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JOINT DISTRIBUTION OF THE EXTREME ROOTS OF A COVARIANCE MATRIX[1]

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1. Introduction and result. The purpose of this note is to find the joint distribution and the distribution of the ratio of the largest root and the smallest root of a sample covariance matrix when the population covariance matrix is a scalar matrix, $\mathbf{\Sigma} = \sigma^2\mathbf{I}$. The main result in this paper is the following

THEOREM. *Let* $\mathbf{S}$ *be a* $(p \times p)$ *matrix having a Wishart distribution* $\mathbf{W}(p, n, \mathbf{I})$, *and* $\lambda_1, \lambda_2, \cdots, \lambda_p$ $(\infty > \lambda_1 > \lambda_2 > \cdots > \lambda_p > 0)$ *be the latent roots of the matrix* $\mathbf{S}$. *Then the distribution of* $x = 1-\lambda_p/\lambda_1$ *is given by*

$$\begin{aligned} f(x) = C(p) \cdot \textstyle\sum_{k=0}^{\infty} \sum_{\kappa} (\Gamma(pn/2+k)/p^k k!) \\ (1) \qquad \cdot \textstyle\sum_{s=0}^{\infty} (((p-1)(p+2)/2+k+s)/s!)x^{(p-1)(p+2)/2+k+s-1} \\ \cdot \textstyle\sum_{\sigma,\delta} g_{\kappa,\sigma}^{\delta} (((p+1-n)/2)_\sigma((p+2)/2)_\delta/(p+1)_\delta) C_\delta(\mathbf{I}_{p-1}) \end{aligned}$$

where $1 > x > 0$, the subscript κ is usual partition of the integer k not more than p parts, the subscript σ and δ are the partitions of the integers s and $k+s$ into not more than $p-1$ parts respectively, the summation $\sum_{\sigma,\delta}$ is over all combinations of these partitions, and the constant

$$C(p) = \pi^{p/2} B_{(p-1)}(p/2, (p+2)/2)/P^{pn/2}\Gamma(p/2)\Gamma_p(n/2).$$

We notice that g-coefficients come from

$$C_\kappa(\mathbf{L})C_\sigma(\mathbf{L}) = \textstyle\sum_\delta g_{\kappa,\sigma}^{\delta} C_\delta(\mathbf{L})$$

tabulated up to the 7th degree in Khatri and Pillai [2].

Consider the sphericity test, $H_0 : \mathbf{\Sigma} = \sigma^2\mathbf{I}$, where σ^2 is unspecified. For the test criteria, we may suggest the likelihood ratio criterion of the geometric mean and the arithmetic mean, $\prod \lambda_i^{1/p}/(\sum\lambda_i/P)$, and also the ratio, $\lambda_p/\lambda_1 \uparrow = 1-X$, of the largest root λ_1 and the smallest root λ_p, identically $(\lambda_1\lambda_p)^{\frac{1}{2}}/((\lambda_1+\lambda_p)/2)$. The joint distribution of the roots λ_1 and λ_p given by (8) is associated with the problems of finding confidence bounds. (See Roy and Gnanadesican [3], and Anderson [1].)

2. Joint distribution and the distribution of the ratio of the largest root and the smallest root. Let $\mathbf{S}$ be the same matrix as before. The joint distribution of the latent roots $\lambda_1, \cdots, \lambda_p$ of the matrix $\mathbf{S}$ is written as follows

$$(2) \qquad f_1(\lambda_1, \cdots, \lambda_p) = C\,|\mathbf{\Lambda}|^{(n-p-1)/2} \exp(\mathrm{tr}(-\tfrac{1}{2}\mathbf{\Lambda})) \textstyle\prod_{i<j}(\lambda_i - \lambda_j)$$

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where the matrix $\Lambda = \text{diag}(\lambda_1, \cdots, \lambda_p)$, $\infty > \lambda_1 > \cdots > \lambda_p > 0$, and the constant

$$C = \pi^{p^2/2}/2^{np/2}\Gamma_p(p/2)\Gamma_p(n/2).$$

Let

$$l_i = (\lambda_1 - \lambda_i)/\lambda_1, \qquad i = 2, \cdots, p.$$

Then we get the joint distribution of the largest root λ_1 and $l_2, \cdots, l_p$

$$\begin{aligned} f_2(\lambda_1, l_2, \cdots, l_p) &= C \cdot \exp(-\tfrac{1}{2}p\lambda_1) \\ &\cdot \textstyle\sum_{k=0}^{\infty} \sum_{\kappa} (\lambda_1{}^{pn/2+k-1}/2^k k!)\, d\lambda_1 \,|\Lambda_l|\; |I-\Lambda_l|^{(n-p-1)/2} C_\kappa(\Lambda_l) \\ &\cdot \textstyle\prod_{i<j}(l_i - l_j) \end{aligned} \tag{3}$$

where $\infty > \lambda_1 > 0$, the matrix $\Lambda_l = \text{diag}(l_p, \cdots, l_2)$, and $1 > l_p > \cdots > l_2 > 0$.

To get the joint distribution of λ_1 and l_p, we have to integrate (3) over the region $l_p > l_{p-1} > \cdots > l_2 > 0$. We use the fact that

$$\begin{aligned} |I-\Lambda_l|^{(n-p-1)/2} C_\kappa(\Lambda_l) &= \textstyle\sum_{s=0}^{\infty} \sum_{\sigma} (p+1-n)_\sigma\, C_\sigma(\Lambda_l) C_\kappa(\Lambda_l)/s! \\ &= \textstyle\sum_{s=0}^{\infty} \sum_{\sigma} \sum_{\delta} (p+1-n)/2)_\sigma\, g_{\sigma,\kappa}^{\delta}\, C_\delta(\Lambda_l)/s! \end{aligned}$$

Let $r_i = l_i/l_p$, $i = 2, \cdots, p-1$. Integrating with respect to $r_2, \cdots, r_{p-1}$, we can express the part involving the subscript in the formula (3) as follows:

$$\begin{aligned} &\textstyle\sum_{s=0}^{\infty} \sum_{\sigma} \sum_{\delta} (g_{\kappa,\sigma}^{\delta} (p+1-n)/2)_\sigma\, l_p{}^{(p-1)(p+2)/2+k+s-1}/s!) \\ &\quad \textstyle\int_{1>r_{p-1}>\cdots>r_2>0} |\Lambda_r|\, C_\delta({}^1\Lambda_r) \prod_{i=2}^{p-1}(1-r_i) \prod_{i<j}(r_i - r_j) \prod_{i=2}^{p-1} dr_i. \end{aligned} \tag{4}$$

Evaluating the above integration by the lemma due to Sugiyama [4], we get

$$\begin{aligned} &\frac{\Gamma_{(p-1)}((p-1)/2)}{\pi^{(p-1)^2/2}} \\ &\cdot \textstyle\sum_{s=0}^{\infty} \sum_{\sigma} \sum_{\delta} (g_{\kappa,\sigma}^{\delta}((p+1-n)/2)_\sigma\, l_p{}^{(p-1)(p+2)/2+k+s-1}/s!) \\ &\cdot ((p-1)(p+2)/2+k+s) \\ &\cdot (\Gamma_{(p-1)}((p+2)/2, \delta)\Gamma_{(p-1)}(p/2)/\Gamma_{(p-1)}(p+1, \delta))C_\delta(I_{p-1}). \end{aligned} \tag{5}$$

Let $(a)_\delta = \prod_{i=1}^{p-1}(a-(i-1)/2)_{\delta_i}$, $\delta = (\delta_1, \cdots, \delta_{p-1})$ such that $\delta_1 \geqq \cdots \geqq \delta_{p-1} \geqq 0$ and $\sum_{i=1}^{p-1} \delta_i = k+s$. Since $(a)_\kappa = \Gamma_p(a, \kappa)/\Gamma_p(a)$, we obtain from (5) and (3) the following joint distribution of λ_1 and l_p:

$$\begin{aligned} f_3(\lambda_1, l_p) &= C(2) \cdot \exp(-\tfrac{1}{2}p\lambda_1) \textstyle\sum_{k=0}^{\infty} \sum_{\kappa} (\lambda_1{}^{pn/2+k-1}/2^k k!) \\ &\cdot \textstyle\sum_{s=0}^{\infty} (((p-1)(p+2)/2+k+s)/s!) l_p{}^{(p-1)(p+2)/2+k+s-1} \\ &\cdot \textstyle\sum_{\sigma,\delta} g_{\kappa,\sigma}^{\delta}(((p+1-n)/2)_\delta((p+2)/2)_\delta/(p+1)_\delta) C_\delta(\mathbf{I}_{p-1}) \end{aligned} \tag{6}$$

where $\infty > \lambda_1 > 0$, $1 > l_p > 0$, and

$$C(2) = \pi^{p/2} B_{(p-1)}(p/2, (p+2)/2)/2^{pn/2}\Gamma(p/2)\Gamma_p(n/2).$$

Now integrating (6) with respect to λ_1, we obtain the distribution of the statistic

$l_p = 1-\lambda_p/\lambda_1 = x$, namely the distribution $f(x)$ in the theorem. Since $l_p = (\lambda_1-\lambda_p)/\lambda_1$, from (6) we have the joint distribution of λ_1 and λ_p

$$f_4(\lambda_1, \lambda_p) = C(2)\cdot\exp(-\tfrac{1}{2}p\lambda_1)\textstyle\sum_{k=0}^{\infty}\sum_{\kappa}(\lambda_1{}^{pn/2+k-2}/2^k k!)$$
$$\cdot\textstyle\sum_{s=0}^{\infty}(((p-1)(p+2)/2+k+s)/s!)(1-\lambda_p/\lambda_1)^{(p-1)(p+2)/2+k+s-1}$$
$$\cdot\textstyle\sum_{\sigma,\delta} g_{\kappa,\sigma}^{\delta}(((p+1-n)/2)_\delta/((p+1/2)_\delta/(p+1)_\delta)C_\delta(\mathbf{I}_{p-1})$$

where $\infty > \lambda_1 > \lambda_p > 0$. We note that if $(p+1-n)/2$ is an integer, the summation of s will be terminated in a finite number of terms.

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