on the Borel sets of $Y$. $P_Y$ as a subset of $E_Y{}^*$ (the dual of $E_Y$) is compact metric in its weak topology [6]. Further by Helly's theorem

$$\tau : v \to \int_Y K(x, y)\, dv(y)$$

is continuous from $P_Y$ into $E_X$. Thus $\tau(P_Y) = B$ is compact in $E_X$. Trivially it is convex. Let $T$ be the convex cone generated by $B$, i.e. $T = \{\lambda f : \lambda \geqq 0, f \in B\}$. We

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will show that $T = C$. To see this let $p \in C$, $p \neq 0$. There exists $p_n \to p$, $p_n \neq 0$ where $p_n$ is the form $p_n(x) = \sum_1^{\gamma_n} a_i K(x, a_i)$, $a_i \geqq 0$, $i = 1, 2, \cdots, \gamma_n$. By assumption some $a_i > 0$ and that $\theta_n \cdot p_n \in B$ where $\theta_n = (\sum a_i)^{-1}$. By the compactness of $B$, $\theta_n \cdot p_n$ has a convergent subsequence and without loss of generality $\theta_n \cdot p_n \to p_* \in B$. Since $K(x, y) > 0$, $\|p_*\| > 0$. Also we have $p_n \to p$ and that $\theta_n$'s are bounded. Without loss of generality $\theta_n \to \theta_0 > 0$. Thus $\theta_0 \cdot p = p_*$ and thus $p = \theta_0^{-1} \cdot p_* \in T$. Thus $C \subset T$. Conversely let $q \in T$, $q \neq 0$. For some $\lambda > 0$, $\lambda q \in B$. Thus $q = \lambda^{-1}\tau(v)$ for some $v \in P_Y$. But in $P_Y$ those measures which have finite spectrum are dense [6]. Therefore we have $\lambda^{-1}\tau(v_n) = q_n$, $v_n \to v$, where $v_n$ is a sequence in $P_Y$ where each one has finite spectrum. Now $q_n$ is of the form $q_n(x) = \lambda^{-1} \cdot \sum_1^{\gamma_m} \mu_i K(x, \alpha_i)$ $\mu_i \geqq 0$, $\sum \mu_i = 1$. Thus $q_n \in C$. The continuity of $\tau$ implies $q \in C$. Thus the two cones coincide.

Continuing with our proof let $H = A(B)$ be the image of $B$ under $A$ and by the continuity of $A$, $H$ is compact. Further the range cone is given by $\bigcup_{\lambda \geqq 0} \lambda H$. Let $S$ be the compact convex hull of 0 and $H$. Then the range cone is $\bigcup_{n=1}^{\infty} nS$. Since the range cone has non-null interior in the closed linear span of its elements, by Baire Category theorem some $nS$ and hence $S$ has non-null interior. $S$ being a compact subset of this Banach space, it is finite dimensional. Let $k$ be the dimension of $S$. Then any point of $H$ is the convex linear combination of at most $k + 1$ of its extreme points. Now since $B$ is the closed convex closure of $h_\alpha(x)$, $\alpha \in Y$, the extreme points of $B$ are contained in the compact set $\{h_\alpha(x), \alpha \in Y\}$. The linearity and continuity of $A$ implies that any extreme point of $S$ is the $A$ image of an extreme point of $B$. Consider the kernel $M(x, \alpha)$ where $M(x, \alpha) = A(h_\alpha(x))$. $M(x, \alpha)$ is continuous on $X \times Y$ and $M(x, \alpha)$'s as functions of $x$ generate a finite dimensional space. Thus for this kernel player II has an optimal strategy $v_0$ with finite spectrum. Let $v_0$ be the value of this game with payoff $M(x, \alpha)$. Thus

$$\int_Y M(x, \alpha)\, dv_0(\alpha) \leqq v_0 \quad \text{for all} \quad x.$$

But $A$ maps $C$ into the cone $K$ and thus for any point $h \in B$ where $h(x) = \int_Y K(x, y)\, dv(y)$ for some $v \in P_Y$ we have by linearity and isometry

$$Ah(x) = \int_Y M(x, y)\, dv(y) \quad \text{and} \quad \|Ah\| = \max_x \int_Y M(x, y)\, dv(y)$$
$$= \|h\| = \max_x \int_Y K(x, y)\, dv(y).$$

This shows that

$$\min_v \max_x \int_Y K(x, y)\, dv(y) = \min_v \max_x \int_Y M(x, y)\, dv(y)$$
$$= \max_x \int_Y M(x, y)\, dv_0(y) = v_0.$$

Thus both games have same value $v_0$ and that $v_0$ is optimal also for the original payoff $K(x, y)$. This completes the proof of the main assertion in the theorem. If the cone $C$ possesses an interior point of $\bar{E}_0$, then by the above argument $\bar{E}_0$ is finite dimensional. Let $h_{\alpha_1}, h_{\alpha_2}, \cdots, h_{\alpha_s}$ span $\bar{E}_0$. Let $x, y$ be generic points and

$x_1, x_2, \cdots, x_s, y_1, y_2, \cdots, y_s$ be any fixed set of points with the property (without loss of generality) that the determinant

$$\Delta = \begin{vmatrix} K(x_1, y_1), \cdots, K(x_1, y_s) \\ \vdots \qquad\qquad \vdots \\ K(x_s, y_1), \cdots, K(x_s, y_x) \end{vmatrix} \neq 0.$$

Now $h_y(x) = K(x, y)$ and $h_{\alpha_1} \cdots h_{\alpha_s}$ form a dependent system and that

$$\begin{vmatrix} K(x, y)\, K(x, y_1) & \cdots K(x, y_s) \\ K(x_1, y)\, K(x_1, y_1) & \cdots K(x_1, y_s) \\ \cdots \quad \cdots & \cdots \quad \cdots \\ K(x_s, y)\, K(x_s, y_1) & \cdots K(x_s, y_s) \end{vmatrix} = 0.$$

Expanding the determinant we have

$$\Delta K(x, y) + \sum_i \sum_j K(x, y_i) K(x_j, y) a_{i_j} = 0.$$

Here $a_{i_j}$'s are the suitable cofactors got by deleting the column $i+1$ and row $j+1$ in the above determinant. Since $\Delta \neq 0$, we have

$$K(x, y) = -\sum_i \sum_j K(x, y_i) \cdot K(x_j, y) a_{i_j} / \Delta,$$

i.e. $K(x, y)$ is separable. But in this case we know that player I possesses an optimal strategy with finite spectrum [5]. The following theorem relates the cone $C$ to extreme optimals of player I.

THEOREM 2. *Let for some optimal mixed strategy $v$ for player II the spectrum $\sigma(v)$ of $v$ be the whole space $Y$. Then an optimal strategy $\mu$ with spectrum $X_0$ for player I is extreme if and only if the cone $C$ generates a dense linear manifold in $L_1(X_0, B, \mu)$ where we view elements of $C$ as functions restricted to $X_0$.*

The proof follows closely the proof of this theorem for finite games [1]. Also it follows from a more general theorem of Douglas [3]. Since the dual of $L_1$ is the space of $\mu$-essentially bounded functions, the proof by contradiction uses separation theorem to get such a function and treating this as the Radon–Nykodym derivative we arrive at a contradiction to the extremity of $\mu$ if $C$–$C$ is not dense in $L_1(X_0, B, \mu)$. The converse is trivial.

The following two examples illustrate the existence of isometries for non-separable kernels to which our main theorem is applicable.

EXAMPLE 1. Let $K(x, y) > 0$ be continuous and convex in $x$ for each $y$ in $0 \leqq x, y \leqq 1$. For any $\alpha$ in $0 \leqq \alpha \leqq 1$ let the function $h_\alpha(x) = K(x, \alpha)$ be mapped to the function $g_\alpha(x) = (1-x)K(0, \alpha) + xK(1, \alpha)$. Since the maximum for a convex function is either at 0 or 1 we have $\|h_\alpha\| = \|g_\alpha\|$. Further for any finite $\lambda_1, \lambda_2, \cdots, \lambda_k \geqq 0$, $0 \leqq \alpha_1, \alpha_2, \cdots, \alpha_k \leqq 1$, $\sum \lambda_i k_{\alpha_i}$ is convex and that if we define

$$A: \sum \lambda_i h_{\alpha_i} \to \sum \lambda_i g_{\alpha_i}$$

then we check that $A$ is an isometry and it is linear and continuous in $\bar{C}$. The range of $A$ consists of functions of the form $a+bx$ and that the range Cone has relative interior. Thus continuous kernels on the unit square which are convex in $x$ for each $y$ have finite optimals for player II (of course also for player I).

EXAMPLE 2. Instead of demanding convexity in $x$ we could weaken the condition by demanding $K(x, y) \leqq (1-x)K(0, y)+xK(1, y)$ for all $x, y$ in the above example. The same argument works.

In these two examples one could see the finite optimals directly by domination arguments. It would be interesting to know whether we could construct such isometries of the theorem for some of the known kernels with finite optimals for player II, such as the generalized concave kernels of Karlin and certain Cauchy–Bell shaped kernels.

Thus it seems that the cone $C$ and its topological structure could give us some guidance in our problem of classification.

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