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# ON AN ASYMPTOTIC REPRESENTATION OF THE DISTRIBUTION OF THE CHARACTERISTIC ROOTS OF $S_1S_2^{-1}$ †

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**1. Introduction and summary.** Let $\mathbf{S}_i: p \times p$ $(i = 1, 2)$ be independently distributed as Wishart $(n_i, p, \mathbf{\Sigma}_i)$. Let the characteristic roots of $\mathbf{S}_1\mathbf{S}_2^{-1}$ and $\mathbf{\Sigma}_1\mathbf{\Sigma}_2^{-1}$ be denoted by $l_i$ $(i = 1, 2, \cdots, p)$ and $\lambda_i$ $(i = 1, 2, \cdots, p)$ respectively such that $l_1 > l_2 > \cdots > l_p > 0$ and $\lambda_1 > \lambda_2 > \cdots > \lambda_p > 0$. Then the distribution of $l_1, \cdots, l_p$ can be expressed in the form (Khatri [8])

$$(1.1) \qquad C|\mathbf{\Lambda}|^{-\frac{1}{2}n_1}|\mathbf{L}|^{\frac{1}{2}(n_1-p-1)}\{\textstyle\prod_{i<j}^{p}(l_j-l_i)\}\int_{O(p)}|\mathbf{I}_p+\mathbf{\Lambda}^{-1}\mathbf{HLH}'|^{-\frac{1}{2}(n_1+n_2)}(\mathbf{H}'\,d\mathbf{H})$$

where

$$C = 2^{-p}\pi^{\frac{1}{2}p(p-1)}\{\textstyle\prod_{i=1}^{p}\Gamma(i/2)\}\,\Gamma_p(\frac{1}{2}n_1+\frac{1}{2}n_2)\{\Gamma_p(\frac{1}{2}p)\Gamma_p(\frac{1}{2}n_1)\Gamma_p(\frac{1}{2}n_2)\}^{-1},$$

$$\Gamma_p(t) = \pi^{\frac{1}{4}p(p-1)}\textstyle\prod_{j=1}^{p}\Gamma(t-\frac{1}{2}j+\frac{1}{2}), \quad \mathbf{L} = \operatorname{diag}(l_1, \cdots, l_p), \quad \mathbf{\Lambda} = \operatorname{diag}(\lambda_1, \cdots, \lambda_p)$$

and $(\mathbf{H}'\,d\mathbf{H})$ is the invariant measure on the group $O(p)$. However, this form is not convenient for further development. Also, since

$$(1.2) \qquad I = \int_{O(p)}|\mathbf{I}_p+\mathbf{\Lambda}^{-1}\mathbf{HLH}'|^{\frac{1}{2}(n_1+n_2)}(\mathbf{H}'\,d\mathbf{H}) = C'\sum_{k=0}^{\infty}\frac{1}{k!}\sum_{\kappa}\frac{C_\kappa(-\mathbf{\Lambda}^{-1})C_\kappa(\mathbf{L})(n_1+n_2)_\kappa}{C_\kappa(\mathbf{I}_p)}$$

where $C' = 2^p\pi^{\frac{1}{4}p(p+1)}/\prod_{i=1}^{p}\Gamma(i/2)$ and the zonal polynomial $C_\kappa(\mathbf{T})$ of any $p \times p$ symmetric matrix $\mathbf{T}$ is defined in James [7], where $\kappa$ is a partition of $k$ into not more than $p$ parts, the use of (1.2) in (1.1) gives a power series expansion, but the convergence of this series is very slow. In the one sample case G. A. Anderson [1] has obtained a gamma-type asymptotic expansion for the distribution of the characteristic roots of the estimated covariance matrix. In this paper we obtain a beta-type asymptotic representation of the roots distribution of $\mathbf{S}_1\mathbf{S}_2^{-1}$ involving linkage factors between sample roots and corresponding population roots. If the roots are distinct the limiting distribution as $n_2$ tends to infinity has the same form as that of Anderson [1]. If, moreover, $n_1$ is assumed also large, then it agrees with Girshick's result [4], which was also discussed in Anderson [1].

**2. The asymptotic representation of $I$.** The procedure used to find the expansion of (1.2) is an extension of the method sketched below for the case $p = 2$. In the asymptotic theory it is necessary to assume $l_1 > l_2 > \cdots > l_p > 0$ and $\lambda_1 > \lambda_2 > \cdots > \lambda_p > 0$. For the simplification of notations we let $\mathbf{A} = \mathbf{\Lambda}^{-1}$, i.e.

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$a_i = 1/\lambda_i (i = 1, \cdots, p)$, $0 < a_1 < a_2 < \cdots < a_p < \infty$, and $n = n_1 + n_2$. Thus for $p = 2$, let $O^{\pm}(2) = \{\mathbf{H} \in O(2), |\mathbf{H}| = \pm 1\}$ then

$$I = 2 \int_{O^+(2)} |\mathbf{I}_2 + \mathbf{AHLH}'|^{-\frac{1}{2}n} (\mathbf{H}' d\mathbf{H}). \tag{2.1}$$

Now let

$$\mathbf{H} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \qquad -\pi < \theta \leqq \pi,$$

so that $(\mathbf{H}'d\mathbf{H}) = d\theta$ and

$$I = 4[(1+a_1 l_1)(1+a_2 l_2)]^{-\frac{1}{2}n} \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} [1 + \tfrac{1}{2}c_{12}(1-\cos 2\theta)]^{-\frac{1}{2}n} d\theta \tag{2.2}$$

where $c_{12} = (a_2 - a_1)(l_1 - l_2)/\{(1+a_1 l_1)(1+a_2 l_2)\} > 0$.

The integrand has a maximum of unity at $\theta = 0$ and then decreases to $(1+c_{12})^{-\frac{1}{2}n}$ at $\theta = \pm\frac{1}{2}\pi$. Write (2.2) as

$$4\left[\textstyle\prod_{i=1}^2 (1+a_i l_i)\right]^{-\frac{1}{2}n} \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \exp\{-\tfrac{1}{2}n \log(1+\tfrac{1}{2}c_{12}(1-\cos 2\theta))\}\, d\theta. \tag{2.3}$$

Since the integral is mostly concentrated in a small neighborhood of the origin, for large $n$, we can expand the argument of the exponential function and $\cos 2\theta$ in the following form

$$4\left[\textstyle\prod_{i=1}^2 (1+a_i l_i)\right]^{-\frac{1}{2}n} \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \exp\{-\tfrac{1}{2}nc_{12}\theta^2\} \exp\{\tfrac{1}{6}nc_{12}\theta^4 + \tfrac{1}{4}nc_{12}^2\theta^4 - \cdots\}\, d\theta. \tag{2.4}$$

If the second exponential function in the integrand is expanded and the integration performed term by term then for large $n$ the limits can be set to $\pm\infty$ (see Erdélyi [3]). Thus for large degrees of freedom $I$ is approximately

$$4\left[\textstyle\prod_{i=1}^2 (1+a_i l_i)\right]^{-\frac{1}{2}n} \left(\frac{2\pi}{c_{12}n}\right)^{\frac{1}{2}} \left[1 + \frac{1}{n}\left(\frac{1}{2c_{12}} + \frac{3}{4}\right) + \cdots\right]. \tag{2.5}$$

LEMMA 1. *If* $\mathbf{A}$ *and* $\mathbf{L}$ *are defined as before then* $f(\mathbf{H}) = |\mathbf{I}_p + \mathbf{AHLH}'|$, $\mathbf{H} \in O(p)$ *attains its identical minimum value* $|\mathbf{I}_p + \mathbf{AL}|$ *when* $\mathbf{H}$ *is of the form*

$$\mathbf{H} = \begin{bmatrix} \pm 1 & & & 0 \\ & \pm 1 & & \\ & & \ddots & \\ 0 & & & \pm 1 \end{bmatrix}. \tag{2.6}$$

PROOF.

$$\begin{aligned} df &= d|\mathbf{I}_p + \mathbf{AHLH}'| \\ &= d|\mathbf{I}_p + \mathbf{A}^{\frac{1}{2}}\mathbf{HLH}'\mathbf{A}^{\frac{1}{2}}| \\ &= |\mathbf{I}_p + \mathbf{A}^{\frac{1}{2}}\mathbf{HLH}'\mathbf{A}^{\frac{1}{2}}| \operatorname{tr}\{(\mathbf{I}_p + \mathbf{A}^{\frac{1}{2}}\mathbf{HLH}'\mathbf{A}^{\frac{1}{2}})^{-1}(\mathbf{A}^{\frac{1}{2}}\, d\mathbf{H}\mathbf{LH}'\mathbf{A}^{\frac{1}{2}} + \mathbf{A}^{\frac{1}{2}}\mathbf{HL}\, d\mathbf{H}'\mathbf{A}^{\frac{1}{2}})\} \\ &= |\mathbf{I}_p + \mathbf{A}^{\frac{1}{2}}\mathbf{HLH}'\mathbf{A}^{\frac{1}{2}}|\, 2\operatorname{tr}\{\mathbf{LH}'\mathbf{A}^{\frac{1}{2}}(\mathbf{I}_p + \mathbf{A}^{\frac{1}{2}}\mathbf{HLH}'\mathbf{A}^{\frac{1}{2}})^{-1}\mathbf{A}^{\frac{1}{2}}\mathbf{HH}'\, d\mathbf{H}\}. \end{aligned}$$

Note that $\mathbf{H}'d\mathbf{H}$ is a skew symmetric matrix, therefore, $df = 0$ implies that $\mathbf{LH}'\mathbf{A}^{\frac{1}{2}}(\mathbf{I}_p + \mathbf{A}^{\frac{1}{2}}\mathbf{HLH}'\mathbf{A}^{\frac{1}{2}})^{-1}\mathbf{A}^{\frac{1}{2}}\mathbf{H}$ is a symmetric matrix. But

$$\mathbf{H}'\mathbf{A}^{\frac{1}{2}}(\mathbf{I}_p + \mathbf{A}^{\frac{1}{2}}\mathbf{HLH}'\mathbf{A}^{\frac{1}{2}})^{-1}\mathbf{A}^{\frac{1}{2}}\mathbf{H}$$

is itself a symmetric matrix and $\mathbf{L}$ is a diagonal matrix with distinct positive roots, so $\mathbf{H}'\mathbf{A}^{\frac{1}{2}}(\mathbf{I}_p+\mathbf{A}^{\frac{1}{2}}\mathbf{HLH}'\mathbf{A}^{\frac{1}{2}})^{-1}\mathbf{A}^{\frac{1}{2}}\mathbf{H}$ has to be a diagonal matrix, say $\mathbf{D}$. Thus $\mathbf{A}^{-1}=\mathbf{H}(\mathbf{D}^{-1}-\mathbf{L})\mathbf{H}'$. This can happen only if $\mathbf{H}$ is of the form with $\pm 1$ in one position in a column or a row and zero in other positions. After substituting those stationary values into $f(\mathbf{H})$ we obtain a general form

$$\prod_{i=1}^{p}(1+a_i l_{\sigma_i}), \tag{2.7}$$

where $l_{\sigma_i}$ is any permutation of $l_i (i=1,\cdots,p)$. Since any permutation is a product of transpositions (2.7) attains its minimum value when $l_{\sigma_i}=l_i (i=1,2,\cdots,p)$. Or $f(\mathbf{H})$ attains its identical minimum value $|\mathbf{I}_p+\mathbf{AL}|$ when $\mathbf{H}$ is of the form of (2.6).

The above lemma enables us to claim that, for large $n$, the integrand of $I$ is negligible except for small neighborhoods about each of these matrices of (2.6) and $I$ consists of identical contributions from each of these neighborhoods so that

$$I \cong 2^p \int_{N(\mathbf{I})} |\mathbf{I}_p+\mathbf{AHLH}'|^{-\frac{1}{2}n}(\mathbf{H}'\,d\mathbf{H}), \tag{2.8}$$

where $N(\mathbf{I})$ is a neighborhood of the identity matrix on the orthogonal manifold. Since any proper orthogonal matrix can be written as the exponential of a skew symmetric matrix we transform $I$ under

$$\mathbf{H}=\exp\mathbf{S}, \qquad \mathbf{S} \text{ a } p\times p \text{ skew symmetric matrix}, \tag{2.9}$$

so that $N(\mathbf{I})\to N(\mathbf{S}=\mathbf{0})$. The Jacobian of this transformation has been computed by G. A. Anderson [1],

$$J=1+\frac{p-2}{24}\operatorname{tr}\mathbf{S}^2+\frac{8-p}{4\times 6!}\operatorname{tr}\mathbf{S}^4+\cdots. \tag{2.10}$$

Direct substitution of (2.9) into $|\mathbf{I}_p+\mathbf{AHLH}'|^{-\frac{1}{2}n}$ yields

$$\begin{aligned} (2.11)\quad &|\mathbf{I}_p+\mathbf{AHLH}|^{-\frac{1}{2}n} \\ &=|\mathbf{I}_p+\mathbf{AL}+\mathbf{ASL}-\mathbf{ALS}+\mathbf{ALS}^2/2+\mathbf{AS}^2\mathbf{L}/2-\mathbf{ASLS}+\cdots|^{-\frac{1}{2}n} \\ &=|\mathbf{I}_p+\mathbf{AL}|^{-\frac{1}{2}n}|\mathbf{I}_p+(\mathbf{I}_p+\mathbf{AL})^{-1}(\mathbf{ASL}-\mathbf{ALS}+\mathbf{ALS}^2/2+\mathbf{AS}^2\mathbf{L}/2 \\ &\quad -\mathbf{ASLS}+\cdots)|^{-\frac{1}{2}n}. \end{aligned}$$

LEMMA 2. *For any $p\times p$ matrix $\mathbf{B}$ and its characteristic roots $b_i (i=1,2,\cdots,p)$, if $\max_{1\leqq i\leqq p}|b_i|<1$ then*

$$|\mathbf{I}_p+\mathbf{B}|^{-\frac{1}{2}n}=\exp\{-\tfrac{1}{2}n\operatorname{tr}(\mathbf{B}-\tfrac{1}{2}\mathbf{B}^2+\tfrac{1}{3}\mathbf{B}^3-\cdots)\}. \tag{2.12}$$

PROOF.

$$\begin{aligned} |\mathbf{I}_p+\mathbf{B}|^{-\frac{1}{2}n} &=\exp\{-\tfrac{1}{2}n\log\textstyle\prod_{i=1}^{p}(1+b_i)\} \\ &=\exp\{-\tfrac{1}{2}n\textstyle\sum_{i=1}^{p}(b_i-\tfrac{1}{2}b_i^2+\tfrac{1}{3}b_i^3-\cdots)\} \\ &=\exp\{-\tfrac{1}{2}n\operatorname{tr}(\mathbf{B}-\tfrac{1}{2}\mathbf{B}^2+\tfrac{1}{3}\mathbf{B}^3-\cdots)\}. \end{aligned}$$

Now apply Lemma 2 to (2.11) and group the terms in the following form (we are not about to prove that the roots of $(\mathbf{I}_p+\mathbf{AL})^{-1}(\mathbf{ASL}-\mathbf{ALS}+\cdots)$ are less than unity for $\mathbf{H}\in N(\mathbf{I})$)

(2.13) $$|\mathbf{I}_p+\mathbf{AL}|^{-\frac{1}{2}n}\exp\left[-\tfrac{1}{2}n(\operatorname{tr}\{\mathbf{S}^2\}+\operatorname{tr}\{\mathbf{S}^3\}+\cdots+\operatorname{tr}\{\mathbf{S}^k\}+\cdots)\right]$$

where $\{\mathbf{S}^k\}$ is the group of terms of order of $\mathbf{S}^k$. With $\mathbf{R}=(\mathbf{I}_p+\mathbf{AL})^{-1}$, it can be shown that

(2.14) $$\operatorname{tr}\{\mathbf{S}^2\}=\operatorname{tr}\left[\mathbf{R}(\mathbf{ALS}^2-\mathbf{ASLS})-\tfrac{1}{2}(\mathbf{RASL}-\mathbf{RALS})^2\right]$$

or simply

(2.15) $$\operatorname{tr}\{\mathbf{S}^2\}=\textstyle\sum_{i<j}^{p}c_{ij}s_{ij}^2$$ where

(2.16) $$c_{ij}=(a_j-a_i)(l_i-l_j)/\{(1+a_il_i)(1+a_jl_j)\}>0.$$

Direct substitution into (2.8) yields

(2.17) $$I\cong 2^p\textstyle\prod_{i=1}^{p}(1+a_il_i)^{-\frac{1}{2}n}\int_{N(\mathbf{S}=\mathbf{0})}\exp\{-\tfrac{1}{2}n\sum_{i<j}^{p}c_{ij}s_{ij}^2\}\exp\{-\tfrac{1}{2}n(\operatorname{tr}\{\mathbf{S}^3\}+\cdots)\}J\prod_{i<j}^{p}ds_{ij}.$$

For large $n$ the limits for each $s_{ij}$ can be put to $\pm\infty$ and we finally have the first term of the expansion of $I$ approximately

(2.18) $$2^p\textstyle\prod_{i=1}^{p}(1+a_il_i)^{-\frac{1}{2}n}\prod_{i<j}^{p}\{2\pi/(nc_{ij})\}^{\frac{1}{2}}.$$

No proof has been given to show that (2.18) is an asymptotic representation for $I$. Hsu's extension of Laplacés method (as used in Anderson [1]) can be applied to prove that the representation is asymptotic.

LEMMA 3. (Hsu). *Let $\Phi(u_1,\cdots,u_m)$ and $g(u_1,\cdots,u_m)$ be real functions on an m-dimensional closed domain D such that*

(i) *$g>0$ on D.*
(ii) *$\Phi(g)^n$ is absolutely integrable over D, $n=0,1,2,\cdots$.*
(iii) *All partial derivatives $g_{u_i}$ and $g_{u_iu_j}$ exist and are continuous, $i,j=1,2,\cdots,m$.*
(iv) *$g(u)$ has an absolutely maximum value at an interior point $\xi$ of D, so that all $g_{u_i}(\xi)=0$, and $|-g_{u_iu_j}(\xi)|>0$.*
(v) *$\Phi$ is continuous at $\xi$ and $\Phi(\xi)\neq 0$. Then for n large*

$$\int_D\Phi(g)^n du_1\cdots du_m\sim[\Phi(\xi)(g(\xi))^n/(\Delta(\xi_1,\cdots,\xi_m))^{\frac{1}{2}}](2\pi/n)^{\frac{1}{2}m}$$

*where $g(u)=\exp\{\psi(u)\}$ and $\Delta(u_1,\cdots,u_m)=|-\psi_{u_iu_j}|$.*

This lemma is used to prove that we have an asymptotic representation for $I$.

THEOREM 1. *Under the same conditions as before*

$$I\sim 2^p\textstyle\prod_{i=1}^{p}(1+a_il_i)^{-\frac{1}{2}n}\prod_{i<j}^{p}\{2\pi/(nc_{ij})\}^{\frac{1}{2}}.$$

PROOF. After substituting $\mathbf{H}=\exp\mathbf{S}$ in (2.8) we can write $I$ as approximately

$$2^p\textstyle\int_{N(\mathbf{S}=\mathbf{0})}\{\exp[-\tfrac{1}{2}\log|\mathbf{I}_p+\mathbf{AHLH}'|]\}^n(1+\tfrac{1}{24}(p-2)\operatorname{tr}\mathbf{S}^2+\cdots)\prod_{i<j}^{p}ds_{ij}$$

so that

$$g = \exp[-\tfrac{1}{2}\log|\mathbf{I}_p+\mathbf{AHLH}'|], \qquad \Phi = 1+\tfrac{1}{24}(p-2)\operatorname{tr}\mathbf{S}^2+\cdots,$$
$$\psi = (-\tfrac{1}{2})\log|\mathbf{I}_p+\mathbf{AHLH}'|$$

and $D = N(\mathbf{S}=\mathbf{0})$. Also $\xi$ corresponds to the point $\mathbf{S}=\mathbf{0}$ and it is clear that all conditions of Lemma 3 are satisfied. To find $|-\psi_{s_{m_1n_1}s_{m_2n_2}}(\mathbf{S}=\mathbf{0})|$ we essentially use

$$h_{ii} = 1-\tfrac{1}{2}\textstyle\sum_{j=1}^{p} s_{ij}^2 \tag{2.19}$$ and

$$h_{ij} = s_{ij}+\tfrac{1}{2}\textstyle\sum_{k=1}^{p} s_{ik}s_{kj}(i\neq j), \tag{2.20}$$

since we are to differentiate the elements of $\mathbf{H}=\exp\mathbf{S}$ at most twice and then set each $s_{ij}$ to zero. With $\psi = (-\frac{1}{2})\log f$ where $f=|\mathbf{I}_p+\mathbf{AHLH}'|$ we have $\partial^2\psi/\partial s_{mn}^2 = (-\frac{1}{2})[f\partial^2 f/\partial s_{mn}^2-(\partial f/\partial s_{mn})^2]/f^2$ and $\partial^2\psi/\partial s_{m_2n_2}\partial s_{m_1n_1} = (-\frac{1}{2})[f\partial^2 f/\partial s_{m_2n_2}\partial s_{m_1n_1}-(\partial f/\partial s_{m_2n_2})(\partial f/\partial s_{m_1n_1})]/f^2$. Now we make use of (2.19) and (2.20) and it can be shown that in evaluating at $\mathbf{S}=\mathbf{0}$, $\partial f/\partial s_{mn} = \partial^2 f/\partial s_{m_2n_2}\partial s_{m_1n_1} = 0$; $\partial^2 f/\partial s_{mn}^2 = 2(a_n-a_m)(l_m-l_n)\prod_{i\neq m,i\neq n}^{p}(1+a_il_i)$ for $m<n$. Therefore, it follows that $-\partial^2\psi/\partial s_{mn}^2 = c_{mn}$; $\partial^2\psi/\partial s_{m_2n_2}\partial s_{m_1n_1} = 0$, and the lemma shows

$$I \sim 2^p[|\mathbf{I}_p+\mathbf{AL}|^{-\frac{1}{2}n}\textstyle\prod_{i<j}^{p} c_{ij}^{-\frac{1}{2}}](2\pi/n)^{\frac{1}{4}p(p-1)}.$$

THEOREM 2. *The asymptotic distribution of the roots,* $l_1>l_2>\cdots>l_p>0$, *of* $\mathbf{S}_1\mathbf{S}_2^{-1}$ *for large degrees of freedom* $n=n_1+n_2$ *when the roots of* $\mathbf{\Sigma}_1\mathbf{\Sigma}_2^{-1}$ *are* $\lambda_1>\lambda_2>\cdots>\lambda_p>0$ *and* $a_i=1/\lambda_i(i=1,\cdots,p)$, *is given by*

$$C2^p\prod_{i<j}^{p}(l_j-l_i)\prod_{i=1}^{p}[(l_i)^{\frac{1}{2}(n_1-p-1)}(a_i)^{\frac{1}{2}n_1}(1+a_il_i)^{-\frac{1}{2}(n_1+n_2)}]\prod_{i<j}^{p}\left[\frac{2\pi}{c_{ij}(n_1+n_2)}\right]^{\frac{1}{2}}. \tag{2.21}$$

The asymptotic formula shows that the joint distribution function of the roots of $\mathbf{S}_1\mathbf{S}_2^{-1}$ is sensitive only to those adjacent roots which are close to each other.

**3. Comparison in limiting cases.** The asymptotic distribution of the characteristic roots of $\mathbf{S}_1\mathbf{S}_2^{-1}$ given in (2.21) can be rewritten as

$$F_1(A)\textstyle\prod_{i<j}^{p}(l_i-l_j)^{\frac{1}{2}}\prod_{i=1}^{p}[l_i^{\frac{1}{2}(n_1-p-1)}(1+a_il_i)^{-\frac{1}{2}(n_1+n_2-p+1)}]\prod_{i=1}^{p}dl_i \tag{3.1}$$

where $F_1(A)$ (also $F_2(A)$ and $F_3(A)$ below) depends on $a_i$'s but not on $l_i$'s. If we make $g_i = n_2l_i(i=1,2,\cdots,p)$ and let $n_2$ tend to infinity then (3.1) reduces to the limiting form

$$F_2(A)\textstyle\prod_{i=1}^{p} g_i^{\frac{1}{2}(n_1-p-1)}\exp[-\frac{1}{2}\sum_{i=1}^{p} a_ig_i]\prod_{i<j}^{p}(g_i-g_j)^{\frac{1}{2}}\prod_{i=1}^{p}dg_i. \tag{3.2}$$

Moreover, let $l_i^* = g_i/n_1$ $(i=1,2,\cdots,p)$ then (3.2) becomes

$$F_3(A)\textstyle\prod_{i=1}^{p} l_i^{*\frac{1}{2}(n_1-p-1)}\exp[-\frac{1}{2}n\sum_{i=1}^{p} a_il_i^*]\prod_{i<j}^{p}(l_i^*-l_j^*)^{\frac{1}{2}}\prod_{i=1}^{p}dl_i^*. \tag{3.3}$$

It can be easily shown that

$$F_3(A) = \{(\tfrac{1}{2}{}_in)^{\frac{1}{2}n_1p-\frac{1}{4}p(p-1)}/\textstyle\prod_{j=1}^{p}\Gamma[(n_1-j+1)/2]\}\prod_{i<j}^{p}(a_j-a_i)^{-\frac{1}{2}}\{\prod_{i=1}^{p}a_i^{\frac{1}{2}n_1}\}$$

and that (3.3) agrees with Anderson's [1] asymptotic distribution for the characteristic roots of $n_1^{-1}\mathbf{S}_1$ when $\mathbf{S}_1$ is distributed as Wishart $(n_1, p, \mathbf{\Sigma}_1)$. Therefore, for $n_1$ large enough, by analogy to Anderson [1] $\prod_{i<j}^{p}(l_i^* - l_j^*)^{\frac{1}{2}}$ tends to $\prod_{i<j}^{p}(\lambda_i - \lambda_j)^{\frac{1}{2}}$ and the resulting independent chi-square distributions tend to normals. This agrees with Girshick's [4] asymptotic normality as noted by Anderson [1].

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