The Annals of Mathematical Statistics
1970, Vol. 41, No. 2, 353–362

ON A CLASS OF INFINITE GAMES RELATED TO LIAR DICE[1]

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1. Introduction and summary. In John Christopher's novel [2], the fate of the whole world is involved in the outcome of a game of Liar Dice. This paper presents a mathematical analysis of a class of zero-sum two-person games related to Liar Dice as described formally in Bell [1]. Karlin [4] provides the necessary background to the theory of games.

Player I receives a number x chosen at random from a uniform distribution on the interval $(0, 1)$. He then chooses a number y in $(0, 1)$ and claims that the number x he received is at least y. Player II, not being informed of x, must *accept* or *challenge* I's claim. If he accepts I's claim, he loses an amount $b(x, y)$, a given function of x and y. If he challenges I's claim, he wins one if in fact $x < y$, and loses one if $x \geqq y$.

The function $b(x, y)$ may be understood to represent player II's expected loss in some other game that is played after II accepts I's claim. For example, if when II accepts I's claim, he must draw a number z from the uniform distribution on $(0, 1)$ and win one if $z > y$ and lose one if $z \leqq y$, then II's expected loss is $b(x, y) = 2y-1$. As another example, if when II accepts I's claim he must draw a number z from the uniform distribution on $(x, 1)$, and win or lose one according as $z > y$ or $z \leqq y$, then II's expected loss is

$$\begin{aligned} b(x, y) &= -1 && \text{if} \quad x \geqq y \\ &= \frac{2y-x-1}{1-x} && \text{if} \quad x \leqq y. \end{aligned}$$

These two examples are taken later to illustrate the general theory. The use of a general $b(x, y)$ allows treatment of situations wherein the basic game is played again whenever II accepts I's claim, with the roles of the players reversed and with the distribution of the future x dependent upon the past x and y.

Our solution to the general problem (Theorem 2) requires rather strong conditions on $b(x, y)$. The general problem is therefore not to be considered completely solved. However, when $b(x, y)$ is independent of the variable x, a complete solution is possible under the sole requirement that $b(y)$ (no longer a function of x) be nondecreasing in y. This is presented in Theorem 1 and the subsequent remarks.

2. The case $b(x,y)$ independent of x. We first consider the case $b(x,y)$ independent of x, and denote this function simply by $b(y)$.

A behavioral strategy for I may be represented by a transition function $F(y \mid x)$, with the interpretation that if I receives x, he states that he received at least y where

Received November 15 1968.

[1] The preparation of this paper was supported in part by NSF Grant No. GP-8049.

y is chosen at random in the interval $(0, 1)$ according to the distribution function $F(y \mid x)$. A behavioral strategy for *II* may be represented by a (measurable) function $p(y)$ on $[0, 1]$ into $[0, 1]$, with the interpretation that if *I* tells *II* he received y at least, *II* accepts the statement with probability $p(y)$, and challenges the statement with probability $1-p(y)$. With a choice of F by *I* and g by *II*, *II*'s expected loss is

$$(1) \qquad L(F, p) = \int_0^1 \int_0^1 \{p(y)b(y) + (1-p(y))(2I(x \geqq y) - 1)\}\, dF(y \mid x)\, dx,$$

where $I(x \geqq y)$ represents the indicator function of the set $\{(x, y) : x \geqq y\}$. ($I(x \geqq y)$ is one if $x \geqq y$ and zero if $x < y$.)

THEOREM 1. *Assume that* $b(y)$ *is nondecreasing, and* $-1 < b(y) \leqq +1$ *for* $0 < y \leqq 1$. *There exists a unique number* c, $0 \leqq c < 1$, *such that*

$$(2) \qquad \int_c^1 \frac{2}{1+b(y)}\, dy = 1.$$

The value of the game exists and is equal to $1-2c$. *The strategy*

$$(3) \qquad \begin{aligned} &F(\cdot \mid x) \text{ degenerate at } x && \text{if} \quad x \geqq c \\ &F(y \mid x) = \frac{1}{c}\int_c^y \frac{1-b(z)}{1+b(z)}\, dz, \quad y \in (c, 1) && \text{if} \quad x < c \end{aligned}$$

is minimax for I. The strategy

$$(4) \qquad \begin{aligned} p(y) &= 1 && \text{if} \quad y \leqq c \\ &= \frac{1+b(c)}{1+b(y)} && \text{if} \quad y > c \end{aligned}$$

is minimax for II.

PROOF. Since $2\int_x^1 (1+b(y))^{-1}\, dy$ is decreasing and continuous in x, with value zero at $x = 1$, and value $2\int_0^1 (1+b(y))^{-1}\, dy \geqq 1$ (since $b(y) \leqq 1$) at $x = 0$, the existence and uniqueness of c, $0 \leqq c < 1$, satisfying (2) is clear.

Suppose *I* uses the strategy: if $x \geqq c$, choose $y = x$; if $x < c$, choose $y \in (c, 1)$ according to the distribution with density $f(y)$. For this strategy,

$$(5) \qquad \begin{aligned} L(F, p) &= \int_0^c \int_c^1 \{p(y)(b(y)+1) - 1\} f(y)\, dy\, dx + \int_c^1 \{1 - p(x)(1-b(x))\}\, dx \\ &= 1 - 2c + \int_c^1 p(y)\{c(b(y)+1)f(y) - (1-b(y))\}\, dy. \end{aligned}$$

If $c = 0$, then $b(y) \equiv 1$, so that $L(F,p) = 1-2c = 1$. If $c \neq 0$, $L(F,p)$ will be a constant $1-2c$ for all strategies p for *II* provided

$$(6) \qquad f(y) = \frac{1}{c}\frac{1-b(y)}{1+b(y)} \qquad c < y < 1.$$

This is, in fact, a probability density if c is chosen to satisfy (2). Thus the lower value of the game is at least $1-2c$, this being attained by the strategy (3).

Now suppose $p(y)$ is known to I. He would choose for given x the distribution $F(y \mid x)$ to give all its mass to those y for which the integrand in (1) achieves its maximum. This maximum is

$$M(x) = \max(\sup_{y \leqq x}\{1-p(y)(1-b(y))\}, \sup_{y>x}\{p(y)(1+b(y))-1\}). \tag{7}$$

II must choose $p(y)$ to minimize the integral of this expression with respect to x over $(0, 1)$. If $p(y)$ is given by (4), $1-p(y)(1-b(y))$ is nondecreasing in y so that

$$\sup_{y \leqq x}\{1-p(y)(1-b(y))\} = 1-p(x)(1-b(x)), \tag{8}$$

a nondecreasing function of x. For $p(y)$ given by (4), $p(y)(1+b(y))-1$ is nondecreasing in y for $y \leqq c$, and constant equal to $b(c)$ for $y > c$. Thus, for all x,

$$\sup_{y>x}\{p(y)(1+b(y))-1\} = b(c). \tag{9}$$

Since (8) and (9) are equal at $x = c$,

$$\begin{aligned} M(x) &= b(c) && \text{for} \quad x \leqq c \\ &= 1-p(x)(1-b(x)) && \text{for} \quad x > c. \end{aligned}$$

II's maximum expected loss using (2) and (4) is

$$\begin{aligned} \textstyle\int_0^1 M(x)\,dx &= cb(c)+\textstyle\int_c^1 \{1-p(x)(1-b(x))\}\,dx \\ &= 1-2c. \end{aligned}$$

This, being equal to a value attainable by I, is thus the value of the game, completing the proof.

REMARK 1. The totality of II's optimal strategies may be found as follows. Let $-1 \leqq \alpha \leqq b(1)$. We first find II's optimal strategies subject to the restriction

$$\sup_{0 \leqq y \leqq 1}\{p(y)(1+b(y))-1\} = \alpha. \tag{10}$$

Let β be any number, $0 \leqq \beta \leqq 1$, such that $b(y) \geqq \alpha$ for $y > \beta$ and $b(y) \leqq \alpha$ for $y < \beta$. Subject to (10), $M(x)$ defined by (7) is bounded below:

$$\begin{aligned} M(x) &\geqq \alpha && \text{for} \quad x \leqq \beta \\ M(x) &\geqq 1-\frac{(1+\alpha)(1-b(x))}{1+b(x)} && \text{for} \quad x > \beta, \end{aligned} \tag{11}$$

and this bound is attainable by, say, $p(y) = (1+\alpha)/(1+b(y))$ for $y > \beta$, and $p(y) = 1$ for $y \leqq \beta$. We search for all p which achieve this lower bound for almost all x.

For the upper inequality of (11) to be valid with equality for almost all $x \leqq \beta$, it must be true that $\sup_{y \leqq x}\{1-p(y)(1-b(y))\} \leqq \alpha$ for almost all $x \leqq \beta$, which in turn implies

$$p(y) \geqq \frac{1-\alpha}{1-b(y)} \quad \text{for all} \quad y < \beta. \tag{12}$$

For the lower inequality in (11) to hold with equality for almost all $x > \beta$, it must be true that $\sup_{y \leqq x}\{1-p(y)(1-b(y))\} \leqq 1-(\alpha+1)(1-b(x))/(1+b(x))$ for almost all $x > \beta$, which in turn implies

$$\sup_{\beta \leqq y \leqq x}\{1-p(y)(1-b(y))\} \leqq 1-(\alpha+1)\frac{1-b(x)}{1+b(x)} \quad \text{for almost all} \quad x > \beta.$$

This implies at points $y > \beta$ of continuity of $b(y)$ that $p(y) = (1+\alpha)/(1+b(y))$. But more generally we have

$$\text{(13)} \qquad \frac{1-b(y+0)}{1-b(y)}\,\frac{1+\alpha}{1+b(y+0)} \leqq p(y) \leqq \frac{1+\alpha}{1+b(y)} \quad \text{for all} \quad y > \beta,$$

and at the point $y = \beta$ itself we must have

$$\text{(14)} \qquad \frac{1-b(\beta+0)}{1-b(\beta)}\cdot\frac{1+\alpha}{1+b(\beta+0)} \leqq p(\beta) \leqq \min\left(1, \frac{1+\alpha}{1+b(\beta)}\right).$$

Conversely, any p satisfying (12), (13) and (14) is easily seen to attain the bound in (11) for almost all x.

The maximum loss guaranteed by any such rule is

$$\int_0^1 M(x)\,dx = \alpha\beta + \int_\beta^1 \left(1-(1+\alpha)\frac{1-b(x)}{1+b(x)}\right)dx$$
$$= 1-2\beta+(1+\alpha)\left[1-\int_\beta^1 \frac{2}{1+b(x)}\,dx\right].$$

This exceeds the value of the game, $1-2c$, by

$$2(c-\beta)+(1+\alpha)\int_c^\beta \frac{2}{1+b(x)}\,dx = 2\int_c^\beta \frac{\alpha-b(x)}{1+b(x)}\,dx.$$

This is zero if and only if

$$\text{(15)} \qquad b(c-0) \leqq \alpha \leqq b(c+0).$$

In summary, *II*'s optimal strategies are given in (12), (13) and (14) with α satisfying (15) and β any number such that $b(y) \leqq \alpha$ for $y < \beta$ and $b(y) \geqq \alpha$ for $y > \beta$.

REMARK 2. The game still has a value and optimal strategies under the sole condition that $b(y)$ be nondecreasing. We state the value and describe the optimal strategies without proof.

If $b(y) \leqq -1$ for all y, the game is trivial with value $b(1)$ (*I* always claims $y = 1$, and *II* always accepts). Similarly, if $b(y) \geqq +1$ for all y, the game is trivial with value $+1$ (*I* always tells the truth, and *II* always challenges). We assume henceforth that for some value of y, $-1 < b(y) < +1$.

Let

$$a_0 = \inf\{y : b(y) > -1\}$$
$$a_1 = \sup\{y : b(y) < +1\}.$$

We distinguish two cases.

Case 1. $\int_{a_0}^{a_1} 2/(1+b(y))\,dy > a_1$. This case is much like Theorem 1. There is a unique c, $a_0 < c < a_1$, such that

$$\int_c^{a_1} \frac{2}{1+b(y)}\,dy = a_1.$$

The value of the game is $1-2c$. An optimal strategy for I is: let $F(y \mid x)$ be degenerate at x if $x > c$, whereas if $x < c$, let $F(y \mid x)$ be the distribution on (c, a_1) with density $f(y) = (1-b(y))/[c(1+b(y))]$, $c < y < a_1$. An optimal strategy for II is: challenge if $y \geqq a_1$; accept if $y \leqq c$; if $c \leqq y \leqq a_1$, accept with probability $(1+b(c))/(1+b(y))$.

Case 2. $\int_{a_0}^{a_1} 2/(1+b(x))\,dx \leqq a_1$. The value of the game is $1-2a_0$. An optimal strategy for I is: let $F(y \mid x)$ be degenerate at x if $x > a_0$, whereas if $x < a_0$, let $F(y \mid x)$ be the distribution on (a_0, a_1) with density $f(y) = (1-b(y))/[c_0(1+b(y))]$, $a_0 < y < a_1$, where $c_0 = \int_{a_0}^{a_1} (1-b(y))/(1+b(y))\,dy$. Optimal for II is simply: accept if $y < a_0$, and challenge if $y \geqq a_0$.

EXAMPLE. If $b(y) = 2y-1$, then equation (2) gives $c = e^{-1} = .36788\cdots$. The value of the game is $1-2e^{-1} = .26424\cdots$. Thus, the game is in I's favor by about 26 cents of a dollar bet. Optimal for I is: if $x > e^{-1}$, call $y = x$, and if $x \leqq e^{-1}$ choose y at random from the distribution with density

$$\begin{aligned} f(y) &= e\left(\frac{1}{y}-1\right) && e^{-1} < y < 1 \\ &= 0 && \text{otherwise.} \end{aligned}$$

Thus, if $x \leqq e^{-1}$, numbers y just above e^{-1} are much more likely to be called than numbers y just below 1. Optimal for II is: accept I's claim if $y \leqq e^{-1}$, and if $y > e^{-1}$, accept I's claim with probability e^{-1}/y.

The theory of the next section shows that there is an optimal pure strategy for player I.

3. The general case. The solution to the general case given here requires rather strong regularity conditions on $b(x,y)$. These conditions are imposed mainly to insure that player I have an optimal strategy in a certain class of pure strategies.

A pure strategy for I may be represented by a measurable function $f(x)$ on $[0,1]$ into $[0,1]$, the interpretation being that if I receives x, he claims $y = f(x)$. The payoff is

$$(16) \quad L(f,p) = \int_0^1 [p(f(x))b(x,f(x)) + (1-p(f(x)))(2I(x \geqq f(x))-1)]\,dx$$

where $p(y)$, as in Section 2, represents the probability that II accepts a claim of y by I.

We consider here strategies of the following form. *For some* β, $0 < \beta < 1$,

(i) $f(x) = x$ *for* $x > \beta$, *and*

(ii) *for* $x \leqq \beta$, $f(x)$ *is absolutely continuous and increasing, with* $f(0) = \beta$ *and* $f(\beta) = 1$. If I uses a strategy of this form, the payoff is

$$(17)\quad L(f,p) = \int_0^\beta [p(f(x))(1+b(x,f(x)))-1]\,dx + \int_\beta^1 [1-p(x)(1-b(x,x))]\,dx.$$

If we let $g(y)$ represent the inverse function to $f(x)$ for $x \in [0,\beta]$, $f(g(y)) = y$, $f'(g(y))g'(y) = 1$ and make the change of variable $z = f(x)$ in the first integral of (17), we obtain

$$(18)\qquad L(f,g) = 1-2\beta + \int_\beta^1 p(z)[(1+b(g(z),z))g'(z)-(1-b(z,z))]\,dz.$$

Now, if $g(z)$ and β are chosen to satisfy the differential equation with boundary conditions,

$$(19)\qquad g'(z) = \frac{1-b(z,z)}{1+b(g(z),z)}, \qquad g(\beta) = 0,\ g(1) = \beta,$$

for almost all $z > \beta$, then the payoff to I is the constant $1-2\beta$, no matter what strategy is used by II. We investigate conditions placed on $b(x,y)$ that, first of all, insure the existence of solutions to (19), and second, entail the existence of a strategy for II that guarantees him a loss not exceeding $1-2\beta$.

We list the conditions below. Let T denote the triangle $T = \{(x,y): 0 \leqq x < y \leqq 1\}$.

A1. $-1 \leqq b(x,y) \leqq +1$	for all	$x \leqq y$
$b(x,y) > -1$	for all	$x < y$
$b(x,x) < 1$	for all	$x < 1$.

A2. $b(x,x)$ is nondecreasing in x.

A3. $b(x,y)$ is continuous on T.

A4. $b(x,y)$ is nonincreasing in x for fixed $y \geqq x$.

A5. $b(x,y)$ is nondecreasing in y for fixed x.

A6. $b_2(x,y) = (\partial/\partial y)b(x,y)$ exists in T and $b_2(x,y)/(1+b(x,y))$ is nondecreasing in x for fixed $y > x$.

Note that the only restrictions that are placed on $b(x,y)$ for $x > y$ occur in A5. If $b(x,y)$ depends only on y, conditions A2 through A6 reduce to the condition that the derivative of b with respect to y exist on the interval $[0,1]$ and be nonnegative.

LEMMA. *Under assumptions A1, A2, A3 and A4, there exists a unique number* c, $0 < c < 1$, *and a unique absolutely continuous solution of the equation*

$$(20)\qquad g'(y) = \frac{1-b(y,y)}{1+b(g(y),y)} \quad \text{a.e.}$$

such that $g(1) = c$ *and* $g(c) = 0$.

PROOF. Let $H(x,y) = (1-b(y,y))/(1+b(x,y))$. Then H is continuous in x for each fixed y and measurable in y for each x, on the set $T = \{(x,y): 0 \leqq x < y \leqq 1\}$. Furthermore, H is bounded in the regions $\{(x,y): 0 \leqq x \leqq y-\varepsilon, y \leqq 1\}$, $\varepsilon > 0$. From the Carathéodory existence theorem, there exist local solutions of (20) at each point $(g(y), y)$ in T, which may be extended to the boundary of T (see e.g. [3] Chapter 2, Section 1).

In addition, $H(x,y)$ is positive on $\{(x,y): 0 \leqq x < y < 1\}$ and nondecreasing in x for fixed y there. This implies that no two solutions of (20) can cross. In fact, if g_1 and g_2 are solutions to (20) both defined at y_1 and y_2, $y_1 < y_2$, and if $\Delta = g_2(y_1) - g_1(y_1) > 0$, then $g_2(y_2) - g_1(y_2) \geqq \Delta$. To see this, note that for $y_1 \leqq y \leqq y_2$,

$$g_i(y) = g_i(y_1) + \int_{y_1}^{y} H(g_i(z), z)\, dz$$

for $i = 1, 2$, so that for all $y \in [y_1, y_2]$,

$$g_2(y) - g_1(y) = \Delta + \int_{y_1}^{y} [H(g_2(z), z) - H(g_1(z), z)]\, dz. \tag{21}$$

Since $H(x,y)$ is nondecreasing in x for fixed y, it is sufficient to show that $g_2(z) \geqq g_1(z)$ for all $z \in [y_1, y_2]$. But if $g_2(z) < g_1(z)$ for some $z \in [y_1, y_2]$, there would be a number $y_3 = \inf\{z: g_2(z) < g_1(z), z \in [y_1, y_2]\}$ with $g_2(y_3) = g_1(y_3)$ and $g_2(y) \geqq g_1(y)$ for $y_1 < y < y_3$. This obviously contradicts (21).

This monotone property of the difference of two solutions to (20), $g_2(y) - g_1(y)$, implies that there is a unique solution to (20) with the boundary condition $g(1) = \beta$, for any β $0 < \beta < 1$. Denote this solution by $g_\beta(y)$.

Since H is positive on $\{(x,y): 0 \leqq x < y < 1\}$, and bounded away from zero in the regions $\{(x,y): 0 \leqq x < y < 1-\varepsilon\}$, $\varepsilon > 0$, the solution $g_\beta(y)$ vanishes for sufficiently small β (i.e. hits the lower boundary of T) at some point, call it $h(\beta): g_\beta(h(\beta)) = 0$. Clearly $h(\beta) \to 1$ as $\beta \to 0$. In addition, h is nonincreasing and continuous from the monotone property of the difference of two solutions. Hence, there exists a unique number c, $0 < c < 1$, such that $h(c) = c$. This number c and the solution $g_c(y)$ satisfy the conclusion of the lemma.

THEOREM 2. *Assume conditions* A1 *through* A6, *and let c and $g(y)$ satisfy the conclusions of the lemma. The value of the game exists and is equal to* $1-2c$. *Player I has an optimal pure strategy: If $x > c$, put $y = x$; if $x \leqq c$ put $y = f(x)$, where f is the inverse function to g. Player II has the following minimax strategy.*

$$\begin{aligned} p(y) &= p(c) && \textit{for} \quad y \leqq c \\ &= p(c)\exp\left\{-\int_c^y \frac{b_2(g(z), z)}{1 + b(g(z), z)}\, dz\right\} && \textit{for} \quad y > c \end{aligned} \tag{22}$$

where

$$p(c) = 2\left[(1 + b(c,1))\exp\left\{-\int_c^1 \frac{b_2(g(z), z)}{1 + b(g(z), z)}\, dz\right\} + (1 - b(c,c))\right]^{-1}. \tag{23}$$

PROOF. From the discussion following (17), it was seen that I's suggested strategy above gives him an expected payoff of $1-2c$ no matter what II does. That II's strategy (22) guarantees him an expected loss not greater than $1-2c$ remains to be shown.

We first show that $p(c)$ defined by (23) lies between zero and one inclusive, from which we may conclude that (22) is, in fact, a strategy for *II*. From A1 it is clear that $p(c) > 0$. From A6

$$\int_c^1 \frac{b_2(g(z),z)}{1+b(g(z),z)}\,dz \leqq \int_c^1 \frac{b_2(c,z)}{1+b(c,z)}\,dz = \log\frac{1+b(c,1)}{1+b(c,c)}$$

so that

$$p(c) \leqq 2\left[(1+b(c,1))\frac{1+b(c,c)}{1+b(c,1)}+(1-b(c,c))\right]^{-1} = 1.$$

We also note that from A5, $b_2(x,y) \geqq 0$ in T so that $p(y)$ of (22) is nonincreasing.

The best that *I* can do against any strategy of *II* is to call y equal to some $f(x)$ which maximizes the integrand of (16). Thus, if *II* uses strategy (22), his expected loss is at most

$$(24)\quad \int_0^1 \max(\sup_{y\leqq x}\{1-p(y)(1-b(x,y))\}, \sup_{y>x}\{p(y)(1+b(x,y))-1\})\,dx$$
$$= \int_0^1 \max(1-p(x)(1-b(x,x)), \sup_{y>x}\{p(y)(1+b(x,y))-1\})\,dx$$

using A5, and the fact that $p(y)$ is nonincreasing. We will show that

$$(25)\quad \sup_{y>x}\{p(y)(1+b(x,y))-1\} = p(f(x))(1+b(x,f(x)))-1$$
$$\geqq 1-p(x)(1-b(x,x)) \qquad \text{for} \quad x<c, \quad \text{and}$$

$$(26)\qquad \sup_{y>x}\{p(y)(b(x,y)+1)-1\} \leqq 1-p(x)(1-b(x,x)) \qquad \text{for} \quad x>c,$$

so that (24) becomes the integral (17) with $\beta = c$, which, with our given $f(x)$, is equal to $1-2c$, as we have seen.

To prove (25) and (26), we evaluate

$$\frac{\partial}{\partial y}(p(y)(1+b(x,y))-1) = p(y)b_2(x,y)+p'(y)(1+b(x,y)).$$

If $y<c$, then $p'(y)=0$, and this partial derivative is nonnegative from A5. If $y>c$, then

$$\frac{\partial}{\partial y}(p(y)(1+b(x,y))-1) = p(y)(1+b(x,y))\left[\frac{b_2(x,y)}{1+b(x,y)}-\frac{b_2(g(y),y)}{1+b(g(y),y)}\right].$$

Thus we have from A6,

$$\begin{aligned}\frac{\partial}{\partial y}(p(y)(1+b(x,y))-1) &\geqq 0 \quad \text{if} \quad x>g(y) \quad (\text{or} \quad y<f(x))\\ &= 0 \quad \text{if} \quad = \qquad\qquad =\\ &\leqq 0 \quad \text{if} \quad < \qquad\qquad > \quad .\end{aligned}$$

In particular,

$$(27)\quad \sup_{y>c}(p(y)(1+b(x,y))-1) = p(f(x))(1+b(x,f(x)))-1 \quad \text{if} \quad x \leqq c,$$

and

(28) $$\sup_{y>x}(p(y)(1+b(x,y))-1) = p(1)(1+b(x,1))-1 \quad \text{if} \quad x \geqq c.$$

At $x = c$, the right side of (26) is equal to the right side of (28) by the definition of $p(c)$ and $p(1)$. But the right side of (28) is nonincreasing in x from A4, whereas the right side of (26) is nondecreasing in x from A2 and the definition of $p(y)$, which is nonincreasing from A1 and A5. This, together with (28), proves (26).

From A4, if $x \leqq c$, (27) is nonincreasing in x. This proves the last inequality in (25) since the right side is nondecreasing in x, the left side is nonincreasing in x, and the left and right sides are equal at $x = c$. In addition, for fixed $x < c$

$$\begin{aligned}\sup_{x<y<c}(p(y)(1+b(x,y))-1) &= p(c)(1+b(x,c))-1 \\ &\leqq p(f(x))(1+b(x,f(x)))-1\end{aligned}$$

from (27), completing the proof of (25).

EXAMPLE. If $b(x,y) = -1$ for $x \geqq y$, and $b(x,y) = (2y-x-1)/(1-x)$ for $x \leqq y$, the differential equation (20) becomes

$$g'(y) = \frac{1-g(y)}{y-g(y)} \qquad g(c) = 0, \qquad g(1) = c.$$

It is easier to solve for the inverse function f, which satisfies the equation

$$f'(x) = \frac{f(x)-x}{1-x} \qquad f(0) = c, \qquad f(c) = 1.$$

This equation has, as the general solution,

$$f(x) = \frac{k}{1-x} + \frac{1+x}{2}.$$

The boundary condition $f(0) = c$, gives $k = c-\frac{1}{2}$ so that

(29) $$(x) = \frac{c-\frac{1}{2}}{1-x} + \frac{1+x}{2},$$

while the boundary condition $f(c) = 1$, gives $c^2-4c+2 = 0$, whose solution between zero and one is $c = 2-2^{\frac{1}{2}} = .586\cdots$. The value of the game is $1-2c = -.172\cdots$. Thus, the game is in *II*'s favor by about 17 cents of a dollar bet. Optimal for *I* is: if $x < c$ call $y = f(x)$ where f is given by (29); if $x \geqq c$ call $y = x$. To find *II*'s optimal strategy (22), we need to compute

$$\begin{aligned}\int_c^y \frac{b_2(g(z),z)}{1+b(g(z),z)}\,dz = \int_c^y \frac{1}{z-g(z)}\,dz = \int_c^y \frac{g'(z)}{1-g(z)}\,dz &= -\log\frac{1-g(y)}{1-g(c)} \\ &= -\log(1-g(y)).\end{aligned}$$

Thus

$$p(c) = \frac{2}{2(1-g(1))+2} = \frac{1}{2-c} = \frac{1}{2^{\frac{1}{2}}} = .707\cdots$$

and II's optimal strategy is

$$\begin{aligned} p(y) &= 2^{-\frac{1}{2}} && \text{if} \quad y \leqq c \\ &= 2^{-\frac{1}{2}}(1-g(y)) && \text{if} \quad y > c \end{aligned}$$

where $g(y)$, the inverse function to f, is

$$g(y) = y-(y^2-2y+2c)^{\frac{1}{2}}.$$

REFERENCES

[1] BELL, R. C. (1960). *Board and Table Games from Many Civilizations.* Oxford University Press.
[2] CHRISTOPHER, JOHN (1964). *The Possessors.* Simon and Schuster, New York, 133, 235.
[3] CODDINGTON, E. A. and LEVINSON, N. (1955). *Theory of Ordinary Differential Equations.* McGraw-Hill, New York.
[4] KARLIN, SAMUEL (1959). *Mathematical Methods and Theory in Games, Programming, and Economics*, **1, 2**. Addison-Wesley, Reading.