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ANOTHER LOOK AT DOOB'S THEOREM

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1. Introduction. Doob's theorem in probabilistic potential theory states that if f is the limit of a decreasing sequence of excessive functions, then f differs from its excessive regularization on at most a semi-polar set. This result is a generalization to probabilistic potential theory of an important theorem of Cartan in classical potential theory. The purpose of this note is to isolate the property of a super-mean-valued function f which enables one to conclude that f differs from its regularization on at most a semi-polar set. This leads to a slight generalization and, at the same time, a new (and simple) proof of Doob's theorem. Moreover our method enables us to conclude in a number of important cases that the exceptional set is actually *polar* rather than semi-polar.

2. The main results. All terminology and notation are the same as in [1]. In particular we fix once and for all a standard process $X = (\Omega, \mathscr{M}, \mathscr{M}_t, X_t, \theta_t, P^x)$ with state space $(E, \mathscr{E})$ and, for simplicity, we assume that $\mathscr{M} = \mathscr{F}$ and $\mathscr{M}_t = \mathscr{F}_t$ for all t. Recall that a universally measurable nonnegative function f is called α-super-mean-valued provided that $P_t^\alpha f \leqq f$ for all t, and that for such an f, the function $\bar{f} = \lim_{t\downarrow 0} P_t^\alpha f$ exists and is the largest α-excessive function dominated by f. One calls $\bar{f}$ the (α-excessive) regularization of f and it is easy to see that $\{\bar{f} < f\}$ is of potential zero. See ([1] page 81 and (II–3.17)).

DEFINITION 2.1. A nonnegative function f is *strongly* α-super-mean-valued provided

(a) f is nearly Borel measurable.
(b) $P_T^\alpha f \leqq f$ for all stopping times T.

Note that if f is the limit of a decreasing sequence of α-excessive functions $\{f_n\}$, then f is strongly α-super-mean-valued since each f_n satisfies Definition 2.1 (a) and (b) and these properties are preserved under the taking of decreasing limits. We will show that if f is strongly α-super-mean-valued, then $\{\bar{f} < f\}$ is semi-polar. In light of the above remark this is a generalization of Doob's theorem. We come now to our key observation.

PROPOSITION 2.2. *Let f be a strongly α-super-mean-valued function and assume that $\bar{f}$ is finite. Given $\varepsilon > 0$ let $A_\varepsilon = \{f - \bar{f} \geqq \varepsilon\}$ and let T be the hitting time of A_ε. Then for each x*

$$\bar{f}(x) \geqq E^x\{e^{-\alpha T}\bar{f}(X_T)\} + \varepsilon E^x\{e^{-\alpha T}\}. \tag{2.3}$$

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PROOF. Since the proof is exactly the same for all α, we will write out the details under the assumption that $\alpha = 0$. First of all fix y not in A_ε. (Note that A_ε is of potential zero so that its complement is finely dense.) Let $\{K_n\}$ be an increasing sequence of compact subsets of A_ε such that if $T_n = T_{K_n}$ then $T_n \downarrow T$ almost surely P^y where $T = T_{A_\varepsilon}$. Let $D_n = D_{K_n} = \inf\{t \geqq 0 : X_t \in K_n\}$. Then

$$T_n = T_{n+1} + D_n \circ \theta_{T_{n+1}}; \qquad T_n = T + D_n \circ \theta_T$$

since $K_n \subset K_{n+1} \subset A_\varepsilon$ for all n. Consequently using the strong Markov property and Definition 2.1 (b)

$$E^y\{f(X_{T_n}) \mid \mathscr{F}_{T_{n+1}}\} = E^{X(T_{n+1})}\{f(X_{D_n})\} \leqq f(X_{T_{n+1}}).$$

In the present situation $\{\mathscr{F}_{T_n}\}$ is a *decreasing* sequence of σ-algebras and so $Y = \lim_n f(X_{T_n})$ exists almost surely P^y. See ([3] page 137). Similarly $E^y\{f(X_{T_n}) \mid \mathscr{F}_T\} \leqq f(X_T)$. Using Fatou's lemma for conditional expectations ([3] page 122) and the fact that $\mathscr{F}_T = \cap \mathscr{F}_{T_n}$ almost surely P^y we obtain $Y \leqq f(X_T)$ almost surely P^y. But $f(X_{T_n}) - \bar{f}(X_{T_n}) \geqq \varepsilon$ almost surely on $\{T_n < \infty\}$ and so using the right continuity of $t \to \bar{f}(X_t)$ and the above inequality we have $f(X_T) - \bar{f}(X_T) \geqq \varepsilon$ almost surely P^y on $\{T < \infty\}$. Therefore

$$P^y[X_T \notin A_\varepsilon;\ T < \infty] = 0 \quad \text{if} \quad y \notin A_\varepsilon. \tag{2.4}$$

Now fix x. Then it is easy to see that $P_t f(x) = P_t \bar{f}(x)$ for all but countably many values of t, and so we can choose a strictly decreasing sequence $\{t_n\}$ of positive numbers tending to zero such that $P^x\{f(X_{t_n}) \neq \bar{f}(X_{t_n})\} = 0$ for each n. Define $R_n = t_n + T \circ \theta_{t_n}$. Then using (2.4) and the fact that $X_{t_n} \notin A_\varepsilon$ almost surely P^x, we have

$$P^x\{X_{R_n} \notin A_\varepsilon;\ R_n < \infty\} = E^x\{P^{X(t_n)}[X_T \notin A_\varepsilon;\ T < \infty]\} = 0.$$

Therefore

$$E^x\{f(X_{R_n})\} \geqq E^x\{\bar{f}(X_{R_n}) + \varepsilon;\ R_n < \infty\} = E^x\{\bar{f}(\quad_{R_n})\} + \varepsilon P^x(R_n < \infty),$$

while

$$E^x\{f(X_{R_n})\} = E^x\{E^{X(t_n)}\{f(X_T)\}\} \leqq E^x\{f(X_{t_n})\} = E^x\{\bar{f}(X_{t_n})\}.$$

Combining these inequalities yields

$$E^x\{\bar{f}(X_{t_n})\} \geqq E^x\{\bar{f}(X_{R_n})\} + \varepsilon P^x(R_n < \infty).$$

As $n \to \infty$, $t_n \downarrow 0$ and $R_n \downarrow T$, and this last inequality becomes ($\bar{f}$ is excessive)

$$\bar{f}(x) \geqq E^x\{\bar{f}(X_T)\} + \varepsilon P^x(T < \infty),$$

which is (2.3) when $\alpha = 0$. Thus Proposition 2.2 is established.

COROLLARY 2.5. *Let f be strongly α-super-mean-valued. Then $\{\bar{f} < f\}$ is semi-polar.*

PROOF. Since $f \wedge n$ is strongly α-super-mean-valued the argument in the first paragraph of the proof of (II–3.6) of [1] shows that it suffices to prove Corollary 2.5 when f is bounded. Thus it suffices to show that each A_ε is thin when f is bounded. But if x is regular for A_ε, (2.3) implies that $\bar{f}(x) \geqq \bar{f}(x) + \varepsilon$ which is a contradiction since $\|\bar{f}\| \leqq \|f\| < \infty$. Thus each A_ε is thin and Corollary 2.5 is established.

REMARK 2.6. The inequality (2.3) implies a sharper result if $\alpha = 0$ and if $\bar{f}$ is finite. Namely that almost surely the path $t \to X_t$ is in A_ε for at most *finitely* many values of t. To see this let $T_0 = 0$, $T_1 = T$, and $T_{n+1} = T_n + T \circ \theta_{T_n}$ be the iterates of $T = T_{A_\varepsilon}$. Iterating (2.3) with $\alpha = 0$ yields $\bar{f}(x) \geqq \varepsilon \sum_{k=1}^{n} P^x(T_k < \infty) + E^x\{\bar{f}(X_{T_n})\}$. In particular $\sum_{k=1}^{\infty} P^x(T_k < \infty) < \infty$ and so the desired result obtains. If $\alpha > 0$ the same argument shows that $T_k \to \infty$ almost surely.

We say that an α-excessive function f is *α-invariant* provided that (i) f is finite and (ii) $P_K{}^\alpha f = f$ for all compact subsets K of E. Note that $f = 0$ is always α-invariant for all $\alpha \geqq 0$. Probably the only case of interest in the following result is when $\alpha = 0$.

COROLLARY 2.7. *Let f be strongly α-super-mean-valued and assume that $\bar{f}$ is α-invariant. Then $\{\bar{f} < f\}$ is polar.*

PROOF. It suffices to show that each A_ε is polar. Let K be a compact subset of A_ε. Then since $T_K \geqq T = T_{A_\varepsilon}$ we obtain from Proposition 2.2

$$\begin{aligned} \bar{f}(x) &\geqq E^x\{e^{-\alpha T}\bar{f}(X_T)\} + \varepsilon E^x(e^{-\alpha T}) \\ &\geqq P_K{}^\alpha \bar{f}(x) + \varepsilon E^x(e^{-\alpha T_K}) \\ &= \bar{f}(x) + \varepsilon E^x(e^{-\alpha T_K}). \end{aligned} \tag{2.8}$$

Consequently K, and hence A_ε, is polar.

For the next corollary we need to recall a definition and to introduce an auxiliary hypothesis on the process X. An α-excessive function f is *regular* provided that almost surely $t \to f(X_t)$ is continuous wherever $t \to X_t$ is continuous on $[0, \zeta)$. It is easy to see that if f is regular then f is *quasi-left-continuous* in the sense that if $\{T_n\}$ is an increasing sequence of stopping times with limit T, then $f(X_{T_n}) \to f(X_T)$ almost surely on $\{T < \zeta\}$. (See, for example, ([1] page 192); the hypothesis there that f is finite is irrelevant.) If X is a special standard process (i.e. satisfies (IV–4.1) of [1]), then these two properties are equivalent.

We next state a special assumption that we will impose on the process X. This condition is closely related to Hunt's hypothesis (B) ([2] page 78).

ASSUMPTION 2.9. Let K be a compact thin set and $x \notin K$. Then there exists an increasing sequence of stopping times $\{T_n\}$ which increases to T_K strictly from below almost surely P^x, that is, almost surely P^x, $\lim T_n = T_K$ and $T_n < T_K$ for all n.

Suppose X has continuous paths and that $\{G_n\}$ is a decreasing sequence of open sets such that $G_n \supset \bar{G}_{n+1} \supset K$ and $\cap G_n = K$. If $x \notin K$ then $\{T_{G_n}\}$ increases to T_K strictly from below almost surely P^x on $\{T_K < \infty\} = \{T_K < \zeta\}$. (Note that $\lim T_{G_n}$ may be finite on $\{T_K = \infty\}$.) If, in addition, X is *special standard* (satisfies (IV–4.1) of [1]), then one can modify the sequence $\{T_{G_n}\}$ to obtain a sequence $\{T_n\}$ which increases to T_K strictly from below almost surely P^x. See (IV–4.38) of [1]. Thus Assumption 2.9 is satisfied if X has continuous paths and is special standard. A sufficient condition that X be special standard is that for each $\alpha > 0$ the α-excessive functions are lower semi-continuous. Also if X satisfies the hypotheses of Section VI–2 of [1], then Assumption 2.9 holds. See (VI–2.9) of [1].

We are now in a position to state our final result.

PROPOSITION 2.10.[2] *Let f be strongly α-super-mean-valued and let $\bar{f}$ be quasi-left-continuous. Then if Assumption* 2.9 *holds,* $\{\bar{f} < f\}$ *is polar.*

PROOF. As in the proof of Corollary 2.1 it suffices to consider the case of bounded f. (Note that $\overline{f \wedge n} = \bar{f} \wedge n$ is quasi-left-continuous if $\bar{f}$ is.) If $\alpha = 0$ then f is strongly β-super-mean-valued for all $\beta > 0$ and the β-regularization of f is the same as the 0-regularization of f. Thus without loss of generality we may assume that $\alpha > 0$ and that f is bounded. Let K be a compact subset of A_ε. Plainly it suffices to show that K is polar. Evidently K is thin since A_ε is thin. Fix $x \notin K$ and let $\{T_n\}$ be as in Assumption 2.9. From Proposition 2.2 with T replaced by T_K (see (2.8)) and with the aid of the strong Markov property and the fact that $T_n + T_K \circ \theta_{T_n} = T_K$ almost surely P^x on $\{T_K < \infty\}$ and hence everywhere, we obtain

$$E^x\{e^{-\alpha T_n} \bar{f}(X_{T_n})\} \geqq E^x\{e^{-\alpha T_K} \bar{f}(X_{T_K})\} + \varepsilon E^x(e^{-\alpha T_K}) .$$

Letting $n \to \infty$ and using the quasi-left-continuity of $\bar{f}$ and the fact that $\alpha > 0$, this becomes (note that $T_K = \infty$ on $T_K \geqq \zeta$)

$$P_K{}^\alpha \bar{f}(x) \geqq P_K{}^\alpha \bar{f}(x) + \varepsilon E^x(e^{-\alpha T_K}).$$

Thus $x \to E^x(e^{-\alpha T_K})$ vanishes off K and hence everywhere since it is α-excessive and K is thin. Therefore K is polar, completing the proof of Proposition 2.10.

The following example shows that Proposition 2.10 is *not* true for general standard processes. Let $E = (-\infty, 0] \cup [1, \infty)$. Starting from $x \geqq 1$ the process is translation to the right at unit speed, 0 is an exponential holding point from which the process jumps to $\{1\}$, and starting from $x < 0$ the process is translation to the right at unit speed until it reaches the holding point 0. Let $f(x) = 1$ if $x \leqq 0$ or if $x = 1$ and $f(x) = 0$ if $x > 1$. Then f is strongly super-mean-valued. In fact, it is the decreasing limit of the sequence $\{f_n\}$ of excessive functions defined by $f_n(x) = 1$ if $x \leqq 0$ or if $1 \leqq x < (n+1)/n$ and $f_n(x) = 0$ if $x \geqq (n+1)/n$. Clearly the regularization $\bar{f}$ of f is given by $\bar{f}(x) = 1$ if $x \leqq 0$ and $\bar{f}(x) = 0$ if $x \geqq 1$. Moreover $\bar{f}$ is continuous and hence regular. But $\{\bar{f} < f\} = \{1\}$ which is thin but not polar. Of course Assumption 2.9 is not satisfied by this process; take $K = \{1\}$. On the other hand this is a Hunt process, and hence special standard, because $\zeta = \infty$. (The fact that this process is quasi-left-continuous depends on the fact that the hitting time of $\{1\}$ is totally inaccessible. This is easily proved using the fact that 0 is an exponential holding point. See, for example, the argument on page 68 of [1].)

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[2] A similar result has been obtained by P. A. Meyer. Private communication.