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FURTHER RESULTS ON MINIMUM VARIANCE UNBIASED ESTIMATION AND ADDITIVE NUMBER THEORY

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1. Introduction. Patil [4] and [5] has investigated problems of existence of minimum variance unbiased (MVU) estimators of parametric functions for the univariate and multivariate power series distributions (PSD) in terms of the number theoretic structure of their respective ranges. A uniform technique of obtaining the MVU estimators, when they exist, has also been provided. We propose to present here certain additional investigations in this area. Introduction of the D and E sets helps develop an interestingly neat picture of the seemingly complicated structural situations and problems. We shall use essentially the same notation and terminology as in Patil [4] and [5]. The next section provides supplemental notation and terminology and quotes certain results that will be used in the text.

2. Notation and terminology. (i) Let I_s denote the s-fold cartesian product of the set I of nonnegative integers with itself. In general, let $\prod_{i=1}^{s} T_i$ denote the cartesian product of the s sets $T_1, T_2, \cdots, T_s$.

(ii) Let $A \subset I_s$ and $B \subset I_s$. A is called a *basis* for B if the n-fold sum of A defined by $n[A] = \{\sum_{i=1}^{n} \mathbf{a}_i, \mathbf{a}_i \in A\}$ is equal to B for some n. In such a case, n is called an *order* of the *basis* A for B.

(iii) Let $T = \prod_{i=1}^{s} T_i$, $T_i \subset I$. The displaced set $D(T)$ is defined as $D(T) = T - \{(\min(T_1), \min(T_2), \cdots, \min(T_s))\}$. Thus, $D(T)$ is the difference between T and the singleton $\{(\min(T_1), \min(T_2), \cdots, \min(T_s))\}$. It is clear that $D(T) = D(\prod_{i=1}^{s} T_i) = \prod_{i=1}^{s} D(T_i)$.

(iv) For an arbitrary subset T of I_s and $\mathbf{x} \in I_s$, $D_x(T)$ denotes the difference set $T - \{\mathbf{x}\}$.

(v) Let $T \subset I_s$. $\mathbf{a} \in T$ is called a lower boundary point (lbp) of T if there is no $\mathbf{x} \in T, \mathbf{x} \neq \mathbf{a}$, such that the ith components satisfy the inequality $x_i \leqq a_i$, for $i = 1, 2, \cdots, s$. The lower boundary (LB) of T is defined to be the set consisting of the lbp's of T and is denoted by LB(T). It is clear that, if $\mathrm{LB}(T) \ni \mathbf{a} \neq \mathbf{b} \in \mathrm{LB}(T)$, then $a_i < b_i$ and $a_j > b_j$ for *some* pair $i \neq j$.

(vi) Let $\mathbf{0} \in A \subset I_s$. Then, the Kvarda–Schnirelmann density $d(A)$ of the set A is defined to be

$$d(A) = \underset{R}{\mathrm{glb}} \frac{A(R)}{I_s(R)}$$

where the glb is taken over all finite subsets R of I_s, excluding $\{0\}$ and the empty set, with the property that if $\mathbf{r} \in R$, $\mathbf{x} \in I_s$ and $x_i \leqq r_i$, $i = 1, 2, \cdots, s$, then $\mathbf{x} \in R$ and

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where $B(R)$ = the number of elements other than the null element (origin) in $B \cap R$. It is clear that $0 \leqq d(A) \leqq 1$ and that $D(A) = 1$ iff $A = I_s$. Also, when $s = 1$, the Kvarda–Schnirelmann density is identical with the Schnirelmann density (Kvarda [2] and Schnirelmann [7]).

(vii) KVARDA'S THEOREM (Kvarda [2]). *If α is the Kvarda–Schnirelmann density of $A \subset I_s$ and $\alpha > 0$, then there exists an integer $n > 0$ such that $n[A] = I_s$.*

(viii) An s-variate random vector $\mathbf{x}$ with probability function

$$p(\mathbf{x}; \boldsymbol{\theta}) = a(\mathbf{x}) \prod \theta_i^{x_i} / f(\boldsymbol{\theta}) \qquad \mathbf{x} \in T$$

is said to have the s-variate PSD with range $T \subset I_s$ and the series function $f(\boldsymbol{\theta}) = \sum a(\mathbf{x}) \prod \theta_i^{x_i}$, where the coefficient function $a(\mathbf{x}) > 0$ for $\mathbf{x} \in T$. $\prod$ stands for the product on $i = 1, 2, \cdots, s$ and the s-dimensional parameter space Ω consists of the parametric vectors $\boldsymbol{\theta} = (\theta_1, \theta_2, \cdots, \theta_s)$ with nonnegative components for which $f(\boldsymbol{\theta})$ is positive, finite and differentiable.

3. Univariate power series distributions. To begin with, we offer an alternative interesting proof to Theorem 2 of Patil [4].

THEOREM 1. *A necessary and sufficient condition for the parameter θ of a* PSD *with range T to be* MVU *estimable is that the displaced set $D(T)$ be a basis of I, i.e., $n[D(T)] = I$ for some n.*

PROOF. We know from the Lemma in Patil [4] page 1051 that a necessary and sufficient condition for the parameter θ of a PSD with range T to be MVU estimable on the basis of a single observation is that $D(T) = I$.

Now, the sample sum $z = \sum_{i=1}^n x_i$ is a complete and sufficient statistic for θ and has a PSD with parameter θ and range $n[T]$. Therefore a necessary and sufficient condition for the parameter θ of a PSD with range T to be MVU estimable on the basis of a random sample of size n is that $D(n[T]) = I$. But,

$$\begin{aligned} D(n[T]) &= n[T] - \{\min(n[T])\} = \{\textstyle\sum_{i=1}^n x_i - \min(n[T]),\ x_i \in T\} \\ &= \{\textstyle\sum_{i=1}^n x_i - n \cdot \min(T),\ x_i \in T\} = \{\textstyle\sum_{i=1}^n [x_i - \min(T)],\ x_i \in T\} \\ &= n[D(T)]. \end{aligned}$$

Hence, the statement of the theorem.

THEOREM 2. *If the parametric function $g(\theta)$ admits a power series expansion in θ, a sufficient condition for $g(\theta)$ to be* MVU *estimable for a sample size n is that the parameter θ of the* PSD *be* MVU *estimable for the sample size n.*

PROOF. Since θ is MVU estimable for the sample size n, $D(n[T]) = I$, i.e. $n[T] + \{1\} \subset n[T]$ and, therefore $n[T] + \{r\} \subset n[T]$ for arbitrary positive integer r, since, $n[T] + \{k\} = n[T] + \{1\} + \{k-1\}$ and $A \subset B$ implies $A + \{k\} \subset B + \{k\}$.

Thus, it follows that the index-set of the product $g(\theta) f_n(\theta)$ is a subset of $n[T]$, the index-set of the series function $f_n(\theta) = [f(\theta)]^n$ of $z = \sum_{i=1}^n x_i$, the sample sum. Hence, the statement of the theorem follows from Theorem 4 in Patil [4].

REMARK 1. The condition in Theorem 2 that $g(\theta)$ admit a power series expansion cannot be replaced by a weaker condition that $g(\theta)f_n(\theta)$ have a power series expansion. For example, consider the case when the series function $f(\theta) = \theta^2+\theta^3+\cdots$, for $0<\theta<1$, $g(\theta) = 1/\theta$ and $n = 1$. Here, θ is actually MVU estimable for every n; whereas, $g(\theta)$ is *not* for any n.

REMARK 2. That MVU estimability of θ is not a necessary condition for MVU estimability of $g(\theta)$ having a power series expansion is shown by the example, when the series function $f(\theta) = \theta^2+\theta^4+\theta^6+\cdots$, for $0<\theta<1$, $g(\theta) = \theta^2$ and $n = 1$. It is clear that, while θ^2 is MVU estimable for every n, θ is *not* for any n.

REMARK 3. If T is finite, then neither θ nor a power series function $g(\theta)$, and in particular the series function $f(\theta)$, is MVU estimable for any sample size. An interesting example, when T is infinite but $f(\theta)$ is not MVU estimable for any sample size, is given by $T = \{1, 2, 2^2, 2^3, \cdots\}$. We have $g(\theta) = f(\theta)$ and $W[g(\theta)f_n(\theta)] = (n+1)[T]$ is not a subset of $n[T] = W[f_n(\theta)]$ for any finite n. To prove this, we note that $2^{n+1}-1 = 1+2+\cdots+2^n \in (n+1)[T]$, but $\notin n[T]$, because of the uniqueness of the representation of a number as a polynomial in 2 with nonnegative coefficients.

THEOREM 3. *The parameter θ of a* PSD *with range T is* MVU *estimable for $n \geqq [[1/\alpha]]$, where $[[m]]$ denotes the smallest integer not less than m and, where, $\alpha = d(D(T))$.*

PROOF. The proof is clear from the following two theorems of the additive number theory. (i): A necessary and sufficient condition for a set A of nonnegative integers to be identical with I is that $d(A) = 1$. (ii): If $d(A) = \alpha > 0$, then $d(n[A]) \geqq \inf(n\alpha, 1)$, (Mann [3]).

4. Multivariate PSD's with cartesian products for the range.

THEOREM 4. *For $i = 1, 2, \cdots, s$, a necessary and sufficient condition for the parameter θ_i of the multivariate* PSD *with range $T = \prod_{i=1}^{s} T_i$, $T_i \subset I$, to be* MVU *estimable is that the set $D(T_i)$ be a basis of I, i.e. $n[D(T_i)] = I$ for some n.*

PROOF. Without loss of generality, let $i = 1$. Next we note that

$$n[T] = \prod_{i=1}^{s} n[T_i].$$

Also, we have from Theorem 4 in Patil [5] that

$$\begin{aligned}\theta_1 \text{ is estimable } &\Leftrightarrow n[T]+\{(1, 0, 0, \cdots, 0)\} \subset n[T] \quad \text{for some } n.\\ &\Leftrightarrow (n[T_1]+\{1\})X\prod_{i=2}^{s} T_i \subset \prod_{i=1}^{s} n[T_i]\\ &\Leftrightarrow n[T_1]+\{1\} \subset n[T_1]\\ &\Leftrightarrow D(n[T_1]) = I\\ &\Leftrightarrow n[D(T_1)] = I.\end{aligned}$$

Hence the statement of the theorem.

THEOREM 5. *A necessary and sufficient condition for the parametric function* $\prod_{i=1}^{s}\theta_i$ *of the multivariate* PSD *with range* $T=\prod_{i=1}^{s}T_i$, $T_i\subset I$, *to be* MVU *estimable for a sample size n is that the* θ_i*'s are individually estimable for the sample size n for* $i=1,2,\cdots,s$.

PROOF. We have from Theorem 4 in Patil [5] that for sample size n

$$\begin{aligned}
\textstyle\prod_{i=1}^{s}\theta_i \text{ is estimable } &\Leftrightarrow n[T]+\{(1,1,\cdots,1)\}\subset n[T]\\
&\Leftrightarrow \textstyle\prod_{i=1}^{s}(n[T_i]+\{1\})\subset\prod_{i=1}^{s}n[T_i]\\
&\Leftrightarrow n[T_i]+\{1\}\subset n[T_i] \qquad \text{for } i=1,2,\cdots,s.\\
&\Leftrightarrow D(n[T_i])=I \qquad \text{for } i=1,2,\cdots,s.\\
&\Leftrightarrow n[D(T_i)]=I \qquad \text{for } i=1,2,\cdots,s.
\end{aligned}$$

Hence, the present theorem follows in view of the preceding Theorem 4.

THEOREM 6. *If the parametric function* $g(\boldsymbol{\theta})$ *admits a power series expansion in* $\theta_1,\theta_2,\cdots,\theta_s$, *a sufficient condition for* $g(\boldsymbol{\theta})$ *to be* MVU *estimable for a sample size n is that the parametric function* $\prod_{i=1}^{s}\theta_i$ *of the multivariate* PSD *with range* $T=\prod_{i=1}^{s}T_i$, $T_i\subset I$, *be* MVU *estimable for the sample size n.*

PROOF. For sample size n,

$$\begin{aligned}
\textstyle\prod_{i=1}^{s}\theta_i \text{ is estimable } &\Rightarrow n[T_i]+\{1\}\subset n[T_i] \qquad \text{for } i=1,2,\cdots,s.\\
&\Rightarrow n[T_i]+\{r_i\}\subset n[T_i]\\
&\qquad \text{for arbitrary } r_i\in I \text{ for } i=1,2,\cdots,s.\\
&\Rightarrow \textstyle\prod_{i=1}^{s}(n[T_i]+\{r_i\})\subset\prod_{i=1}^{s}n[T_i]\\
&\qquad \text{for arbitrary } r_i\in I.\\
&\Rightarrow \textstyle\prod_{i=1}^{s}n[T_i]+(r_1,r_2,\cdots,r_s)\subset\prod_{i=1}^{s}n[T_i]\\
&\qquad \text{for arbitrary } (r_1,r_2,\cdots,r_s).
\end{aligned}$$

Thus it follows that the index-set of the product $g(\boldsymbol{\theta})\cdot f_n(\boldsymbol{\theta})$ is a subset of $n[T]$, the index-set of the series function $f_n(\boldsymbol{\theta})=[f(\boldsymbol{\theta})]^n$. Hence, the statement of the theorem follows from Theorem 7 in Patil [5].

THEOREM 7. *The parametric function of a multivariate* PSD *with range* $T=\prod_{i=1}^{s}T_i$, $T_i\subset I$, *is* MVU *estimable for* $n\geqq\max_i[[1/\alpha_i]]$, *where* $[[m]]$ *denotes the smallest integer not less than m and, where* $\alpha_i=d(D(T_i))$, $i=1,2,\cdots s$.

PROOF. Follows from Theorem 5, Theorem 4 and Theorem 3.

THEOREM 8. *A necessary and sufficient condition for the parametric function* $\prod_{i=1}^{s}\theta_i$ *of a multivariate* PSD *with range* $T=\prod_{i=1}^{s}T_i$, $T_i\subset I$, *is* MVU *estimable is that the displaced set* $D(T)$ *be a basis of* I_s, *i.e.* $n[D(T)]=I_s$.

PROOF. Since $\prod_{i=1}^{s} \theta_i$ is estimable, we have from Theorem 5 and Theorem 4, that

$$D(n[T_i]) = I \quad \text{for} \quad i = 1, 2, \cdots, s.$$

$$\therefore \quad \prod_{i=1}^{s} d(n[T_i]) = I_s.$$

But, by definition, $\prod_{i=1}^{s} D(n[T_i]) = D(\prod_{i=1}^{s} n[T_i])$, which, in turn, $= D(n[T])$, since $n[T] = n[\prod_{i=1}^{s} T_i] = \prod_{i=1}^{s} n[T_i]$. Thus $D(n[T]) = n[D(T)] = I_s$. Hence, the statement of the theorem.

REMARK 4. The statement of Theorem 8 is interesting in that it raises the question of defining a density for "multivariate" sets, in general, on the lines of the Schnirelmann density defined for "univariate" sets. While the question seems to be still open, by and large, in the literature of the additive number theory, the concept of the Kvarda–Schnirelmann density and the associated Kvarda's theorem as quoted in Section 2 are adequate for our present purpose. Thus, we do have also a multivariate analog of Corollary 4 in Patil [4] as follows:

COROLLARY 1. *A sufficient condition for the parametric function $\prod_{i=1}^{s} \theta_i$ of a multivariate* PSD *with range $T = \prod_{i=1}^{s} T_i$, $T_i \subset I$, to be* MVU *estimable is that the Kvarda–Schnirelmann density of the set $D(T)$ is positive.*

5. Multivariate PSD's with arbitrary range. For the multivariate PSD with no restrictions on its range T, Patil [5] has discussed the MVU estimation in terms of additive number theoretic structure of suitably constructed "univariate" sets. In this section, we attempt to obtain results which are of a more constructive nature and thus enjoy more of the operational and practical value. The main result of this section is contained in Theorem 10 which is a generalization of Theorem 1 to the multivariate PSD's. We start with Theorem 9 which corresponds to the Lemma in Patil [4] page 1051.

THEOREM 9. *A necessary and sufficient condition for the parameters $\theta_1, \theta_2, \cdots, \theta_s$ of the multivariate* PSD *to be* MVU *estimable on the basis of a random sample of size* 1 *is that $E_a = D_a(T) \cap I_s = I_s$ for every $\mathbf{a} \in A =$* LB(T).

PROOF. From Theorem 4 in Patil [5], we have that, θ_i is MVU estimable on the basis of a single observation, if and only if, $\mathbf{x} \in T \Rightarrow \mathbf{x} + \mathbf{e}_i \in T$, $i = 1, 2, \cdots, s$, where $\mathbf{e}_i$ is the ith basis vector $(0, 0, \cdots, 1, 0, \cdots, 0)$.

Now, suppose that $\mathbf{x} \in T \Rightarrow \mathbf{x} + \mathbf{e}_i \in T$ for $i = 1, 2, \cdots, s$. Then, given $\mathbf{a} \in A \subset T$, we can prove by induction that $\mathbf{a} + \mathbf{r} \in T$ for every $\mathbf{r} \in I_s$. Hence, $I_s \subset D_a(T)$.

Conversely, suppose that $I_s \subset D_a(T)$ for all $\mathbf{a} \in A$. Now let $\mathbf{x} \in T$ and let an lbp $\mathbf{a} \in A$ such that $a_i \leqq x_i$, $i = 1, 2, \cdots, s$. Then $\mathbf{x} - \mathbf{a} \in I_s$. This implies that $\mathbf{x} - \mathbf{a} + \mathbf{e}_i \in I_s \subset D_a(T)$, i.e., $\mathbf{x} + \mathbf{e}_i \in T$ for $i = 1, 2, \cdots, s$.

Thus $I_s \subset D_a(T)$ for all $\mathbf{a} \in A$, if and only if, $\mathbf{x} \in T \Rightarrow \mathbf{x} + \mathbf{e}_i \in T$ for $i = 1, 2, \cdots, s$. Hence the Theorem.

The following lemma which is interesting in itself also shows that the criterion

$I_s \subset D_a(T)$ in the last theorem needs to be applied only to a finite number of points $\mathbf{a}$.

LEMMA 1. *$A = \text{LB}(T) \subset I_s$ contains a finite number of points.*

PROOF. For $s = 1$, $A = \{\min(T)\}$, a singleton. Assuming that the result of the lemma is true for $1, 2, \cdots, s-1$, we will show its validity for s. To begin with, define $A[x_1] = \{x_2 : (x_1, x_2) \in A = \text{LB}(T) \subset I_s\}$. Clearly, $\mathbf{x}_1 \in I_r$ iff $A(\mathbf{x}_1] \subset I_{s-r}$ for arbitrary $1 \leqq r < s$. Also, we observe that, $\mathbf{x} \neq \mathbf{y}$, $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2) \in A$ and $\mathbf{y} = (\mathbf{x}_1, \mathbf{y}_2) \in A \subset I_s$ iff $\mathbf{x}_2 \neq \mathbf{y}_2$, $\mathbf{x}_2 \in \text{LB}(A[\mathbf{x}_1])$ and $\mathbf{y}_2 \in \text{LB}(A[\mathbf{x}_1]) \subset I_{s-r}$ for arbitrary $1 \leqq r < s$.

To show that $A = \text{LB}(T) \subset I_s$ is finite, let $\mathbf{a} \in A$. It is clear that

$$A = (\textstyle\bigcup_r \bigcup_{i_r} A_{i_r}) U\{\mathbf{a}\}$$

where, $\mathbf{i}_r = (i_1 < i_2 < \cdots < i_r)$ is a subset of $(1, 2, \cdots, s)$, $1 \leqq r < s$, implying that the unions are finite in number, and where, with $\{i_1 < i_2 < \cdots i_r < i_{r+1} < \cdots < i_s\} = \{1, 2, \cdots, s\}$.

$$A_{i_r} = \{\mathbf{x} \in A : x_{i_1} \leqq a_{i_1}, x_{i_2} \leqq a_{i_2}, \cdots, x_{i_r} \leqq a_{i_r}, x_{i_{r+1}} > a_{i_{r+1}}, \cdots, x_{i_s} > a_{i_s}\}.$$

To show that A_{i_r} is finite, note that $(x_{i_1}, x_{i_2}, \cdots, x_{i_r})$ can take at most $\prod_{j=1}^{r}(a_{i_j}+1)-1$ values. Consider its typical value $(t_1, t_2, \cdots t_r) = \mathbf{t}_r$. Then $A_{i_r} = U_{t_r} A_{i_r}^{t_r}$ where

$$A_{i_r}^{t_r} = \{\mathbf{x} \in A : x_{i_1} = t_1, x_{i_2} = t_2, \cdots, x_{i_r} = t_r, x_{i_{r+1}} > a_{i_{r+1}}, \cdots, x_{i_s} > a_{i_s}\}.$$

In view of the initial observation, we shall be through if we have $\text{LB}\, A_{i_r}^{t_r}[\mathbf{t}_r] \subset I_{s-r}$ finite, which is assumed under induction hypothesis for I_k where $k = 1, 2, \cdots, s-1$.

Now, from Theorem 9 we conclude that $\theta_1, \theta_2, \cdots, \theta_s$ are MVU estimable on the basis of a random sample of size n iff $I_s \subset D_b(n[T])$ for every $\mathbf{b} \in \text{LB}(n[T])$. Theorem 11 below provides a necessary and sufficient condition to be satisfied by T for $\theta_1, \theta_2, \cdots, \theta_s$ to be MVU estimable. We need the following lemmas.

LEMMA 2. *If $A = \text{LB}(T) \subset I_s$ and $A_n = \text{LB}(n[T])$, then $A_n \subset n[A]$.*

PROOF. For $n = 1$, the lemma is obvious. Setting up the induction hypothesis on $n = m$, we shall show the lemma to be true for $n = m+1$. Let $\mathbf{b} \in A_{m+1} = \text{LB}((m+1)[T]) \subset (m+1)[T] = m[T]+T$ so that $\mathbf{b} = \mathbf{c}+\mathbf{a}$ where $\mathbf{c} \in m[T]$ and $\mathbf{a} \in T$. To show that $\mathbf{c} \in A_m \subset m[A]$ and $\mathbf{a} \in A$ and hence $\mathbf{b} = \mathbf{c}+\mathbf{a} \in (m+1)[A]$, suppose that $\mathbf{c} \neq A_m$. Then there exists $\mathbf{x} \in m[T]$ such that $\mathbf{x}_i \leqq \mathbf{c}_i$ for all i implying that $x_i + a_i \leqq c_i + a_i = b_i$ for all i and hence that $\mathbf{b} \notin A_{m+1}$. Therefore, $\mathbf{c} \in A_m \subset m[A]$ by induction hypothesis. Similarly it can be shown hat $\mathbf{a} \in A$.

LEMMA 3. *If $\{E_i : i = 1, 2, \cdots, r\}$ are* bases *for I_s, then there exists an integer n, such that $\sum_{i=1}^{r} n_i[E_i] = I_s$, where the nonnegative integers n_i add up to n.*

PROOF. Let the *order* of the *basis* E_i be m_i. Then $n = rm$ with $m = \max(m_1, m_2, \cdots, m_r)$ has the desired property.

THEOREM 10. *Let* $A = \{\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_r\} = \mathrm{LB}(T) \subset I_s$. *Let* $E_i = D\mathbf{a}_i(T) \cap I_s$, $i = 1, 2, \cdots, r$. *For the parameters* $\theta_1, \theta_2, \cdots, \theta_s$ *of the multivariate* PSD *to be* MVU *estimable, it is sufficient that each* E_i *is a* basis *for* I_s.

PROOF. By Lemma 1, A is finite. Let $A = \{\mathbf{a}_i : i = 1, 2, \cdots, r\}$. Let $\mathbf{b} \in A_n = \mathrm{LB}(n[T])$. From Lemma 2 and Lemma 3, there exist n and $n_1, n_2, \cdots, n_r$ such that $\mathbf{b} = \sum_{i=1}^{r} n_i \mathbf{a}_i$ and $\sum_{i=1}^{r} n_i[E_i] = I_s$. Thus for $\mathbf{c} \in I_s$, we have, $\mathbf{c} = \sum_{i=1}^{r} w_i = \sum_{i=1}^{r} \sum_{j=1}^{n_i} x_{ij}$, where, $w_i \in n_i[E_i]$, $x_{ij} \in E_i$ and $w_i = \sum_{j=1}^{n_i} x_{ij}$ implying that $x_{ij} + \mathbf{a}_i \in T$, from which follows $\sum\sum x_{ij} + \sum n_i \mathbf{a}_i \in n[T]$; i.e., $\mathbf{c} + \mathbf{b} \in n[T]$; i.e., $\mathbf{c} \in D_b(n[T])$. Now, the theorem is clear because $\theta_1, \theta_2, \cdots, \theta_s$ are MVU estimable iff $I_s \subset D_b(n[T])$ for every $\mathbf{b} \in A_n$.

COROLLARY 2. *If the range* T *of the multivariate* PSD *is such that the Kvarda–Schnirelmann density of* $E_a = D_a[T] \cap I_s$ *is positive for every* $\mathbf{a} \in \mathrm{LB}(T)$, *then* $\theta_1, \theta_2, \cdots, \theta_s$ *are* MVU *estimable.*

Proof is obvious from Kvarda's Theorem and Theorem 10.

LEMMA 4. *Let* R_s *denote s-dimensional Euclidean space. If* $F_i \subset R_s$, $i = 1, 2, \cdots, r$ *are such that* $I_s \subset n_i[F_i]$ *for positive integers* n_i, *then there exists an integer n such that* $I_s \subset \sum n_i[F_i]$, *where* $\sum n_i = n$.

PROOF. Similar to that of Lemma 3.

THEOREM 11. *Let* $A = \mathrm{LB}(T) \subset I_s$. *For the parameters* $\theta_1, \theta_2, \cdots, \theta_s$ *of the multivariate* PSD *to be* MVU *estimable on the basis of a random sample of size n, it is necessary and sufficient that,* $I_s \subset n[D\mathbf{a}(T)]$ *for all* $\mathbf{a} \in A$.

PROOF. To prove the necessity, we first note that $I_s \subset D_b(n[T])$ for all $\mathbf{b} \in A_n$. Now consider an arbitrary $\mathbf{a} \in A$. Since $n\mathbf{a} \in n[T]$, there exists a $\mathbf{b} \in A_n$ such that $\mathbf{b} \leqq n\mathbf{a}$, implying $\mathbf{r} + (n\mathbf{a} - \mathbf{b}) \in I_s \subset D_b(n[T])$, where $\mathbf{r} \in I_s$. Therefore, $\mathbf{r} + n\mathbf{a} \in n[T]$, i.e., $\mathbf{r} \in D_{na}(n[T])$, i.e., $I_s \subset D_{na}(n[T])$. Now, it can be verified that $D_{na}(n[T]) = n[D_a(T)]$, from which the result follows.

To prove the sufficiency, the proof runs analogous to that of Theorem 10, using $D_{a_i}[T]$ instead of E_i.

REMARK 5. It may be interesting to note that sufficiency of Theorem 10 follows from the sufficiency of Theorem 11 in view of the fact that $I_s = n[D_a(T) \cap I_s]$ implies $I_s \subset n[D_a(T)]$ since $D_a(T) \cap I_s \subset D_a(T)$.

With our present approach, it should be interesting to prove a result which can be independently obtained by Rao–Blackwellization.

The next theorem constitutes this result. Here, the Rao–Blackwellization would work by considering the conditional expectation of the available MVU estimator based on a subsample of size n, conditioning being on the entire sample of size $n_1 \geqq n$.

THEOREM 12. *For the multivariate* PSD, *if* $\theta_1, \theta_2, \cdots, \theta_s$ *are* MVU *estimable for a random sample of size n, then* $\theta_1, \theta_2, \cdots, \theta_s$ *are* MVU *estimable for a random sample of size* $n_1 \geqq n$.

PROOF. We have to show that $I_s \subset D_b(n_1[T]$ for every $\mathbf{b} \in A_{n_1}$. Let $n_1 = n+r$. First of all, observe that $A_{n_1} \subset A_n + A_r$. Let $\mathbf{b} = \mathbf{c}+\mathbf{d}$, where $\mathbf{c} \in A_n$ and $\mathbf{d} \in A_r$. Now, if $\mathbf{t} \in I_s$, $\mathbf{t} \in D_c(n[T])$ as a result of the given hypothesis. Therefore, $\mathbf{t} \in n[T] + \{\mathbf{d}\} - \{\mathbf{c}+\mathbf{d}\} \subset n_1[T] - \{\mathbf{b}\} = D_b(n_1[T])$.

In the conclusion, we prove a theorem which generalizes Theorem 2 to the multivariate PSD.

THEOREM 13. *If the parametric function $g(\boldsymbol{\theta})$ admits a power series expansion in $\theta_1, \theta_2, \cdots, \theta_s$ then a sufficient condition for $g(\boldsymbol{\theta})$ to be* MVU *estimable on the basis of a random sample of size n is that $\theta_1, \theta_2, \cdots, \theta_s$ be* MVU *estimable on the basis of a random sample of size n.*

PROOF. Let $\theta_1, \theta_2, \cdots, \theta_s$ be MUV estimable on the basis of a random sample of size n, and let $x \in n[T]$, so that by Theorem 4 in Patil [5], $\mathbf{x}+\mathbf{e}_i \in n[T]$ for $i = 1, 2, \cdots, s$. Then by induction we can easily prove that $\mathbf{x}+\mathbf{r} \in n[T]$. Thus $W[g(\boldsymbol{\theta})f_n(\boldsymbol{\theta})] \subset W[f_n(\boldsymbol{\theta})]$ and the statement of the theorem follows from Theorem 7 in Patil [5].

6. Applications. In statistical ecological work with plant populations or insect populations, situations can arise (see Rao [6] where observations on the random vector $(x_1, x_2, \cdots, x_s)$ with $x_i \in I$ are ignored when one or more of the observed component x_i's are zero. Normally, one prefers to ignore such observations in view of the difficulty in ascertaining whether the zero-count for the component arose as a result of the "absence" of, or, as a result of the "total damage" to, the species represented by the component.

The multivariate PSD's like the multivariate negative multinomial, the multivariate logarithmic series and the multivariate Poisson with independent components have found applications in ecological research. It is quite conceivable that these distributions truncated in one or more of the "axes" of their ranges would be suitable, as models, when zero-counts would be under doubt. For the MVU estimation problems, then, results of this paper, in particular of Section 4 and Section 5, would apply.

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