

## ABSTRACTS OF PAPERS

(Abstracts of papers presented at the Central Regional meeting, Dallas, Texas, April 8–10, 1970.  
Additional abstracts have appeared in previous issues.)

**124-5. On the Behrens–Fisher problem.** J. S. MEHTA AND R. SRINIVASAN, Temple University.

We consider here the relative merits of several statistics available for the Behrens–Fisher problem of testing two means with unknown and possibly unequal population variances. The statistics considered are due to Banerjee, Fisher, Pagurova, Wald and Welch. They are compared on the basis of the stability of their sizes, and the magnitude of their powers for variations in the nuisance and non-centrality parameters and for various combinations of sample sizes. Recommendations are made on the basis of these comparisons. It is shown that if the sample sizes are both larger than seven, then the solutions due to Pagurova and Welch are very good in the above sense. However, for smaller values of the sample sizes none of the solutions stabilize the size adequately. For these values certain modifications of Pagurova's solution are presented. (Received January 19, 1970.)

**124-6. Model building for prediction in regression analysis based on repeated significance tests.** W. J. KENNEDY AND T. A. BANCROFT, Iowa State University.

This paper considers two different sequential procedures for model building in regression. The bias and mean square error of predicted  $y$  are derived for each procedure assuming unknown error variance  $\sigma^2$ . For each procedure the relative efficiency of the respective sequential procedure to the procedure retaining all predictors is defined to be the inverse ratio of the arithmetic average of the respective mean square errors and is obtained to make comparisons and recommendations regarding probability levels for the preliminary tests. The results provide a usable extension of those results obtained in the two papers by H. J. Larson and T. A. Bancroft [Sequential model building for prediction in regression analysis I. *Ann. Math. Statist.* **34** (1963); and Biases in prediction by regression for certain incompletely specified models. *Biometrika* **50** (1963)]. Confidence limits for the expected value of predicted  $y$  are derived by extending the results of B. M. Bennett [On the use of preliminary tests in certain statistical procedures. *Ann. Inst. Statist. Math.* **8** (1955)]. (Received January 26, 1970.)

**124-7. A multivariate sequential discrimination procedure.** CAMPBELL B. READ, Southern Methodist University and Southwestern Medical School of the University of Texas.

An extension of a procedure by Hall and Baker for testing the mean of a normally distributed rv with unknown variance is made to the case of the mean of a multinormally distributed rv with covariance matrix known except for a scalar multiplier. The test replaces the multiplier by an estimator, otherwise following the steps of a Wald SPRT. The error probabilities have known bounds; approximations to the OC function and to the ASN are presented. (Received January 29, 1970.)

**124-8. Sequential confidence interval for the regression coefficient based on Kendall's tau.** MALAY GHOSH AND P. K. SEN, University of North Carolina.

A robust procedure is considered for the problem of providing a bounded length confidence interval for the regression coefficient (in a simple regression model) based on Kendall's tau. The

problem of estimating the difference in the location parameters in the two-sample case may be viewed as a special case of our problem. It is shown that the estimate of the regression coefficient based on Kendall's tau (see Sen, *J. Amer. Statist. Assoc.* **63** (1968), 1379-1389) possesses certain desirable properties. Comparison with the procedure based on the least squares estimator is also made. (Received January 30, 1970.)

**124-9. Confidence regions for  $F$  with minimum expected area (preliminary report).**  
G. P. STECK, Sandia Laboratories.

Procedures are now available for minimizing the expected area of confidence regions for  $F$  based on inverting one-sample Kolmogorov type tests of  $H: F = F_0$  against  $A: F \neq F_0$ . Current empirical and theoretical work shows that the Kolmogorov regions, i.e.,  $F_n \pm \lambda$ , are not optimum in this sense. Similar results are obtained in the two-sample case. (Received February 3, 1970.)

**124-10. On stochastic games II.** ASHOK MAITRA AND T. PARATHASARATHY, Indian Statistical Institute, Calcutta and Case Western Reserve University.

In this paper, we consider positive stochastic games, when the State and action spaces are all infinite. We prove under certain conditions the positive stochastic game has a value and that the maximizing player has an  $\epsilon$ -optimal Stationary Strategy and the minimizing player has an optimal Stationary Strategy. (Received February 3, 1970.)

**124-11. On ordered families and their order statistics.** RUDOLF BORGES AND GERALD ROGERS, New Mexico State University.

Pfanzagl (*Ann. Math. Statist.* **35** 1216) considers three order relations on a class of distribution functions  $F(x; w)$  on the real line with parameter  $w \in W \subset Re$ : (1)  $F(x; w)$  nonincreasing in  $w$ ; (2) For  $w < w'$ ,  $F(x; w')/F(x; w)$  nondecreasing in  $x$  or  $(1 - F(x; w'))/(1 - F(x; w))$  nondecreasing in  $x$ ; (3)  $(F(y; w') - F(x; w'))/(F(x; w) - F(x; w'))$  nondecreasing in both  $x$  and  $y$  whenever  $F(y; w) > F(x; w)$ . Our theorem is: *If  $G(x; w)$  is the df of the  $r$ th smallest order statistic from a population with df  $F(x; w)$ , then the family  $\{F\}$  has order 1 iff the family  $\{G\}$  has order 1;  $\{F\}$  has order 2 implies  $\{G\}$  has order 2;  $\{F\}$  has order 3 implies  $\{G\}$  has order 3.* The proof is accomplished by means of three lemmas on functional inequalities which may be of some independent interest. (Received February 11, 1970.)

**124-12. A double sampling plan for comparing two variances.** AARON GOLDMAN AND R. K. ZEIGLER, University of Nevada and Los Alamos Scientific Laboratory.

A method making calculations possible for the operating characteristic curve of a double sampling plan involving the  $F$  ratio is presented. In addition, comparisons to match two sets of single sampling plans, and a formula for obtaining the average sample size (ASN) are given. (Received February 11, 1970.)

**124-13. Asymptotic linearity of random rank statistics in regression parameter.**  
HIRA LAL KOUL, Michigan State University.

A rank statistic based on random number of observations is called random rank statistic. Suppose  $\Pr [Y_i \leq y] = F(y - \theta c_i)$ ,  $1 \leq i \leq N_r$ ,  $N_r, r \geq 1$  a sequence of positive integer valued random variables.  $\{c_i\}$  are some known constants. We consider Wilcoxon type random signed rank statistic,  $S_{N_r}(\theta)$ , based on ranks of  $\{|Y_i - \theta c_i|, 1 \leq i \leq N_r\}$ , and signs of  $\{Y_i - \theta c_i, 1 \leq i \leq N_r\}$  and prove, under suitable conditions on  $F$ ,  $\{c_i\}$  and  $\{N_r\}$ , the asymptotic linearity of  $S_{N_r}(\theta N_r^{-1/2})$ ,

uniformly in  $|\theta| \leq a$ , in probability. We also prove the asymptotic normality of  $S_{N_r}(0)$  and hence that of  $S_{N_r}(\theta N_r^{-1/2})$ . The results are also true for a larger class of rank statistics. (Received February 11, 1970.)

**124-14. Sequential rules for estimating the location parameter of an exponential distribution.** ASIT P. BASU, IBM Corporation.

Let us assume that observations are obtained sequentially from a population with density function  $f(x) = \sigma^{-1} \exp(-(x-\mu)/\sigma)$ ,  $x > \mu$ ,  $\mu \geq 0$ ,  $\sigma > 0$ . Also assume that observations are obtained at random. In this paper we consider two sequential rules (corresponding to two different loss functions) for estimating the location parameter of the two-parameter exponential distribution when the corresponding scale parameter is unknown. The rules are compared with corresponding "optimum fixed sample procedures" with known  $\sigma$  and are found satisfactory. (Received February 11, 1970.)

**124-15. The bivariate doubly noncentral  $t$  distribution.** MARAKATHA KRISHNAN, Temple University.

Let  $(X_1, X_2)$  follow a bivariate normal distribution with means  $\delta$ , variance  $\sigma^2$  and correlation  $\rho$ ; let  $(Y_1, Y_2)$  follow independently a bivariate noncentral chi distribution with  $\nu$  degrees of freedom, noncentrality parameter  $\lambda$  and correlation coefficient  $\rho$  (M. Krishnan, SIAM Review 9, 1967). Then the joint distribution of  $(X_1\sqrt{\nu}/Y_1, X_2\sqrt{\nu}/Y_2)$  is defined to follow a bivariate doubly noncentral  $t$  with  $\nu$  d.f., n.p.  $\lambda$  and c.c.  $\rho$ :  $\text{Biv}(t'' | \nu, \delta, \lambda, \rho)$ . This is an extension to the usual bivariate  $t$  (C. W. Dunnett and M. Sobel, *Biometrika* 41, 1954; M. M. Siddiqui, *Ann. Math. Statist.* 38, 1967) where it is assumed that  $Y_1 = Y_2 = Y$  and  $\lambda = 0$ . The pdf of  $\text{Biv } t''$  is derived. An application to this distribution involving the sample means and variances from two correlated nonhomogeneous normal populations, is examined. (Received February 11, 1970.)

**124-16. Efficient estimation of the mean of an exponential distribution when an outlier is present.** PRAKASH C. JOSHI, University of North Carolina.

In this paper, the problem of estimating the mean  $\sigma$  of  $n$  independent observations  $x_1, x_2, \dots, x_n$ ,  $n-1$  of which are from  $p(x, \sigma) = (1/\sigma) \exp(-x/\sigma)$ ,  $x \geq 0$ ,  $\sigma > 0$  and one of which is from  $p(x, \sigma/\alpha)$ ,  $0 < \alpha \leq 1$  is considered. Attention has been restricted to the class of estimators of the form  $T_m = (\sum_{i=1}^{m-1} x_{(i)} + (n-m+1)x_{(m)})/(m+1)$ , where  $X_{(1)} \leq X_{(2)} \leq \dots \leq X_{(n)}$  represent the order statistics obtained by rearranging the random variables  $X_1, X_2, \dots, X_n$  in a non-decreasing order of magnitude. An expression for the Mean Square Error of  $T_m$  is obtained and  $m^*$ , the optimum value of  $m$  in the sense that the MSE of  $T_{m^*}$  is smallest, is tabled. An iterative procedure for the estimation of  $\sigma$  is provided. (Received June 11, 1970.)

**124-17. A new bivariate Burr distribution.** FRED C. DURLING, D. B. OWEN, AND J. WANZER DRANE, Medical University of South Carolina and Southern Methodist University.

A new bivariate Burr distribution,  $F(x, y) = 1 - (1 + x^{b_1})^{-p} - (1 + y^{b_2})^{-p} + (1 + x^{b_1} + y^{b_2} + r x^{b_1} y^{b_2})^{-p}$ ,  $x, y \geq 0$ ,  $0 \leq r \leq p+1$ ;  $F(x, y) = 0$  elsewhere, is developed and investigated mathematically. Two interesting cases of the distribution are presented when the parameter  $r = 0, 1$ . For the limiting case,  $r = 0$ ,  $F(x, y)$  reduces to the bivariate case of the multivariate Burr distribution developed by Takahasi (1965). When  $r = 1$ ,  $F(x, y) = F(x) \cdot F(y)$ , the independent case. The relationship of the bivariate Burr distribution and its marginals to the Pearson curves is also discussed. (Received February 12, 1970.)

**124-18. On the uniform minimum variance unbiased estimator for the logarithmic parameter.** IRIS MOORE AND S. K. KATTI, Florida State University and University of Missouri, Columbia.

Let  $x_1, x_2, \dots, x_n$  be random samples from the logarithmic population with parameter  $\theta$ . Also let  $Y$  be the sum of the  $x_i$ 's and  $Q_n(i)$  be the coefficient of  $z^i$  in  $\{-\ln(1-z)\}^n$ ,  $|z| < 1$ . Then  $\hat{\theta}(Y) = Q_n(Y-1)/Q_n(Y)$  is the unique minimum variance unbiased estimator of  $\theta$ . We have derived the following additional results on this estimator: (i) There exists an integer  $I_n$  such that  $\hat{\theta}(Y)$  exceeds 1 whenever  $Y$  exceeds  $I_n$ . (ii)  $\hat{\theta}(Y)$  converges to 1 as  $Y$  approaches infinity. (iii) There exists an  $I_n^*$  such that  $\hat{\theta}(Y)$  is monotone increasing for  $Y$  less than  $I_n^*$  and monotone decreasing otherwise.

The variance of  $\hat{\theta}(Y)$  is compared to the Cramér-Rao and the second Bhattacharyya bounds. It is shown that the ratios of the variance with these bounds go to infinity as  $\theta$  approaches 1. This indicates the unreliability of the bounds as approximations to the true variance of the unbiased estimators. (Received February 12, 1970.)

**124-19. An approximation of the Poisson distribution by the normal distribution of order  $1/2$ .** RUDOLF BORGES. New Mexico State University and Justus Liebig University.

Let  $X$  be a Poisson random variable with expectation  $\lambda$ . Then (\*)  $Y = (\frac{2}{3})\lambda^{-1/6}[(X + \frac{1}{6})^{2/3} - \lambda^{2/3}]$  is for  $\lambda \rightarrow \infty$  asymptotically standardized normally distributed with an error of order  $1/\lambda$ . This follows from the following general result. Define (\*\*)  $Z = \lambda^{1/2}f'(1)^{-1}[f((X+c)/\lambda) - f(1)]$  with strictly monotone increasing and three times differentiable function  $f$ . Then  $Z$  is for  $\lambda \rightarrow \infty$  asymptotically standardized normally distributed with an error of order  $\lambda^{-1/2}$ . The error is of order  $1/\lambda$ , if and only if both  $f'(1)/f''(1) = -\frac{1}{3}$  and  $c = \frac{1}{6}$ . The constant  $c = \frac{1}{6}$  might be replaced by a correction for bias. Hence the transformation (\*) is preferable to the well-known transformations (\*\*) with  $f(x) = x$  or  $x^{1/2}$  or  $\log x$ . We mention that the error terms of order  $\lambda^{-1/2}$  of these transformations are essentially in proportion to  $\frac{1}{6}$ ,  $-\frac{1}{12}$ , and  $-\frac{1}{3}$ . Anscombe used the transformation (\*) in 1960. (Received February 13, 1970.)

**124-20. Asymptotic properties of the solution of a system of renewal equations.** KENNY S. CRUMP, Louisiana Tech.

Let  $F(t)$ ,  $t \geq 0$ , be an  $n \times n$  matrix of right-continuous nondecreasing functions which are not all concentrated at the origin, let  $Z(t) = (z_i(t))$ ,  $t \geq 0$ , be an  $n$ -dimensional column vector of measurable functions and let  $M(t) = (M_1(t), \dots, M_n(t))^T$ ,  $t \geq 0$ , be a column vector that satisfies the renewal equation  $M(t) = Z(t) + F * M(t)$  where  $*$  is defined as matrix multiplication except elements are convolved rather than multiplied. Such systems of equations arise, among other places, in connection with the moments of multitype branching processes. Let  $\alpha$  be the number which makes the largest eigenvalue of the matrix  $(\int \exp(-\alpha t) F_i f_i(dt))$  equal to one. If each  $e^{-\alpha t} z_i(t)$  is directly Riemann integrable and  $F(\infty)$  is an irreducible matrix then  $M_i(t) \sim c_i e^{\alpha t}$  as  $t \rightarrow \infty$  except when  $F(t)$  satisfies certain lattice conditions, in which case the limit must be modified slightly. If the assumption that  $F(\infty)$  is irreducible is dropped then various statements of the form  $M_i(t) \sim d_i t^{r_i} e^{B_i t}$  as  $t \rightarrow \infty$  hold where  $B_i \leq \alpha$  and  $r_i$  is an integer less than or equal to  $n$ . The techniques used are generalizations of those employed by William Feller in his book *An Introduction to Probability Theory and its Applications* 2. (Received February 16, 1970.)

**124-21. Some two-sample nonparametric tests for life distributions.** RICHARD A. JOHNSON AND G. K. BHATTACHARYYA, University of Wisconsin.

Two-sample nonparametric test procedures are studied in an attempt to find tests which have reasonable power over a class of life distributions which includes the Weibull and gamma. Locally

best tests are obtained for alternatives of the form  $\{f(x; \theta, p) = 1/\theta f(x/\theta, p), \theta > 0$  (fixed) where  $1/\theta f(x/\theta, 1) = \theta^{-1} \exp(-x/\theta)\}$  which includes changes in the shape parameter for the gamma and Weibull as well as the scale parameter in the exponential. This form of alternative, in the one-sample problem, was considered by Bickel and Doksum (*Ann. Math. Statist.* **40** (1969) 1216–1235). It is shown that the test for a change of shape in the gamma has scores which are the negative of the expected values of order statistics in the reduced extreme-value distribution (type I). Small sample power is calculated as well as several Pitman ARE's. The results suggest that the Weibull scores should only be used when the underlying population is Weibull. (Received February 16, 1970.)

**124-22. An inference problem suggested by risk-theory** (preliminary report).

ROBERT B. MILLER, University of Wisconsin.

The Pearson Type III distribution often appears in discussions of risk theory (see Seal, H. L., (1969), *Stochastic Theory of a Risk Business*. Wiley, New York, for a bibliography). The method of moments and the method of maximum likelihood are commonly used to estimate the parameters of this distribution. However, these point estimation approaches fail to yield much insight into the behavior of the Pearson Type III. The purpose of this paper is to obtain some additional insight by (i) plotting contours of its likelihood function and (ii) studying the effect of different prior distributions on the parameters by plotting contours of the resulting posterior distributions. (Received February 17, 1970.)

**124-23. Another look at variance and variance components.** JAMES N. ARVESON,

Purdue University.

The present paper is an attempt to clarify some earlier results on the variance of variance component estimates, considered by Tukey (*Ann. Math. Statist.* **28** 43–56) and Hooke (*Ann. Math. Statist.* **27** 80–98). The techniques used in the above papers are those of polykays and bipolykays. In the present paper, Hoeffding's U-statistic approach is used. The results obtained in the present paper are the same as cited above, however, in the two-sample case, one is able to readily extend results to the unbalanced case. This in some degree generalizes recent work of Harville (*Ann. Math. Statist.* **40** 408–416) to a situation without the normality assumption. Different weighting schemes are examined however. (Received February 17, 1970.)

**124-24. Probabilities of large deviations of sums of random variables.** P. K. BHATTACHARYA,

University of Arizona.

$\{X_n, n = \dots, -1, 0, 1, \dots\}$  is a strictly stationary stochastic process,  $S_N = X_1 + \dots + X_N$  and  $F_N$  is the empirical distribution function of  $X_1, \dots, X_N$ . (i) If  $P[0 \leq X_n \leq 1] = 1$  and  $\lim_{n \rightarrow \infty} E[X_n | X_0, X_{-1}, \dots] = E[X_n] = m$  uniformly in the sample sequences, then for every  $a > 0$  there is  $K(a) > 0$  so that  $P[S_N - Nm > Na] \leq e^{-N/K(a)}$ . (ii) If for each real  $x$ ,  $\lim_{n \rightarrow \infty} P[X_n \leq x | X_0, X_{-1}, \dots] = P[X_n \leq x] = F(x)$  uniformly in the sample sequences, then for every  $a > 0$  there is  $K(a) > 0$  so that  $P[\sup_x |F_N(x) - F(x)| > a] < e^{-N/K(a)}$ . (Received February 17, 1970.)

**124-25. Infinite divisibility and variance mixtures of the normal distribution.**

DOUGLAS KELKER, Washington State University.

The infinite divisibility of variance mixtures of  $n(0, \sigma^2)$  can be partially characterized in terms of the mixing distribution  $G$ . It is well known that if  $G$  is infinitely divisible, so is the mixture. The  $G$  for Student's  $t$  distribution is infinitely divisible for 3, 5, and 7 degrees of freedom. But  $G$  need not be infinitely divisible for the mixture to be infinitely divisible. If a non-degenerate  $G$  has bounded support, then the mixture is not infinitely divisible. All scale parameter mixtures of the Cauchy distribution are infinitely divisible. If  $G(u)$  or  $u^{-3/2}G(1/u)$  is completely monotone, then the mixture is infinitely divisible. (Received February 17, 1970.)

**124-26. Estimating the solution of a stochastic difference equation.** KENNETH N. BERK, Illinois State University.

Best linear unbiased estimates are discussed assuming data  $y_k, k = T, \dots, t, y_k = a_1 f_1(k) + \dots + a_J f_J(k) + n_k$ , where  $f_1, \dots, f_J$  are the  $J$  solutions of a linear homogeneous difference equation of order  $J$ , the coefficients  $a_1, \dots, a_J$  are unknown, and  $\{n_k\}$  is a sequence of zero-mean random variables with variances. Estimates are considered for future values of  $\{y_t\}$  and for a process  $\{x_t\}$  which differs from  $\{y_t\}$  by a zero-mean process  $\{u_t\}$  of known covariance. The method is based on discussion of the zero-mean process  $\{s_t\}$  obtained by applying the difference operator to  $\{y_t\}$ , as applied by Box, Jenkins, and Bacon [Models for forecasting seasonal and nonseasonal time series. *Spectral Analysis of Time Series* 271–312, Wiley (1967)] and Couts, Grether, and Nerlove [Management Science 13 (1966) 1–21]. In the case that  $\{s_k\}$  is weakly stationary, the results provide for the prediction of trend processes using the theory of weakly stationary processes. Assuming that  $\{u_k\}$  is also stationary, orthogonal to  $\{x_k\}$ , an explicit expression is given for the estimate of  $x_k$ . This generalizes Whittle's [Prediction and Regulation, Van Nostrand (1963) p. 95] theorem for processes with  $n$ th difference stationary. The result is then applied to derive the estimators of exponentially discounted least squares, as generalized by Berk [*J. Math. Anal. Appl.*, to appear] to allow nonorthogonal noise. (Received February 17, 1970.)

**124-27. The Hájek–Rényi inequality for sampling from a finite population.** P. K. SEN, University of North Carolina.

In simple random sampling (without replacement) from a finite population, it is shown that the well-known Hájek–Rényi inequality holds for a broad class of symmetric estimators, namely, U-Statistics. This enables one to provide suitable rates for the almost sure convergence of these statistics. Also, in a special case, this yields an inequality comparable to Kolmogorov's for dependent summands, treated in Hájek and Šidák [*Theory of Rank Tests* (1967), Academic Press, New York, pages 184–186]. (Received February 18, 1970.)

**124-28. Sampling plans based on the Polya distribution for an OTPD specification on outgoing quality.** R. T. HILL AND V. P. SINGH, IBM Components Division.

This paper is concerned with the development of a new approach in quality control which we call the Outgoing Tolerable Percentage Defective (OTPD). It is applicable to a manufacturing process continuously producing large lots of varying quality. The OTPD is defined as a specified percentage defective which is to be exceeded in a shipment of product to a customer with no more than a small specified risk ( $\beta$ ). Considering the process distribution to be of the Beta form, the number of defects in a random sample is given by the Polya distribution. Attributes sampling plans are derived based on the OTPD concept and the Polya distribution. For different values of the parameters of the process distribution, various tables are generated which give sample sizes and accept numbers in accordance with certain OTPD specifications. (Received February 24, 1970.)

**124-29. Some near-minimal resolution IV plans of the  $2^m \times 3$  type,  $m \leq 15$ .** D. A. ANDERSON AND J. N. SRIVASTAVA, University of Wyoming and Colorado State University.

We present designs of resolution IV of the  $2^m \times 3$  type, in which  $N$ , the number of runs, equals  $4(m+1)$ . These designs allow the estimation of the main effects  $A_1, \dots, A_m, B$  and  $B^2$ , and the interaction effects  $A_1 B, \dots, A_m B, A_1 B^2, \dots, A_m B^2$ . Since every resolution IV design of the  $2^m \times 3$  series must allow the estimability of these effects ( $(3m+2)$  in number), our plans involve only  $(m+2)$

extra runs. However, we have also shown that we can divide the  $4(m+1)$  runs into  $(m+1)$  blocks of 4 runs each, such that the block design thus obtained still allows the estimation of the above  $(3m+2)$  effects. However, since  $(m+1)$  degrees of freedom now get confounded with blocks, we have effectively only  $3m+3$  observations, giving 1 degree of freedom for error. This justifies the term "near-minimal" in the title. Furthermore, all of our designs are balanced (w.r.t. the  $m$  2-level factors), except three designs, which are partially balanced. Also, for each design, we obtained the  $(3m+2) \times (3m+2)$  covariance matrix of the estimates of the parameters. Looking at these matrices, we conjecture that in the class of all designs with  $4(m+1)$  runs, our designs minimize the trace, the determinant and the largest root of the covariance matrix of the estimates of the parameters. (Received February 24, 1970.)

**124-30. On the asymptotic efficiency of estimates.** GEORGE G. ROUSSAS, University of Wisconsin.

Under the same set of conditions used in previous papers (see, e.g., *Ann. Math. Statist.* **36** 978–992 and *Ann. Math. Statist.* **40** 1207–1215), we consider several classes of sequences of estimates of a real-valued parameter and establish an upper bound for the asymptotic probability that a certain normalized version of the estimate lies in a given class of sets. (Received February 24, 1970.)

**124-31. Bayesian optimisation in sampling finite populations.** J. N. K. RAO AND P. D. GHANGURDE, University of Manitoba and Prairie View A & M College.

The problem of optimum allocation in sampling finite populations, when prior information in the form of a prior distribution is available, is considered. The following cases are investigated: (a) stratified sampling (with known or unknown strata sizes), (b) two-stage sampling, (c) non-response problem. Two different allocations are derived in each case using two different approaches: a Bayesian posterior analysis and a Bayesian preposterior analysis. The solutions are based on a recent approach of Hartley and Rao (*Biometrika* (1968) 547–559), the essential feature of which is a new definition of likelihood based on the assumption that a character  $y$  is measured on a known scale of measurement so that the character may attain a finite set of known scale-points on the  $y$ -scale. Emphasis is given to data-based priors. The solutions are distribution-free and also free from the assumptions of infinite populations and/or known variances. (Received February 25, 1970.)

**124-32. Robust estimation in a simple exponential model.** P. V. RAO, UNIVERSITY OF FLORIDA.

Let  $y_1, y_2, \dots, y_n$  be independent random variables with  $y_j = 1 - \exp(-\rho x_j) + \varepsilon_j$ , where  $\rho > 0$ ,  $\varepsilon_j$  has a cdf  $F_j$  with  $F_j(0) = \frac{1}{2}$ , and the  $x_j$  are known constants. Also, let  $\alpha_j = -x_j^{-1} \ln(1 - y_j)$ ,  $j = 1, 2, \dots, n$  and  $\hat{\rho}_n = \text{med. } \{\alpha_1, \alpha_2, \dots, \alpha_n\}$ . In this paper, some small sample and large sample properties of  $\hat{\rho}_n$  are derived under suitable regularity conditions on the  $F_j$ . In particular, conditions for asymptotic normality and consistency of  $\hat{\rho}_n$  (as an estimator for  $\rho$ ) are given. (Received March 4, 1970.)

**124-33. Some distribution-free tests for multivariate linear hypotheses based on statistically equivalent blocks.** RASHID AHMAD, University of Wisconsin—Milwaukee.

Consider the multivariate linear model  $X_{ijk} = \mu + \alpha_i + \beta_j + \gamma_{ij} + \varepsilon_{ijk}$  ( $1 \leq i \leq r$ ,  $i \leq j \leq c$ ,  $1 \leq k \leq n$ ;  $\alpha = \mathbf{0} = \beta$ ,  $\gamma_i = \mathbf{0}$  for all  $i$  and  $\gamma_{\cdot j} = \mathbf{0}$  for all  $j$ ). Assuming  $p$ -variate normal distribution the statistical techniques for inference are highly developed in the literature (see Anderson

(1958)). Recently Chatlerjee and Sen (1964) (*Calcutta Statist. Assoc. Bull.* **13** 18–58), Anderson (1966) *Multivariate Analysis* ed. P. R. Krishnaiah, Academic Press, New York) and Bell and Smith (1969) *Multivariate Analysis 2* ed. P. R. Krishnaiah, Academic Press, New York) proposed some nonparametric tests for one-sample, two-sample and  $k$ -sample problems. In particular, Anderson (1966) based his tests on Statistically Equivalent Blocks (SEB). In this paper assuming an arbitrary  $p$ -dimensional continuous distribution it is shown that SEB-order statistics random vector  $\mathbf{X}_{(.)} = (\mathbf{X}_{(1)}, \dots, \mathbf{X}_{(N)})$  and SEB-ranks random vector  $\mathbf{R}_{(.)} = (R_1, \dots, R_N)$  are mutually independent and consequently Friedman (1937), Kruskal-Wallis (1952) and van der Waerden (1953) type distribution-free tests are proposed. (Received March 5, 1970.)

**124-34. Some information theory aspects of statistical inference.** TAKIS PAPAIOANNOU, Iowa State University. (Invited).

A small sample statistical information theory based on three measures of information is presented. Twelve statements are used to delimit the concept of statistical information. Four basic properties are imposed on functional measures of information: nonnegativity, additivity under independent observations, maximal information and invariance under sufficient transformations. The Fisherian information measure,  $I_X^F(\theta)$ , is examined along with aspects of conditional information. The trace and determinant of  $I_X^F(\theta)$  are suggested as information measures in the multi-parameter case. Metrics or almost metrics on spaces of probability measures  $P_\theta$ ,  $\theta \in \Theta$ , and 1-1 mappings,  $k$ , from  $\Theta$  onto itself are used to construct new information measures. The modified Kullback-Leibler functional measure of information is given by  $I_X^{KL}(\theta; k) = \int f(x, \theta) \ln [f(x, \theta) / f(x, k(\theta))] d\lambda$  provided that  $P_\theta \ll P_{k(\theta)}$  for all  $\theta$ . It is shown to possess the desired properties and to be invariant under parametric transformations. The generalized Bhattacharyya functional measure is given by  $I_X^B(\theta; k) = -\ln \rho_X^*(P_\theta, P_{k(\theta)})$ , where  $\rho_X^*(P_\theta, P_{k(\theta)}) = \int [f(x, \theta)]^{1/p} [f(x, k(\theta))]^{1/q} d\lambda$ ,  $p, q \geq 0$ ,  $1/p + 1/q = 1$ . The following propositions on  $\rho_X^*$  are shown to hold: (i) for any statistic  $T$ ,  $\rho_X^* \leq \rho_T^*$  with equality iff  $T$  is sufficient for  $\theta$ , (ii)  $\rho_{X_1 X_2}^* = \rho_{X_1}^* \cdot \rho_{X_2}^*$  if  $X_1, X_2$  are independent. It is found that  $I_X^B(\theta; k)$  satisfies the basic properties of information theory. It is free of regularity conditions and invariant under reparametrizations. It covers all models covered by the previous measures and, moreover, non-regular models like the uniform  $(\theta-1, \theta+1)$  family or the Pitman trapezoidal distributions. (Received March 13, 1970.)

(Abstracts of papers presented at the Eastern Regional meeting, Chapel Hill, North Carolina, May 5–7, 1970. Additional abstracts have appeared in previous issues.)

**125-13. On the equivalence of resolvable Kirkman-Steiner triple systems and sets of mutually orthogonal Latin squares.** A. HEDAYAT AND B. L. RAKTOE, Cornell University and University of Guelph.

It is shown that for every resolvable Kirkman-Steiner triple system of order  $v \equiv 3 \pmod{6}$ , there exists at least one pair of orthogonal Latin squares of order  $v$ . It is also shown that the combinatorial structures of these orthogonal Latin squares are different from those of corresponding orders already available in the literature. Thus these orthogonal Latin squares may prove to be useful in the search for non-prime power order projective planes. The sufficiency of  $v \equiv 3 \pmod{6}$  for the existence of a resolvable Kirkman-Steiner triple system of order  $v$  has been shown, though in a rather long paper, by D. K. Ray-Chaudhuri and R. M. Wilson [Solution of Kirkman's schoolgirl problem, *Proc. Amer. Math. Soc. Symposium on Combinatorics* (1968)]. We have demonstrated that the result obtained in this paper is very suggestive towards an alternative and possibly shorter proof for the existence and construction of a resolvable Kirkman-Steiner triple system via orthogonal Latin squares. (Received January 15, 1970.)



**125-14. On the effect of grouping on order statistics.** JOSEPH L. GASTWIRTH AND DONALD R. MCNEIL, The Johns Hopkins University.

The aim of this paper is two-fold: (1) To study the usefulness of two different formulas for the effect of grouping on the order statistics from a sample from a continuous distribution function. (2) To evaluate an approximation to the grouping effect in large samples by using the asymptotic distribution of the order statistics in the ungrouped case in place of their exact distribution in a formula for the grouping effect. (Received January 29, 1970.)

**125-15. Minimax designs.** V. N. MURTY, Pennsylvania State University, Capitol Campus.

When an Experimenter is interested in more than one parameter in the regression model, and tries to obtain a design that minimizes the maximum variance of the individual regression coefficients, he is looking for a minimax design with respect to the single parameters, a concept introduced by Elfving (*Design of Linear Experiments Probability and Statistics*, Harold Cramér Volume, edited by Ulf Grenander, John Wiley & Sons, New York, 58-74). In this paper we explicitly present the minimax s.p. designs for the ordinary polynomial regression, when the degree of the polynomial is less than or equal to 12. A general solution of the problem is still open, but the results obtained do indicate the direction in which one could look for a possible general solution. (Received February 6, 1970.)

**125-16. Equivariant procedures in the compound decision problem with finite state component problem.** JAMES HANNAN AND J. S. HUANG, Michigan State University and University of Waterloo.

Let  $(\mathcal{X}, \mathcal{B}, P)$  be a probability measure space for each  $P \in \mathcal{P} = \{F_0, \dots, F_m\}$ ,  $\mathcal{A}$  be an action space and  $L$  be a loss function defined on  $\mathcal{X} \times \mathcal{P} \times \mathcal{A}$  such that for each  $i$ ,  $c_i = \int \sup_a L(x, F_i, a) dF_i(x) < \infty$ . In the compound problem, consisting of  $N$  components each with the above structure, we consider procedures equivariant under the permutation group. With  $\rho_{ij} = \sup_{B \in \mathcal{B}} |F_i(B) - F_j(B)|$  and  $K(\rho) = .5012 \dots \rho(1-\rho)^{-3/2}$ , we show that the difference between the simple and the equivariant envelopes is bounded by (T1)  $\{2K(\rho) \sum_i c_i^2\}^{\frac{1}{2}} N^{-\frac{1}{2}}$  where  $\rho = \sup_{i,j} \rho_{ij}$ , and by (T2)  $2^m \{2K(\rho') \sum_i c_i^2\}^{\frac{1}{2}} N^{-\frac{1}{2}}$  where  $\rho' = \sup \{\rho_{ij} | \rho_{ij} < 1\}$ . The bound (T1) is finite if and only if the  $F_i$  are pairwise non-orthogonal and (T2) is designed to replace it otherwise. (Received February 11, 1970.)

**125-17. An elementary theorem on the probability of large deviations.** TIMOTHY J. KILLEEN AND THOMAS P. HETTMANSPERGER, Pennsylvania State University.

Let  $X_1, X_2, \dots$  be a sequence of real-valued random variables and  $\varphi_1, \varphi_2, \dots$  a sequence of real numbers such that  $\lim_{n \rightarrow \infty} n^{-1} \ln P(X_n \geq \varphi_n) = c$ , where  $c$  is a negative constant. The limit is important for calculating the exact Bahadur efficiency of tests of statistical hypotheses (*Ann. Math. Statist.* **38** (1967) 303-325). If  $X_n$  has a density or probability function, say  $f_n(x)$ ,  $n = 1, 2, \dots$ , sufficient conditions are given so that  $\lim_{n \rightarrow \infty} n^{-1} \ln P(X_n \geq \varphi_n) = \lim_{n \rightarrow \infty} n^{-1} \ln f_n(\varphi_n)$ . The latter limit is often easy to calculate and examples for the  $F$  and hypergeometric distributions are given. (Received February 13, 1970.)

**125-18. Some small sample results for ratio estimators.** P. S. R. S. RAO AND J. N. K. RAO, University of Rochester and University of Manitoba.

Mean square errors of four ratio estimators are compared, assuming the model  $y_i = \alpha + \beta x_i + e_i$  with  $E(e_i | x_i) = 0$ ,  $E(e_i e_j | x_i, x_j) = 0$ ,  $V(e_i | x_i) = \delta x_i^t$ ,  $i \neq j = 1, 2, \dots, n$ ,  $t \geq 0$ ,  $\delta > 0$  and  $x_i$  has

the gamma distribution with parameter  $h$ . The four ratio estimators are: classical, Quenouille's and Mickey's based on splitting the sample at random into  $g (\geq 2)$  groups, and modified Beale's. The optimum choice of  $g$  is investigated for several values of  $t > 0$ —Rao and Webster [*Biometrika* (1966)] considered  $t = 0$ . The biases and stabilities of three estimators of the variance of the classical ratio estimator (including the jack-knife variance estimator) are also compared under the above model with the additional assumption that the  $e_t$  are normally distributed conditional on the  $x_t$ . Our results are exact for any sample size  $n$ . (Received February 16, 1970.)

**125-19. The multi-dimensional age-dependent branching process—subcritical case.**

K. AIYAPPAN NAIR, State University of New York at Buffalo.

Consider a population consisting of  $n$  types of particles living and reproducing independently of others. A particle of type  $i$  lives for a random length of time  $t_i$  which is distributed according to the law  $P\{t_i \leq t\} = G_i(t)$ , and at the time of its death is replaced by a random number of offspring of various types. Let  $N_{ij}$  denote the number of offspring of type  $j$  produced by a particle of type  $i$ ,  $m_{ij} = EN_{ij}$ , and  $Z(t) = (Z_1(t), \dots, Z_n(t))$  denote the number of particles of various types present at time  $t$ . THEOREM. Let the following conditions hold. (i)  $EN_{ij}^2 < \infty$  for all  $(i, j, k)$ ; (ii)  $M$  is an irreducible matrix with Perron-Frobenius root less than one; (iii)  $|I - (m_{ij}G_i^*(\lambda))| = 0$  where  $G_i^*(\lambda) = \int_0^\infty e^{-\lambda t} dG_i(t)$  has a largest real root  $\alpha$ ; (iv)  $\int_0^\infty t^3 e^{-\alpha t} dG_i(t) < \infty$ ,  $i = 1, 2, \dots, n$ ; (v) Density  $g_i(t)$  of  $G_i(t)$  exists and  $e^{-\alpha t}g_i(t)$  and  $te^{-\alpha t}g_i(t)$  are of bounded variation in  $(0, \infty)$  ( $i = 1, 2, \dots, n$ ); (vi)  $te^{-\alpha t}g_i(t) \rightarrow 0$  as  $t \rightarrow \infty$ ,  $i = 1, \dots, n$ . Then the conditional random vector  $Z(t)$  given that  $Z(0) > 0$  converges in distribution to a random vector whose distribution is independent of the type of the initial individual. The proof is based on Haar's Tauberian theorem [Feller, *Ann. Math. Statist.* 12]. (Received February 17, 1970.)

**125-20. A Bayesian analogue of Paulson's Lemma and its use in tolerance region construction when sampling from the multi-variate normal.**

IRWIN GUTTMAN, University of Wisconsin and University of Massachusetts.

A tolerance region  $S = S(\mathbf{x}_1, \dots, \mathbf{x}_n)$ , which is based on the observed values  $\mathbf{x}_i$ ,  $i = 1, \dots, n$  of  $n$  independent observations from  $f(\mathbf{x}|\theta)$ , is said to be of  $\beta$ -expectation, if the posterior expectation of the coverage  $C(S)$  of  $S$ , where  $C(S) = \int_S f(\mathbf{y}|\theta) d\mathbf{y}$ , is  $\beta$ . Suppose we define the predictive distribution of  $\mathbf{Y}$ , (where  $\mathbf{Y}$  also has the distribution  $f$ ), given the sample  $(\mathbf{x}_1, \dots, \mathbf{x}_n)$  as  $h(\mathbf{y}|\mathbf{x}_1, \dots, \mathbf{x}_n) = \int_{\Omega} f(\mathbf{y}|\theta)p(\theta|\mathbf{y}_1, \dots, \mathbf{x}_n) d\theta$ , where  $p(\theta|\mathbf{x}_1, \dots, \mathbf{x}_n)$  is the posterior distribution of  $\theta$  as found by using Bayes' Theorem. The following may then be stated and proved. LEMMA. If on the basis of observed data, a predictive  $\beta$ -confidence region  $S$  is constructed such that  $\int_S h(\mathbf{y}|\mathbf{x}_1, \dots, \mathbf{x}_n) = \beta$ , then  $C(S)$  has posterior expectation  $\beta$ , that is,  $S$  is of  $\beta$ -expectation. This lemma is then utilized to show that if sampling from the  $k$ -variate normal distribution, that the following region is of  $\beta$ -expectation:  $S = \{\mathbf{y} | (\mathbf{y} - \bar{\mathbf{x}})'[(n-1)V/(n-k)]^{-1}(\mathbf{y} - \bar{\mathbf{x}}) \leq (1+n^{-1})kF_{k, n-k; 1-\beta}\}$  where  $(n-1)V = \sum_{i=1}^n (\mathbf{x}_i - \bar{\mathbf{x}})(\mathbf{x}_i - \bar{\mathbf{x}})'$ ,  $\bar{\mathbf{x}} = n^{-1} \sum_{i=1}^n \mathbf{x}_i$  and  $F_{k, n-k; 1-\beta}$  is the point exceeded with probability  $(1-\beta)$  when using the  $F$ -distribution with  $(k, n-k)$  degrees of freedom. (Received March 4, 1970.)

**125-21. A distribution-free test for bivariate symmetry.** MYLES HOLLANDER, Florida State University.

Let  $(X_1, Y_1), \dots, (X_n, Y_n)$  be independent and identically distributed according to the bivariate distribution  $F$ . For testing  $H_0: F(x, y) = F(y, x)$ , all  $(x, y)$ , we consider the statistic  $D_n = \iint [S_n(x, y) - S_n(y, x)]^2 dS_n(x, y)$  where  $S_n(x, y) = (n-1)^{-1} \sum_{j=1}^n \phi(X_j, x)\phi(Y_j, y)$  and  $\phi(a, b) = 1$  if  $a \leq b$ , 0 otherwise. We show that  $nD_n$  is neither distribution-free for finite  $n$ , nor asymptotically distribution-free under  $H_0$ . (This is the case even if  $F$  is restricted to the class of absolutely continuous distributions.) We thus propose a conditionally distribution-free test based on  $nD_n$ . The test is consistent against  $K_1 = \{F: \iint [F(x, y) - F(y, x)]^2 dF(x, y) > 0\}$ . The class  $K_1$  contains  $K_2 = \{F: F$  is absolutely continuous and  $H_0$  is false\} and  $K_3 = \{F: F$  is continuous with support  $R_2$  and  $H_0$  is false\}, but  $K_1$  does not contain  $K_4 = \{F: H_0$  is false\}. (Received March 5, 1970.)

**125-22. Limit processes for cross-spectral functions.** IAN B. MACNEILL, University of Toronto.

The limits in distribution of the sequences of stochastic processes defined by sample co-spectral and quadrature spectral distribution functions are found using the theory of weak convergence of stochastic processes. The limit processes are shown to be Gaussian with independent increments and with covariances defined in terms of the hypothesized spectral densities. The distributions of certain functionals on these limit processes are computed and these are used to obtain the limiting distributions for a variety of goodness-of-fit tests for spectral distribution functions. (Received March 5, 1970.)

**125-23. Weak convergence results for a class of multivariate Markov processes.** SIEGFRIED SCHACH, The Johns Hopkins University.

In this paper we obtain weak convergence results for several variations and generalization of the classical Ehrenfest process. The basic model considered here can be described as follows:  $N$  balls are distributed among  $K$  urns, with  $n_k(t)$  balls being in urn  $k$  at time  $t$ . Balls move among the urns according to the following rules: the probability that a ball shifts from urn  $k$  to urn  $l$  during  $(t, t + \Delta t)$  is  $n_k \lambda_k p_{kl} \Delta t + o(\Delta t)$ , ( $1 \leq k, l \leq K$ ), and the probability of more than one transition during  $(t, t + \Delta t)$  is of order  $o(\Delta t)$ . We are interested in the limiting behavior of suitably normalized versions of the process  $(n_1(t), \dots, n_K(t))$ , as the number of balls gets large. Our main result states that, suitably normalized, these processes converge to what we call multivariate Ornstein-Uhlenbeck processes in the sense of weak convergence of probability measures. In particular, we arrive at a diffusion approximation for  $(n_1(t), \dots, n_K(t))$  as  $N$  gets large. The technique of a "random change of time" enables us to obtain weak convergence results for a large class of discrete-time multivariate Ehrenfest models also. As a special case, we obtain a new proof of Iglehardt's (*Ann. Math. Statist.* (1968)) limit theorems. (Received March 9, 1970.)

**125-24. Optimal allocation of observations for partitioning a set of normal populations.** MILTON SOBEL AND YUNG LIANG TONG, University of Minnesota and University of Nebraska.

Let  $\pi_0$  denote the control population with distribution  $N(\mu_0, \sigma_0^2)$  and  $\pi_1, \dots, \pi_k$  denote the  $k$  experimental populations with distributions  $N(\mu_1, \sigma_1^2)$ . The problem of partitioning the set  $\{\pi_1, \dots, \pi_k\}$  by their locations w.r.t.  $\pi_0$  with equal sample size was considered by one of the authors (*Ann. Math. Statist.* **40** 1300-1324). In this paper we allow the sample size of  $\pi_0$  to differ from the common sample size of  $\pi_1, \dots, \pi_k$ . Let there be  $n_0$  observations taken from  $\pi_0$  and  $n_1$  observations taken from each of  $\pi_1, \dots, \pi_k$ ; we consider the optimal value of  $\alpha = n_0/n_1$  as a function of  $k$  and  $\lambda = \sigma_0^2/\sigma_1^2$  for fixed total sample size  $n = n_0 + kn_1$ . THEOREM 1. *The expected number of populations misclassified is uniformly minimized at  $\alpha = (k\lambda)^{\frac{1}{2}}$ .* THEOREM 2. *The probability of correct decision at the least favorable configuration is maximized at  $\alpha = (k\lambda)^{\frac{1}{2}}$  as  $n \rightarrow \infty$ .* The related problem of optimal allocation for the construction of two-sided rectangular confidence regions for  $\theta = (\theta_1, \dots, \theta_k) = (\mu_1 - \mu_0, \dots, \mu_k - \mu_0)$  is also considered. Let  $\alpha_0$  denote the  $\alpha$  value which will maximize the confidence coefficient for fixed  $n$ ; we have obtained THEOREM 3. (a)  $\alpha_0 \leq (k\lambda)^{\frac{1}{2}}$  for every  $n$ , (b)  $\alpha_0 \rightarrow (k\lambda)^{\frac{1}{2}}$  as  $n \rightarrow \infty$ . When  $\sigma_0^2, \sigma_1^2$  are unknown, a sequential procedure is considered. It can be shown that the allocation under that sequential procedure is asymptotically optimal when  $n$  is large. (Received March 9, 1970.)

**125-25. An exact comparison of the waiting times under three priority rules.** SREEKANTAN S. NAIR AND MARCEL F. NEUTS, Purdue University.

In the article by Nair and Neuts (*Operations Res.* **17** (1969) 466-477) the authors recall a branching process description of the M/G/1 queue. [Kendal, *J. Roy. Statist. Soc. Ser. B* **13** (1951) 151-

185; Neuts, *Duke Math. J.* **36** (1967) 215–231.] Within each generation customers are served in the order of shortest (SPT) or longest (LPT) service times first. A number of comparisons between these two service rules and the first come, first served (FCFS) service rule in regards to expected waiting times in the equilibrium state were carried out. Many questions involving more than expected values can be asked. In order to answer them an exact comparison of the three waiting times as random variables needs to be made. In this paper we study the joint distribution of these three random variables and the covariance matrix of the trivariate distribution under equilibrium condition. In the end we obtain the limiting probability that a customer “does better” under one priority rule than under each of the other two. (Received March 9, 1970.)

**125-26. Characterization of the distribution of a random matrix by rotational invariance.** W. H. OLSON AND V. R. RAO UPPULURI, Oak Ridge National Laboratory.

In quantum mechanics, under certain symmetry conditions, energy is represented by a real symmetric matrix  $X$ . If energy is represented by  $X$  for a first observer, then energy is represented by  $OXO'$  for a second observer with a rotated coordinate system, where  $O$  is the orthogonal matrix relating the axes of the observers. Descriptions based on  $X$  and  $OXO'$  are completely equivalent physically (Wigner, *Group Theory and Its Application to the Quantum Mechanics*). Thus, if a statistical hypothesis is made on  $X$  then it is natural to make the same statistical hypothesis on  $OXO'$ . The following makes this precise and characterizes the possible statistical hypotheses. Let  $\{x_{ij}\}_{i < j}$ ,  $i, j = 1, 2, \dots, n$  be an independent set of random variables on a probability space  $(\Omega, \mathcal{F}, P)$ . Let  $X$  be the  $n \times n$  matrix where  $x_{ij} = x_{ji}$  a.s., and let  $Y_0 = (y_{ij}^0) = OXO'$ , where  $O$  is any orthogonal matrix. Let  $x = (x_{11}, \dots, x_{1n}, x_{22}, \dots, x_{nn})$  and  $y_0 = (y_{11}^0, \dots, y_{1n}^0, y_{22}^0, \dots, y_{2n}^0, \dots, y_{nn}^0)$ . Let  $\mathcal{B}_n$  denote the Borel  $\sigma$ -algebra of subsets of the  $n$ -dimensional Euclidean space  $R_n$ . Then we have the THEOREM.  $P(x \in B) = P(y_0 \in B)$  for all  $B \in \mathcal{B}_{n(n+1)/2}$  and all orthogonal  $O$  if and only if  $x_{ii}$  is normal with mean  $\mu$  variance  $2a^2$  and  $x_{ij}$ ,  $i < j$ , is normal with mean 0 and variance  $a^2$ , for some constants  $\mu$  and  $a^2 > 0$ . This result seems to have been first proxed in this context by Porter and Rosenzweig (*Ann. Acad. Sci. Fenn. Ser. A*, No. 44 (1960)), under more restrictive conditions than those given here. (Received March 10, 1970.)

**125-27. Higher moments of renewal stopping time.** ADHIR KUMAR BASU, Queen's University.

Let  $\{X_n\}$  be independent rv with  $E(X_n) = \mu$ ,  $E(X_n - \mu)^2 = \sigma^2 < \infty$  and  $0 < \mu < \infty$ . Let  $N_c = N = \inf(n \geq 1; S_n > c)$  where  $S_n = x_1 + \dots + x_n$  and  $0 < c < \infty$ . It is known that if  $\{X_n\}$  obey Lindberg Condition (Seigmund [*Ann. Math. Statist.* **40** (1969) 1074–1077]), then  $E(N) = c/\mu + o(c^3)$ ,  $E(N - c/\mu)^2 = c\sigma^2/\mu^3 + o(c)$ . We have proved that if  $\{X_n\}$  be independent rv with  $E(X_n) = \mu > 0$ ,  $E(X_n - \mu)^2 = \sigma^2$ ,  $E(X_n - \mu)^3 = \gamma$ ,  $E(X_n - \mu)^4 = \rho < \infty$ , then  $E(N - c/\mu)^4 = \mu^{-6}3c^2\sigma^4 + o(c^2)$  as  $c \rightarrow \infty$ . If moreover random variables are identical then  $E(N - c/\mu)^{2k} = \sigma^{2k}c^k(2k)!/(k!2^k\mu^{3k}) + o(c^k)$  as  $c \rightarrow \infty$  for all  $k = 1, 2, 3, \dots$ . We conjecture that the last result is true if  $\{X_n\}$  are independent rv satisfying  $L_{2k}$ , the Lindberg Condition of order  $2k$  (see Brown [*Ann. Math. Statist.* **40** (1969) 1236–1249]). (Received March 10, 1970.)

**125-28. On the extensions of Gauss–Markov theorem to subsets of the parameter space under complex multivariate linear models.** J. N. SRIVASTAVA AND LYMAN L. McDONALD, Colorado State University and University of Wyoming.

This paper concerns the problem of linear estimation (without the assumption of normality) under certain general kinds of multiresponse linear models. These include the general incomplete

multiresponse model and its important special case, the hierarchical multiresponse model, and also the multiple design multiresponse model. These were considered in Srivastava *Ann. Inst. Statist. Math.* **19** (1967) 417–437, where the general problem of obtaining the best linear unbiased estimate (BLUE) of general linear functions of the location parameters was investigated. In this paper we continue this study in the direction of obtaining necessary and sufficient conditions for each of the above models to permit the existence of BLU estimates for all elements in a subset of the set of all estimable linear functions of the location parameters. (Received March 10, 1970.)

**125-29. On the principle of inclusion and exclusion.** RUDOLF BORGES, New Mexico State University and Justus Liebig Universität.

It is well known that (\*)  $\sum a_i P(A_i) = 0$  holds for arbitrary (probability) measures of subsets  $A_i$  of  $S$  iff (\*\*)  $\sum a_i I(A_i) = 0$ , when  $I(A_i)$  denotes the indicator (or characteristic function) of the set  $A_i$ . A formal statement of the principle of inclusion and exclusion is presented: Prove condition (\*) at each point  $x$  in  $S$  of definition of the indicators  $I(A_i)$ . Obtain an inclusion and exclusion formula (\*) by the mentioned implication. This formulation of the principle of inclusion and exclusion as a method of proof allows a comparison of it with the method of indicators due to Whitney (1933) and (1932) and to Loève (1942). The proofs of (\*\*) by both methods can be considerably simplified, if one replaces the usual double sums by a single sum as explained by the following example: The proofs of inclusion and exclusion formulae, say  $P(A_1 \cdots A_n) = P(A_1) + \cdots - P(A_1 A_2) - \cdots$ , usually use some binomial identity. Replace this identity by an identity of the following type:  $\sum (-1)^{|K|} = 0$ , where the summation runs over all subsets  $K$  of a finite nonvoid set  $M$ . The number of elements of  $K$  is denoted by  $|K|$ . (Received March 12, 1970.)

**125-30. On the asymptotic optimality of spectral analysis for testing hypotheses about time series.** W. S. LIGGETT, JR. Raytheon Corporation.

Classification of a sample from a zero mean, stationary, Gaussian time series into populations distinguished by characteristics of the spectrum can be done with a decision theoretic procedure or spectral analysis. Decision theory requires that each population be characterized by a probability distribution on the space of spectral density functions. In this paper, we relate the two methods by showing that under many conditions, as the sample length increases, the expected cost of the Bayes test formed from spectral estimates by approximating their sampling distribution by a product of chi-squared distributions approaches the expected cost of the Bayes test formed from the original data. The amount of smoothing that can be used in the spectral estimates depends on the prior knowledge of the order of differentiability of the spectrum. This result is related to but weaker than the notion that spectral estimates are asymptotically sufficient statistics for the second order properties of a stationary Gaussian time series. (Received March 12, 1970.)

**125-31. Recurrence relations for the mixed moments of order statistics.** PRAKASH C. JOSHI, University of North Carolina.

Let  $X_{r:n}$  ( $1 \leq r \leq n$ ) be the order statistics obtained by rearranging in non-decreasing order of magnitude the variates  $X_i$  ( $1 \leq i \leq n$ ) having a common marginal continuous cdf  $P(x)$ . Several recurrence relations between the first moments  $E(X_{r:n})$  and mixed moments  $E(X_{r:n} X_{s:n})$  are summarized by Govindarajulu (*Ann. Math. Statist.* **34** (1963) 633–651). In this paper, we give a simple argument which generalizes some of these recurrence relations. These generalizations then lead to some modifications in the theorems given by Govindarajulu. (Received March 12, 1970.)

**125-32. Markov additive processes.** ERHAN ÇINLAR, Northwestern University.

Let  $(E, \mathcal{E})$  be a measurable space,  $\mathbf{R}^n$  the Euclidean  $n$ -space,  $\mathcal{B}^n$  the Borel sets of  $\mathbf{R}^n$ , and let  $P_{t,s}$  be a Markov transition function on  $(E \times \mathbf{R}^n, \mathcal{E} \times \mathcal{B}^n)$  which is translation invariant in the

second coordinate variables (that is,  $P_{t,s}(x, y; A \times (B + y)) = P_{t,s}(x, 0; A \times B)$ ). By a Markov additive process we mean a two-dimensional Markov process  $(X_t, A_t)$  with such a transition function. It is shown that such a process  $(X_t, A_t)$  exists if and only if the first coordinate process  $X_t$  does. The main result of the paper is the analog for the process  $A_t$  of Levy's decomposition of an additive process. Our method rests on exploiting the translation invariance of  $P_{t,s}$  in the second coordinate variables, showing the existence of a regular version of the conditional probability given  $\sigma\{X_t; t \geq 0\}$ , and showing that  $A_t$  has independent increments when conditioned on the paths of  $X_t$ . These processes generalize Markov renewal processes, to continuous time, and the additive functionals of Markov processes. (Received March 13, 1970.)

**125-33. Sequence-compound estimation of constrained means of  $p$ -variate normal distributions** (preliminary report). V. SUSARLA, Michigan State University.

For  $j = 1, 2, \dots$ , let  $x_j$  be independent  $p$ -variate normals with covariance  $I$  and means  $\theta_j$  constrained to  $r$ -dimensional subspaces  $s_j$  and let  $\omega_j$  denote the coordinates of  $\theta_j$  with respect to an orthonormal basis in  $s_j$ . Given  $\alpha < \infty$ , we exhibit a sequence of estimators  $\{T_j(x_1, \dots, x_j)\}$  such that the expectation of  $n^{-1} \sum_1^j |T_j - \theta_j|^2$  less the  $r$ -dimensional Bayes risk against the empiric distribution of  $\omega_1, \dots, \omega_n$  is  $O(n^{-1/(r+4)})$  uniformly with respect to  $\{\theta_j\}$  in  $\mathbf{X}_j s_j \cap \{|\theta_j| \leq \alpha\}$ . Gilliland [(1966), RM 162, Department of Statistics and Probability, Michigan State Univ.] has obtained the above result for the degenerate case  $p = 1$ . (Received March 13, 1970.)

**125-34. Weak convergence of empirical distribution functions of random variables subject to perturbations and scale factors.** J. SRINIVASA RAO AND JAYARAM SETHURAMAN, Indiana University and Florida State University.

Let  $Z_1, Z_2, \dots$  be nonnegative independent and identically distributed random variables with distribution function  $F(x)$  and density function  $f(x)$ . Let  $\{\alpha_{ni}, i = 1, \dots, n, Z_n, n = 1, 2, \dots\}$  be positive random variables associated with  $\{Z_1, Z_2, \dots\}$ . The empirical distribution functions  $F_n(x)$  and  $\bar{F}_n(x)$  of  $\{Z_i/\alpha_{ni}, i = 1, \dots, n\}$  and  $\{Z_i/\alpha_{ni} Z_n, i = 1, \dots, n\}$ , respectively arise in many situations, for instance in connection with sample spacings (see Sethuraman, J., and Rao, J. S., Pitman efficiencies of tests based on spacings. First International Symposium on Nonparametric Techniques in Statistical Inference (1969)). Consider the following conditions: (A)  $n^{\frac{1}{2}} \max_i |\alpha_{ni} - 1| = o_p(1)$ , (A')  $\max_i |\alpha_{ni} - 1| = o(1)$ , (B)  $xf(x)$  is bounded, (B')  $xf(x)$  tends to 0 as  $x \rightarrow \infty$ , (C) for some  $\alpha > 0$ ,  $x^\alpha(1 - F(x)) \rightarrow 0$  as  $x \rightarrow \infty$  and (D)  $\xi_n = n^{\frac{1}{2}}(Z_n - 1) = O_p(1)$ . Under (A') when the  $\{\alpha_{ni}\}$  are non-random or generally under (A) and (B), the processes  $\{n^{\frac{1}{2}}(F_n(x) - F(x)), x \geq 0\}$  converge weakly to a Gaussian process in  $D[0, \infty]$ . Under the further conditions (B'), (C) and (D),  $\sup_x |n^{\frac{1}{2}}(\bar{F}_n(x) - F(x)) - n^{\frac{1}{2}}(F_n(x) - F(x)) - xf(x)\xi_n| = o_p(1)$ . These results are then applied to problems connected with spacings. (Received March 13, 1970.)

**125-35.  $L_p$  functions—their prediction and approximation using iterated conditional expectations and their representation by Kolmogorov–Arnold' sums** (preliminary report). J. L. DENNY, University of Arizona.

Let  $\{X_n; n = 1, 2, \dots\}$  be a process with known distribution. The problem is to obtain a (non-linear) predictor for  $X_m$  having observed  $X_1, \dots, X_{2k}$ ,  $2k < m$ , subject to the constraint that one can "utilize only  $k$  observations at a time" in constructing the predictor. A method for predicting  $X_m$  is proposed whose convergence properties (as one repeats the number of times one utilizes  $\{X_1, \dots, X_k\}$  and  $\{X_{k+1}, \dots, X_{2k}\}$ ) are consequences of theorems of D. L. Burkholden and Y. S. Chow when  $p \geq 2$ , and of G. C. Rota and of E. M. Stein when  $p > 1$ . Necessary and sufficient conditions are studied for the predictor to converge a.e. to  $X_m$ , and these conditions are related to measure-theoretic versions of theorems of A. N. Kolmogorov and V. I. Arnold' on

representing continuous functions by sums of superpositions. In turn, the latter theorems are studied by means of sigma-algebras  $\mathcal{A}_1$  and  $\mathcal{A}_2$  such that "locally,  $\mathcal{A}_1 \subset \mathcal{A}_2$ ." (Received March 13, 1970.)

**125-36. Irregular designs and sequential analysis of variance: some Monte Carlo results.** ROGER D. H. JONES AND KLAUS HINKELMANN, University of Georgia and Virginia Polytechnic Institute.

This paper compares the ASN of two types of sampling schemes arising in sequential analysis of variance, the method discussed by Johnson [*Ann. Math. Statist.* **24** (1953) 614–623] and the method studied by Jones and Hinkelmann [*Ann. Math. Statist.* **39** (1968) 1779]. The first method results in an orthogonal design, whereas the second method gives rise to a non-orthogonal (irregular) design which, as the authors have pointed out [*Meth. Inf. Med.* **8** (1969) 41–46], occurs quite naturally in certain types of application. For both methods no general formula for the ASN is available. Therefore, Monte Carlo studies were used to compare the two methods. For the one-way and two-way fixed effects model these studies indicate that the ASN can be reduced significantly under both  $H_0$  and  $H_1$  by using irregular designs with the reduction being a monotone increasing function of the number of treatments and the basic non-centrality parameter. (Received March 13, 1970.)

**125-37. Linear programs for the solution of discrete Markovian decision problems.** K. DANIEL AND L. BROOKS, University of Maryland.

This paper is concerned with maximizing the expected total discounted return in a Markovian Decision problem with finite state space and finite action space when the interest rate is small. The expected total return under stationary policies is expanded into a Laurent series in an alternate form than the one given by Miller and Veinott (*Ann. Math. Statist.* (1969) **40**). This series is used to derive a necessary and sufficient condition for a stationary policy to be optimal for all sufficiently small interest rates. The condition is then shown to lead to a finite sequence of linear programs whose solutions determine the optimal expected return uniquely and characterize all stationary optimal policies. Similar results have been derived by the second author for Markovian Renewal programs. (Received March 13, 1970.)

**125-39. The optimum design of a two factor experiment.** R. J. OWEN, University of Michigan. (Invited).

This paper is concerned with a two factor analysis of variance situation, the factors being conveniently referred to as blocks and treatments. One of the treatments has the role of a control. Attention is focused on inference about the treatment parameters, the block parameters being regarded as nuisance parameters. With a general multivariate normal form for the distribution of errors and for the prior distribution on the block and treatment parameters, the posterior distribution of the treatment parameters is derived. With a quadratic loss function an algorithm is derived for the optimum allocation of treatments over a given sample with known blocking. In special cases the optimum allocation can be written down immediately and the algorithm need not be resorted to. (Received March 13, 1970.)

**125-40. On some characterization of probability distributions.** IGNANCY I. KOTLARSKI, Oklahoma State University.

The aim of the paper is the following theorem on some characterization of probability distributions. THEOREM 1. Let  $X_0, (X_1, \dots, X_n), (X_{n+1}, \dots, X_{n+m})$  be three real independent random vectors having dimensions  $1, n, m$ , respectively ( $n \geq 1, m \geq 1$ ), whose characteristic functions do not vanish. Then the joint distribution of  $(Y_1, \dots, Y_{n+m})$  where  $Y_k = X_0 + X_k, k = 1, 2, \dots, n+m$ , determines

the distributions of all the three random vectors  $X_0, (X_1, \dots, X_n), (X_{n+1}, \dots, X_{n+m})$  up to a shift. As an application of Theorem 1 the following characterization of the gamma distribution is given. **THEOREM 2.** Let  $X_0, (X_1, \dots, X_n), (X_{n+1}, \dots, X_{n+m})$  be three real independent random vectors having dimensions 1,  $n, m$  respectively ( $n \geq 1, m \geq 1$ ), where  $P(X_k > 0) = 1, k = 0, 1, \dots, n+m$ . Denote  $Y_k = X_k/X_0, k = 1, \dots, n+m$ . The necessary and sufficient condition for  $X_0, X_1, \dots, X_{n+m}$  to be mutually independent and to be gamma distributed with parameters  $p_k, a, Ga(p_k, a)$  ( $k = 0, 1, \dots, n+m$ ) is that the joint distribution of the vector  $(Y_1, \dots, Y_{n+m})$  is given by the probability density function,  $g(y_1, \dots, y_{n+m}) = \Gamma(p_0 + \dots + p_{n+m})[\Gamma(p_0) \dots \Gamma(p_{n+m})]^{-1} y_1^{p_1-1} \dots y_{n+m}^{p_{n+m}-1} [(1 + y_1 + \dots + y_{n+m})^{p_0 + \dots + p_{n+m}}]^{-1}$  for  $y_k > 0$  ( $k = 1, \dots, m+n$ );  $g(y_1, \dots, y_{n+m}) = 0$  elsewhere.) These theorems are generalizations of theorems published in an article: *Pacific J. Math.* **20** (1967) 69–76. (Received March 16, 1970.)

**125-41. What is probability?** Clifford J. MALONEY, National Institutes of Health.

Two objective explications of probability developed at about the same time in the sixteenth and seventeenth centuries. One, called the a priori definition, was based on characteristics of games of chance by Cardan, Pascal, Fermat and others. The other was developed from observations by John Graunt on Bills of Mortality, and is called the frequency theory. In the present century, an approach stemming from work as early as Bernoulli and based on "degree of rational belief," in which the prerequisites of neither of the objective definitions appears to be available, has been found to be of service. In each case "probability" is viewed as an attribute inhering to individuals from a specified set. By removing the property of probability from the individual members of the set, and associating it with the (completed) infinite sequence of selections from the set on the one hand, and with the ensemble of all possible such infinite sequences of trials associated with a specific selection procedure on the other hand, a foundation for the concept of probability is achieved which appears to be free of the logical limitations of other approaches. (Received March 16, 1970.)

**125-42. Reliability of complex systems with stochastic hazard functions. I.** N. SHIMI, Florida State University.

Consider a system of  $N$  components each of which is subject to failure. The system fails with any component failure and each component is immediately replaced upon its failure by a component of the same type. We also assume that the hazard rates for the different components are realizations of nonnegative stochastic processes  $\{\Phi_{N,i}(t), t \geq 0\} i = 1, 2, \dots, N$ . The components are assumed to be conditionally independent, given the hazard rates. The main result of this paper is to find the limiting distribution of the total number of failures in an arbitrary time interval  $[0, t]$ . If the stochastic processes  $\Phi_{N,i}(\cdot)$  satisfy the conditions: (i) for  $\alpha > 0, N^\alpha \max_{1 \leq i \leq N} \int_0^\alpha \Phi_{N,i}(x) dx \rightarrow 0$  in probability as  $N \rightarrow \infty$ ; (ii)  $\sum_{k=1}^N \int_0^t \Phi_{N,i}(x) dx \rightarrow \Psi(t)$  in probability as  $N \rightarrow \infty$ , then the limiting distribution of the total number of failures in the interval  $[0, t]$  exists and its probability generating function, evaluated at  $s$ , is given by  $E \exp -(1-s)\Psi(t)$ . This result is applied to the case of a complex system exposed to random environment in the form of a random load function. (Received March 16, 1970.)

**125-43. Sharp upper bounds for interval probabilities in terms of normalized moments** (preliminary report). MORRIS SKIBINSKY, University of Massachusetts.

There is for each positive integer  $n$  a 1-1 correspondence between points interior to  $M_n$ , the space of the first  $n$  moments corresponding to the class of all distributions on the unit interval  $[0, 1]$  and sequences  $p_1, p_2, \dots, p_n$  of "normalized" moments with  $0 < p_i = 1 - q_i < 1$ . (See *J. Appl. Probability* **5** (1968) 693–701 for definitions and development of this concept.) Let  $[a, b]$  denote a closed subinterval of  $[0, 1]$ , take  $p = (p_1, p_2, \dots, p_n)$ , and define  $U(p | a, b) = \sup \{P([a, b]) : P \in \mathcal{P}(p)\}$ , where  $\mathcal{P}(p)$  denotes the family of all distributions on  $[0, 1]$  whose first  $n$  normalized moments are



$p_1, \dots, p_n$ . Using results due to J. H. B. Kemperman (*Ann. Math. Statist.* **39** (1968) 93–122), simple analytic expressions are found for  $U(p|a, b)$  for  $n = 2$  (all  $p$  interior to the unit square), and for  $n = 3$  (the expressions for some exceptional  $p$  interior to the unit cube remain to be verified). For example when  $n = 3$ , we have for all  $p$  whose corresponding ordinary moments lie in the union of certain natural simplex subsets of  $M_3$ , that  $U(p|a, b) = \zeta_1 \zeta_2 \zeta_3 / a(\zeta_2 \zeta_3 + \lambda_a^2)$ , where  $\zeta_i = q_{i-1} p_i$  ( $q_0 = 1$ ) and  $\lambda_a = \zeta_1 + \zeta_2 - a$ . Other expressions for  $U$  are simple variants of the above (e.g. with  $b$  replacing  $a$  on the right-hand side or with  $a \lambda_a$  added to both numerator and denominator of the right-hand side). An interesting general symmetry property is observed to hold for the cases which are developed and is conjectured to hold in generality. Namely that  $U(p_1, p_2, p_3, \dots | a, b) = U(q_1, p_2, q_3, \dots | 1-b, 1-a)$ . (Received March 16, 1970.)

**125-44. Distribution-free tests for ordered alternatives in the randomized block model.** WALTER R. PIRIE AND MYLES HOLLANDER, Virginia Polytechnic Institute and Florida State University.

Let  $X$  be the  $n \times k$  random matrix whose elements  $X_{ij}$  represent the  $j$ th observation in the  $i$ th block of a randomized block experiment. Let  $R_{ij}$  be the rank of  $X_{ij}$  in the ranking from least to greatest of  $\{X_{ij}; j = 1, \dots, k\}$  and  $R$  be the  $n \times k$  matrix whose elements are  $R_{ij}$ . The hypothesis of interest  $H_S$  is exchangeability within blocks. Assuming  $X$  has a continuous distribution tests based on  $R$  are distribution-free under  $H_S$ . For a class of normal ordered shift alternatives, the locally most powerful test based on  $R$  is to reject  $H_S$  for large values of  $T_n(R) = \sum_{i=1}^n \sum_{j=1}^k c_j D_{R_{ij}}^k$  where  $D_j^k$  is the expected value of the  $j$ th order statistic of a random sample of size  $k$  from the standard normal distribution, and the  $c_j$ 's are constants which reflect the a priori ordering. (A brief null distribution table is given for the case  $c_j = j$ ,  $j = 1, \dots, k$ .) Pitman efficiencies are presented for  $T_n(R)$  with respect to some of its distribution-free competitors, viz. (i) the most-powerful test of  $H_S$  for a simple normal ordered alternative; (ii) Page's test [*J. Amer. Statist. Assoc.* **58** (1963) 216–230]; (iii) a test proposed by Tukey [*Ann. Math. Statist.* **28** (1957) 987–992]. (Received March 16, 1970.)

**125-45. Some further results on the construction of partially balanced weighing designs with two associate classes.** K. V. SURYANARAYANA, Fayetteville State College.

A paper entitled "Construction of some partially balanced weighing designs with two associate classes," contributed by the author at the 1969 Annual Meeting of the Joint Statistical conference at New York, dealt with the definition and construction of Partially Balanced Weighing Designs with two associate classes, where the study was restricted to the triangular, Latin square and Group divisible schemes. The corresponding study for the case of cyclic scheme is discussed in a joint paper (sent for publication) by I. M. Chakravarti and the author. The latter deals with two special series of this kind. The present paper is an extension of this work and deals with the construction of some new series of two class PBWD's with cyclic scheme. The first main theorem deals with a PBWD described by:  $(v, b, r, p) = ((12t + 1), (12t + 1)t, 6t, 3)$ , with  $t$  initial blocks. The second main theorem deals with the construction of another series of PBWD's where the block size is a varying parameter (along with the number of treatments):  $(v, b, r, p) = ((4tp + 1), (4tp + 1)t, 2tp, p)$ , with  $t$  initial blocks. (Received March 16, 1970.)

**125-46. A test for comparing diversities based on the Shannon formula.** KERMIT HUTCHESON, University of Georgia.

Several indices of dispersion have been suggested by ecologists, the most commonly used being the measure corresponding to the entropy concept. Much has been done recently toward finding exact moments, asymptotic moments, and the distributional properties of  $h = -\sum_{i=1}^s (n_i/n) \ln(n_i/n)$ . Based on the recent work mentioned above, a  $t$ -test is proposed for the purpose of comparing the diversities of populations as measured by " $h$ ." (Received March 16, 1970.)

**125-47. Matrix differentiation after partitioning, and other results.** D. S. TRACY AND R. P. SINGH, University of Windsor.

Various interrelations of  $Y_r$ ,  $Y_c$  (vector representations of matrix  $Y_{m \times n}$ ) in terms of  $I_{(m)}$ ,  $I_{(n)}$  introduced in Tracy and Dwyer [*J. Amer. Statist. Assoc.* **64** (1969) 1576–1594] are studied. This is carried on for differentials  $dY_r$ ,  $dY_c$  and some results of Neudecker [*J. Amer. Statist. Assoc.* **64** (1969) 953–963] involving Kronecker products are presented in more compact form. As applications, shorter proofs of Lemma 3.2.2 and Theorem 8.2.1 in Anderson's *Introduction to Multivariate Statistical Analysis* [Wiley, New York (1958)] are obtained using the rule for identification of matrix derivatives. For a large matrix  $Y$ , differentiation after partitioning into blocks is proposed. Here  $Y_{[r]}$ ,  $Y_{[c]}$  are obtained as block-generalizations of  $Y_r$ ,  $Y_c$  and matrix derivatives like  $\partial Y_{[c]}/\partial X_{[c]}$  are introduced. Differentiation of linear matrix functions  $Y = F(X)$  subsequent to such partitioning is discussed. The theorems so obtained are useful in differentiating large matrices. (Received March 16, 1970.)

**125-48. On the distribution of the latent roots of complex Wishart matrix (non-central case).** TAKESI HAYAKAWA, University of North Carolina and Institute of Statistical Mathematics, Tokyo.

Let  $T$  be an  $m \times n$  ( $m \leq n$ ) arbitrary complex matrix and  $S$  be an  $m \times m$  positive definite Hermitian matrix. Then we define a generalized Hermite polynomial (g.H.p.)  $\tilde{H}_\kappa(T)$  and a generalized Laguerre polynomial (g.L.p.)  $\tilde{L}_\kappa^\gamma(S)$ ,  $\gamma > -1$  by the same way as Hayakawa ((1969) *Ann. Inst. Statist. Math.*) and Constantine ((1967) *Ann. Math. Statist.*), respectively. We can derive some properties of g.H.p. and g.L.p. (i) generating function of g.H.p. and g.L.p. (ii) a relation between g.H.p. and g.L.p. (iii) Mehler's formula for g.H.p. (iv) a relation between g.L.p. and a univariate Laguerre polynomial. By using these properties we can obtain a pdf of latent roots of a complex non-central Wishart matrix, which is different form as James ((1964) *Ann. Math. Statist.*), and pdf of a maximum latent root and of trace which is essentially same as non-central  $\chi^2$  distribution. (Received March 16, 1970.)

**125-49. Queuing processes at fixed-cycle traffic lights.** JAMES G. LITTLE, JR., Cornell University.

In this paper two traffic intersection models are presented. The first model considers the queue in a left turn lane at a fixed-cycle traffic light having no separate left turn phase. For simplicity of analysis a lower bound queue is studied. Expressions are obtained for the steady state probabilities and expectations of the queue length at the end of each cycle and of the maximum queue length during the cycle. The second model considers the virtual waiting time process of straight through traffic at a fixed-cycle traffic light. The model is formulated in both discrete and continuous time versions. In the discrete time model a matrix equation for the steady state probabilities is obtained. A set of integro-differential equations for the transition distribution function is obtained for the continuous time model. Simulation studies were made to supplement the analytic results for both of the intersections studied. (Received March 17, 1970.)

**125-50. Tests of coordinate independence for a bivariate sample on a torus.** EDWARD D. ROTHMAN, University of Michigan.

Blum, Kiefer and Rosenblatt [*Ann. Math. Statist.* **32** (1961) 485–492] consider tests of coordinate independence of a bivariate sample in the plane based on the sample distribution function. In this paper we treat this problem with the plane replaced by a torus. Unfortunately, the tests proposed in Blum, et al. are not invariant with respect to choice of origin on the torus. A test with this property rejects for large values of  $D_n = n [T_n(x, y) + T_n(x, y) dF_n(x, y) - T_n(x, y) dF_{n1}(x) - T_n(x, y) dF_{n2}(y)]^2 dF_n(x, y)$  where  $T_n(x, y) = F_n(x, y) - F_{n1}(x)F_{n2}(y)$ ,  $F_n(x, y)$  is the sample distribution

function for the joint sample and  $F_{n_1}(x)$  and  $F_{n_2}(y)$  are sample distribution functions for the first and second coordinates. The asymptotic distribution function under the null hypothesis of independence is shown to be an infinite convolution of  $X^2$  random variables and the asymptotic distribution under a sequence of alternatives converging to the null hypothesis is found to be an infinite convolution of noncentral  $X^2$ . (Received March 17, 1970.)

**125-51. Individual versus social optimization in queuing systems (preliminary report).**  
NIELS C. KNUDSEN, Cornell University.

In the present paper it is shown that the imposition of tolls on customers arriving at a queuing system may often lead to the attainment of social optimality. The basic model underlying the study is the M/M/s queuing model ( $s \geq 1$ ). It is assumed that each customer who receives service obtains a reward of  $R$ , and that  $h(t)$ , the waiting cost of a customer who spends  $t$  time units in the queuing system, may be any monotone increasing function of  $t$ . Applications are discussed. The paper gives an extension and simplification of proofs of results for the single server queue with linear waiting cost, obtained by P. Naor in his paper [The regulation of queue size by levying tolls. *Econometrica* 37 (1969) 15–24]. (Received March 17, 1970.)

*(Abstracts of papers to be presented at the Annual meeting, Laramie, Wyoming, August 25–28, 1970. Additional abstracts will appear in future issues.)*

**126-2. Two-sided confidence intervals for ranked means.** EDWARD J. DUDEWICZ,  
University of Rochester.

Suppose given  $k (\geq 2)$  populations  $\pi_1, \dots, \pi_k$  such that observations from population  $\pi_i$  have density function  $f(x - \theta_i)$ ,  $x \in \mathcal{D}$ , where the location parameter  $\theta_i$  is unknown ( $1 \leq i \leq k$ ). Assume  $E_f \equiv \int_{-\infty}^{\infty} xf(x) dx < \infty$ . Let the population means be denoted by  $\mu_1, \dots, \mu_k$  and their ranked values by  $\mu_{[1]} \leq \dots \leq \mu_{[k]}$ . Dudewicz [*Naval Res. Logist. Quart.* 17 (1970)] gave upper and lower confidence intervals for  $\mu_{[i]}$  ( $1 \leq i \leq k$ ) and studied existence questions for certain intervals. We use his one-sided intervals and  $P(AB) = P(A) + P(B) - P(A \cup B)$  to obtain a class of two-sided intervals on  $\mu_{[i]}$  ( $1 \leq i \leq k$ ). For the case of  $\mu_{[k]}$ , Lal Saxena and Tong [*J. Amer. Statist. Assoc.* 64 (1969) 296–299] previously proposed another 2-sided interval. These are compared in the case of normal populations, and ours has smaller length for most values of  $k$ . (Received February 25, 1970.)

**126-3. Balanced optimal  $2^m$  fractional factorial designs of resolution  $V$ ,  $m = 7$ .**  
D. V. CHOPRA AND J. N. SRIVASTAVA, Wichita State University and  
Colorado State University.

A balanced  $2^m$  fractional factorial design  $T$  of resolution  $V$  is equivalent to a partially balanced array (PBA)  $T$  with 2 Symbols and Strength 4. A partially balanced array of Strength 4 is a matrix  $T(m \times N)$  with elements 0 and 1 (where  $N$  denotes the number of treatment-combinations in the design) such that if  $T_0$  is any  $(4 \times N)$  submatrix of  $T$ , and  $\alpha$  is any vector with 4 elements (out of which  $i$  elements are nonzero,  $i = 0, 1, 2, \dots, 4$ ), then  $\alpha$  occurs exactly  $\mu_i$  times as a column of  $T_0$ , where the nonnegative integers  $\mu_i$  are independent of  $T_0$ . Given any design  $T$ , we assume that interest lies in the estimation of the general mean, main effects and 2-factor interactions, under the assumption that the higher order interactions are zero. Let  $(V)_T$  denote the variance-covariance matrix of the usual estimate of the parameter vector under the design  $T$ . Then we want to choose  $T$  such that (i)  $T$  is balanced, i.e.  $(V)_T$  is invariant under a permutation of factor symbols, and (ii)  $t \sim (V)_T$  is a minimum. In this paper, for  $m = 7$ , and various practical values of  $N$  arrays satisfying

(i) and (ii) are obtained. The parameters  $\mu' = (\mu_0, \mu_1, \mu_2, \mu_3, \mu_4)$  are: (43113), (53113), (63113), (73113), (73114), (83114), (93114), (43126), (53126), (63126), (73126), (83126), (83127), (83128), (23323), (33323), (43323), (43324), (53324), (53325), (44323), and (23336); the actual arrays along with their efficiencies with respect to the corresponding (possibly non-existent) orthogonal designs will be presented in the paper. (Received March 20, 1970.)

**126-4. A truncated Bernoulli trials test.** DAVID C. BOSSARD, Daniel H. Wagner, Associates.

Let  $p_j(t)$  be the a priori probability that event  $E$  occurs prior to time  $t$  for trial  $j$ . Observe  $N$  independent trials over the time interval  $[0, T]$ , and observe the number,  $N_0$ , of occurrences of  $E$ . The Bernoulli Trials test checks whether  $N_0$  agrees with the theoretical expected number,  $\sum p_j(T)$ . Suppose, however, that the occurrence of  $E$  in trial  $j$  disrupts the trial so that  $p_j(t)$  cannot be determined for  $t > t_j =$  the time  $E$  occurs on trial  $j$  (although  $p_j$  is an a priori function, its evaluation in any particular trial may be related to parameters that describe the trial such as the history of cigarette smoking in a study relating cigarette smoking to mortality). Then the Bernoulli test cannot be performed because  $p_j(T)$  is known only for those trials on which  $E$  did not occur. This paper gives a modification of the test which only requires knowledge of the functions  $p_j, j = 1, \dots, N$ , up to the time that  $E$  occurs (or to  $T$ , if  $E$  does not occur). The new statistic has a distribution with mean 0 and a variance which can be estimated with the given information. The technique for estimating the variance can be used for all moments, and hence to estimate the characteristic function, if desired. (Received March 20, 1970.)

*(Abstracts of papers contributed by title.)*

**70T-24. A theorem on unimodal distribution** (preliminary report). PANKAJ GHOSH, Catholic University of America.

B. V. Gnedenko, and A. N. Kolmogorov defined a distribution function  $F(x)$  to be unimodal if there exists at least one value  $x = a$  such that  $F(x)$  is convex for  $x < a$  and concave for  $x > a$ . The point  $x = a$  is called the vertex of the distribution. In this paper the author has proved a theorem that: A unimodal distribution has either a single vertex or uncountably many vertices. As an example a uniform distribution over the interval  $(0, 1)$  has infinitely many vertices and it is unimodal because one can trivially show that its characteristic function  $f(t)$  can be expressed as  $f(t) = 1/t \int_0^1 g(u) du, -\infty < t < \infty$ , where  $g(u)$  is the characteristic function of degenerate distribution with degeneracy at  $x = 1$ , in other words  $f(t)$  satisfies the Khinchine's theorem. (Received January 26, 1970.)

**70T-25. Multi-sample tests for scale.** M. RAGHAVACHARI, Carnegie-Mellon University.

A test of B. V. Sukhatme for the two-sample scale problem is generalized to the case of testing of equality of scale parameters of  $c$ -populations. It is also shown that the Pitman's A.R.E. of this generalized test relative to other existing tests of this problem is the same as the A.R.E. of Sukhatme's test relative to the well-known tests for the two-sample case. It is also shown that for the multi-sample case and for a general class of distributions the tests proposed by Lehmann, Bartlett and Scheffé are all equivalent to one another in the sense of Pitman's A.R.E. (Received January 27, 1970.)

**70T-26. Asymptotically optimal tests for finite Markov chains.** LUIS B. BOZA, University of California at Berkeley.

The results described in this note extend those of Prof. W. Hoeffding for multinomials (*Ann. Math. Statist.* **36** 369–408), to the case of a discrete time, finite Markov chain, with fixed initial state and stationary transition behavior. Using Whittle's formula (*J. Roy. Statist. Soc. Ser. B* **17** 235–242) a large deviation result for  $\Pr(C \in A)$  is obtained.  $C$  is the transition count matrix of the process up to time  $T$ , and  $A$  is a set of "realizable" transition count matrices at time  $T$ . This result is used as a basis for the asymptotic comparison of any given sequence of tests for the transition probability matrix, with a suitably constructed sequence of likelihood ratio tests. The comparison is done at fixed alternatives, when the sizes of the given tests decrease to zero at a certain rate, as the length of the observed path increases. The criterion for comparison is the ratio of type II error probabilities. As an example, chi-square tests and likelihood ratio tests for a simple hypothesis are compared in the two by two case. The applicability of this method to Anderson and Goodman's model (*Ann. Math. Statist.* **28** 89–110) is briefly discussed. (Received January 27, 1970.)

**70T-27. Estimates of logarithms of interactions for  $S^m$  factorial designs with an underlying exponential distribution (preliminary report).** RUSSELL MAIK, Horace Mann Educators.

The best estimates of the natural logarithms of the interaction degrees of freedom in modified saturated  $S^m$  factorial designs of resolution three and five in which each interaction degree of freedom and each treatment combination is replaced by its natural logarithm, i.e.  $\text{Exp}(\ln \theta) = \ln \theta^* = E' \Delta_0^{-2} (\ln \mathbf{L})$  are obtained. In the above equation,  $\text{Exp}$  denotes expected value,  $E'$  is the design matrix and  $\Delta_0$  is a diagonal matrix related to  $E'$ , while the  $\theta$ 's are treatment combinations and the  $L$ 's are interaction degrees of freedom. Also, each  $\theta_i^*$  corresponds to the scale parameter of a univariate exponential with a location parameter of zero in an experiment in which we observe the first  $k_i$  observations in a sample of size  $n_i$ . Various modified saturated  $2^m$  factorial designs under the above model are examined to determine the  $k_i$ 's when their sum is held constant. If all the  $k_i$ 's are equal in a modified  $S^m$  factorial design under the above model and our design matrix  $E'$  has rank equal to the number of parameters being estimated, one of which is the natural logarithm of the general mean, then the maximum likelihood estimate of  $\ln \mathbf{L}$  is identical to the estimates of  $\ln \mathbf{L}$  obtained by maximizing the likelihood function with respect to the vector  $\ln \theta^*$ . (Received January 29, 1970.)

**70T-28. Some results on invariant sets for translation parameter family of probability measures.** NEIL W. RICKERT AND PRAMOD K. PATHAK, University of Illinois at Chicago Circle.

Let  $\mu$  be a given probability measure on  $(X, \Sigma)$  where  $X$  is some finite dimensional Euclidean space and  $\Sigma$  is the class of Borel sets on  $X$ . A set " $A$ " is called  $\mu$ -invariant iff for all  $\theta \in X$ ,  $\mu(A + \theta) = \mu(A)$ . The class of  $\mu$ -invariant sets is denoted by  $\mathcal{A}(\mu)$ . In this paper, we present the following main results concerning  $\mu$ -invariant sets: (1) Suppose that the characteristic function of  $\mathcal{A}(\mu)$  vanishes on a compact set. Then  $A \in \mathcal{A}(\mu)$  implies that  $\mu(A) = 0$  or 1. (2) If  $\mathcal{A}(\mu)$  is a  $\sigma$ -field, then there exists a closed subgroup  $H \subset X$  such that  $A \in \mathcal{A}(\mu)$  iff for all  $h \in H$ ,  $\lambda[(A + h)\Delta A] = 0$ , where  $\lambda$  denotes the Lebesgue measure on  $(X, \Sigma)$ . (3) We also provide proofs (or counterexamples) of most of the conjectures on  $\mu$ -invariant sets contained in the paper of Basu and Ghosh [*Ann. Math. Statist.* **40** (1969) 162–174]. (Received February 2, 1970.)

**70T-29. Functions of order statistics.** GALEN R. SHORACK, University of Washington.

Let  $0 \leq X_{n1} \leq \dots \leq X_{nn} \leq 1$  be the order statistics of a uniform  $(0, 1)$  sample. Let  $F_n$  be the empirical df and  $F_n^{-1}$  its left continuous inverse. Let  $g$  be a fixed function and let the  $c_{ni}$ 's be fixed constants. Let  $T_n = n^{-1} \sum_1^n c_{ni} g(X_{ni})$ . For  $0 < t < 1$  let  $L_n(t) = n^{\frac{1}{2}}[g(F_n^{-1}(t)) - g(t)]$ . In Shorack (*Ann. Math. Statist.* **40** (1969)) asymptotic normality of  $T_n$  and convergence of the  $L_n$  processes in integral and supremum metrics is established. In the present report these results are greatly improved. Lemma 4.3 of the above reference is replaced by linear bounds and the  $c_{ni}$ 's are handled differently; thus the quantity  $\beta_n$  does not appear in the present paper. The resulting theorems have weaker assumptions, stronger conclusions, more natural statements and shorter proofs. (Received February 3, 1970.)

**70T-30. Functions of ordered "renewal" spacings.** GALEN R. SHORACK, University of Washington.

Let  $0 \leq Y_1 \leq \dots \leq Y_n$  be an ordered sample from a continuous df  $F$  having  $F(0) = 0$ . Let  $T_n = n^{-1} \sum_1^n c_{ni} h(Y_i/\bar{Y})$  where  $h$  is a fixed function and the  $c_{ni}$ 's are fixed constants. For  $0 \leq t \leq 1$  let  $L_n(t) = n^{\frac{1}{2}}[G_n^{-1}(t) - g(t)]$  where  $G_n$  is the empirical df of the  $Y_i/\bar{Y}$  and  $g$  denotes  $F^{-1}$ . Asymptotic normality of  $T_n$  and convergence of the  $L_n$  processes in integral and supremum metrics is established. The technique used is that established in Pyke and Shorack (*Ann. Math. Statist.* **39** (1968)) and also the preceding abstract. Statistics of the form  $T_n$  are used in testing exponentiality. (Received February 3, 1970.)

**70T-31. Characterization of distributions via order statistics.** PREM S. PURI, Purdue University.

Let  $X_1, X_2, \dots, X_n$  be independent and identically distributed random variables (rv) whose common distribution is the same as that of a nonnegative rv  $X$ . Let  $X^{(1)} \leq X^{(2)} \leq \dots \leq X^{(n)}$  be their order statistic. Earlier we considered (see Abstracts in *Ann. Math. Statist.* (1969) **40** page 725 and page 1517) for  $n = 2$  the problem of characterizing all possible distributions of  $X$  with the property that the distributions of  $|X_1 - X_2|$  and  $X$  are identical. Here we consider the general problem of characterizing distributions with the property (P) that the distribution of  $X^{(n)} - X^{(n-1)}$  coincides with that of  $X$  for any fixed  $n \geq 2$ . As before, it is shown that in general such a distribution has to be either purely discrete, or purely continuous or singular and that it cannot be their mixture. Let  $X$  satisfy the above property (P). Then it is shown that  $X$  has a moment generating function. Also it is shown under certain conditions that if  $X$  is a lattice type rv, it must have a "geometric type distribution", and if it is absolutely continuous, its distribution has to be exponential with pdf  $f(x) = \theta e^{-\theta x}$ , for  $x \geq 0$ , and zero elsewhere, with  $\theta > 0$ . The singular case is not considered. However, as before, in this case  $X$  cannot be bounded as it is shown that if  $X$  is bounded, then its distribution is either degenerate with  $P(X=0) = 1$ , or is the one with  $P(X=0) = 1 - P(X=a) = 1/(n)^{(n-1)^{-1}}$ , for some  $a > 0$ . Finally, the more general problem of characterizing the distributions of  $X$  with the property that the distributions of  $(n-r+1)(X^{(r)} - X^{(r-1)})$  and  $X$  are same has also been considered. Here  $r$  is a fixed integer with  $1 \leq r \leq n$  and by convention  $X^{(0)} = 0$ . (Received February 11, 1970.)

**70T-32. A stability of symmetrifications of product measures with few distinct factors.** JAMES HANNAN AND J. S. HUANG, Michigan State University and University of Waterloo.

Let  $\mathcal{P} = \{F_0, \dots, F_m\}$  be a class of probability measures on  $(\mathcal{X}, \mathcal{B})$ . For any signed measure  $\tau$  on  $\mathcal{B}^N$ , let  $\tau^*$  be the average of  $\tau g$  over all  $N!$  permutations  $g$  and let  $\|\tau\| = \sup\{|\tau(C)| : C \in \mathcal{B}^N\}$ .

Let  $d_{ij} = \|F_i - F_j\|$  and  $K(x) = .5012 \cdots x(1-x)^{-3/2}$ . For any nonnegative integral partitions  $N = (N_0, \dots, N_m)$  and  $N' = (N'_0, \dots, N'_m)$  of  $N$ , let  $\delta_i = N'_i - N_i$  and  $\Lambda_i = (N'_i \wedge N_i) + 1$ . With  $\tau = \sum_i F_i^{N_i} - \sum_i F_i^{N'_i}$  and  $n = \#\{i \mid \delta_i \neq 0\} - 1$ , we bound  $\|\tau^*\|^2$  by (T3)  $nK(\hat{d}) \sum_i \delta_i^2 \Lambda_i^{-1}$  with  $\hat{d} = \sup\{d_{ij} \mid \delta_i \neq 0, \delta_j \neq 0\}$  and, if  $\mathcal{P}$  is internally connected by chains with non-orthogonal successive elements, by (T4)  $\frac{1}{2}mK(\hat{d})(\sum_i |\delta_i|)^2 (\sum_i \Lambda_i^{-1})$  with  $\hat{d} = \sup\{d_{ij} \mid F_i \not\perp F_j\}$ . The bound (T3) is finite if and only if the  $F_i$  with  $\delta_i \neq 0$  are pairwise non-orthogonal and (T4) is designed to replace it otherwise. (Received February 11, 1970.)

**70T-33. On a theorem of Bahadur on the rate of convergence of test statistics.**  
 M. RAGHAVACHARI, Carnegie-Mellon University.

Let  $x_1, x_2, \dots, x_n$  be independent and identically distributed random variables whose distribution depends on a parameter  $\theta \in \Theta$ . Let  $\Theta_0$  be a subset of  $\Theta$  and consider the test of the hypothesis that  $\theta \in \Theta_0$ .  $L_n(x_1, \dots, x_n)$  is the level attained by a test statistic  $T_n(x_1, \dots, x_n)$  in the sense that it is the maximum probability under the hypothesis of obtaining a value as large or larger than  $T_n$  where large values of  $T_n$  are significant. Under some assumptions R. R. Bahadur showed that when a non-null  $\theta$  obtains,  $L_n$  cannot tend to zero at a rate faster than  $[\rho(\theta)]^n$  where  $\rho$  is a function defined in terms of Kullback-Liebler information numbers. Bahadur posed the question if this result is true without any assumptions whatsoever and this paper answers this question in the affirmative. Some aspects of the relationship between the rate of convergence of  $L_n$  and the rate of convergence of the size  $\alpha_n$  of the tests are also studied and it is shown that  $n^{-1} \log L_n$  tends in probability to some  $h(\theta)$  when  $\theta \in \Theta - \Theta_0$  obtains if and only if  $n^{-1} \log \alpha_n$  tends to  $h(\theta)$  and  $h(\theta)$  is independent of  $p$  with  $p, 0 < p < 1$  as the given asymptotic power of the test when  $\theta$  obtains. (Received February 16, 1970.)

**70T-35. Binomial group-testing with three outcomes.** S. KUMAR, University of Wisconsin-Milwaukee.

The problem is to classify each of the  $N$  given units into one of the two disjoint categories by means of group-testing. We shall call the two categories good and defective. Test on an individual unit classifies it into one of the two disjoint categories. A simultaneous test on a group of  $x (\geq 2)$  units has one of the three possible outcomes: (i) all the  $x$  units are good, (ii) all the  $x$  units are defective, (iii) there is at least one good unit and at least one defective unit present. It is assumed that the  $N$  units can be represented by independent binomial random variables with probability  $q$  and  $p = 1 - q$  of being good and defective respectively. The problem is to devise a procedure, for known value of  $q$ , which minimizes the expected number of tests  $E(T)$  to classify all the  $N$  units as good or defective. A procedure, which is optimal in a certain class of procedures, is proposed. The lower bounds for the expected number of tests under any procedure are obtained by using information theory and Huffman code. Furthermore we prove the following Theorem: Let  $R$  be an optimal procedure among all procedures and  $E(T | R)$  be the expected number of tests under  $R$ . Then  $E(T | R) \leq (N/2) + Npq$  if  $N$  is even;  $E(T | R) \leq (N+1)/2 + (N-1)pq$  if  $N$  is odd. (Received February 16, 1970.)

**70T-36. On the estimation of means in a bivariate normal distribution with equal marginal variances.** M. AHSANULLAH, Food and Drug Directorate, Tunney's Pasture.

This paper considers the problem of estimation of the mean  $\mu$ , of one of the components of bivariate normal distribution (BND) with equal marginal variances from a sample of size  $n$ . The result of a preliminary test of hypothesis that the means  $\mu_1, \mu_2$  of the two components of BND are equal, is used to define an estimator  $\bar{x}^*$  for  $\mu_1$ . The bias and mean square error of this estimate are studied and the regions in the parameter space in which  $\bar{x}^*$  has small mean square error than

$\bar{x}_1$ , the sample mean of the first component are determined. The relative efficiency of this estimator to the usual estimator is tabulated and the tables can be used to determine a proper choice of significance level of the preliminary test. (Received February 16, 1970.)

**70T-37. A simultaneous confidence set for means and a stepwise procedure of selecting variables on which two multivariate normal populations differ.** NICO F. LAUBSCHER, National Research Institute for Mathematical Sciences.

It is well known that a simultaneous confidence interval may be computed for any linear combination of the differences in mean values between two populations of the variables considered. It is shown that if the overall Hotelling  $T^2$ -test rejects the hypothesis of equal mean vectors, there exists at least one linear combination (which is exhibited) of the differences in population means for which the above confidence interval is significantly different from zero (at the same level as the overall test). It is also shown that for this particular linear combination every term may be interpreted as the contribution of difference in mean on a specific variable towards the value of the overall test statistic. A method which is similar to that of backward elimination in stepwise regression is suggested for determining those variables making the largest contribution towards the overall significance. (Received February 24, 1970.)

**70T-38. On the distributions of the products of order statistics.** LEELA GULATI, Lucknow University.

Let  $X_{(1)} \leq X_{(2)} \leq \dots \leq X_{(n)}$  denote the order statistics from the exponential distribution with a nonnegative range. Then the density of  $Z_{(i)} = X_{(i)} \cdot X_{(i+1)}$  ( $i = 1, 2, \dots, n-1$ ) is  $f(z) = (n!/(i-1)!) \cdot (1/(n-i-1)!) \binom{i-1}{k} (-1)^k K_0(p)$  where  $p^2/4 = z(k+1)(n-i)$  and  $K_0(p)$  is the modified Bessel function of the second kind and of order  $\nu = 0$ . The distributions are useful in the problem of selection and ranking rules if each  $X_{(i)}$  is a  $j$ th order statistic. For that let  $X_{(0)}, X_{(1)}, \dots, X_{(p)}$  denote the  $j$ th order statistics from a continuous distribution. Define  $Y_{(i)} = X_{(0)} \cdot X_{(i)}$  ( $i = 1, 2, \dots, p$ ) then  $Y_{(1)}, Y_{(2)}, \dots, Y_{(p)}$  form a sequence of exchangeable random variables. A new selection procedure for selecting the population with largest quantile is proposed using  $Y_{(\max)}$  and  $Y_{(\min)}$ . The asymptotic and exact distributions of various order statistics are derived with specific results for extreme values. (Received February 26, 1970.)

**70T-40. Note on efficiencies of rank tests relative to the  $F$ -test in testing difference of scale.** FRIEDRICH-WILHELM SCHOLZ, University of California at Berkeley.

In the two-sample situation we want to test  $F(x)$  vs.  $G(x) = F(sx)$ ,  $s > 1$ . Chernoff and Savage (*Ann. Math. Statist.* **29** (1958) 972-994) provide a large class of rank tests for which efficacies in the above model can be computed. This note shows that the asymptotic efficiency of all these rank tests relative to the  $F$ -test has infimum zero. This is established by considering the family of densities:  $f_\alpha(x) = C(\alpha)|x|^{-\alpha}(1-|x|)^\alpha$  for  $e_\alpha \leq |x| \leq 1$ ,  $f_\alpha(x) = C(\alpha)e_\alpha^{-\alpha}(1-e_\alpha)^\alpha$  for  $|x| \leq e_\alpha$  and  $f_\alpha(x) = 0$  otherwise, where  $e_\alpha = \exp(-(1-\alpha)^{-1})$  and  $0 < \alpha < 1$ . It follows that the result holds even if attention is restricted to distributions whose densities are uniformly bounded, unimodal, and symmetric. The proof is essentially a generalization of the note by M. Raghavachari (*Ann. Math. Statist.* **36** (1965) 1306-1307) who proved the above result for the normal scores test. (Received March 3, 1970.)

**70T-41. The exact distribution of Wilks' criterion.** A. M. MATHAI AND P. N. RATHIE, University of Montreal and University of Waterloo.

This article gives the exact distribution of Wilks' likelihood ratio criterion for testing linear hypotheses about regression coefficients. The density and the distribution function are obtained



for the most general cases of the parameters and are represented in simple algebraic functions from which the percentage points can be computed. The techniques of partial fractions and inverse Mellin transform are used to obtain the results. Some particular cases are shown to agree with the known results. (Received March 11, 1970.)

**70T-42. The exact distribution of Votaw's criterion.** A. M. MATHAI AND P. N. RATHIE, University of Montreal and University of Waterloo.

This article gives the exact distribution of Votaw's criterion for testing compound symmetry in a multivariate normal distribution. In other words the hypothesis is that the population covariance matrix is bipolar. The exact distribution, in the most general cases, is obtained with the help of inverse Mellin transform, Calculus of residues and the properties of Psi and Riemann Zeta functions. The density and the distribution functions are represented in terms of Meijer's  $G$ -function, Braaksma's  $H$ -function as well as in simple computable functions from which the percentage points can be computed. The particular cases are shown to agree with the known results. (Received March 11, 1970.)

**70T-43. On the minimum variance unbiased estimator of  $\sigma^2$ .** S. A. PATIL AND J. B. PATEL, Tennessee Technological University.

Let  $X_1, X_2, \dots, X_m$ , where  $X_i' = (x_{i1}, x_{i2}, \dots, x_{ip})$ , be a sample of size  $m$  from a  $p$ -variate normal population with mean vector  $\mu$  and common variance  $\sigma^2$  and correlation matrix  $P = \|\rho_{ij}\|$ . Then  $S_1^2, S_2^2, \dots, S_p^2$  are the variance components of the sample. The variance covariance matrix of  $S_i^2$  is  $R$  where  $R = \|R_{ij}\|$ ,  $R_{ij} = \rho_{ij}^2$ . It is shown that  $R$  is positive definite whenever  $P$  is positive definite. The minimum variance unbiased estimator based on the linear combination of  $S_i^2$  is obtained for  $\sigma^2$ . The estimator is expressed in terms of elements of  $R$ . The efficiency of this estimator with the estimator based on the single component is considered. The distribution of the estimator is discussed. For the trivariate case the weights in the estimator are shown to be positive, and if the weights are equal then the  $R_{ij} = \rho^2$  for  $i \neq j$ . The estimators for special cases of the correlation matrix having equal correlation  $\rho_{ij} = \rho$ ,  $i \neq j$ , and the correlation matrix having the form of serial correlation with  $\rho_{ij} = \rho^{|i-j|}$  are considered in some detail. (Received March 16, 1970.)

**70T-44. Optimal experimental design for polynomial regression.** STEPHEN M. STIGLER, University of Wisconsin.

The problem of choosing the optimal design to estimate a regression function which can be well-approximated by a polynomial is considered, and two new optimality criteria are presented and discussed. The use of these criteria is illustrated by a detailed discussion of the case that the regression function can be assumed approximately linear. These criteria, which can be considered as compromises between the incompatible goals of inference about the regression function under an assumed model and of checking the adequacy of the model, are found to yield designs which are superior in many respects to others which have been proposed to deal with this problem, including minimum bias designs. (Received March 17, 1970.)

**70T-45. Rates of convergence of test statistics with applications to Bahadur and Hodges-Lehmann efficiencies.** M. RAGHAVACHARI, Carnegie-Mellon University.

Let  $x_1, x_2, \dots, x_n$  be independent and identically distributed random variables whose distribution depends on a parameter  $\theta$ ,  $\theta \in \Theta$ . Let  $\Theta_0$  be a subset of  $\Theta$  and assume that large values of a statistic

$T_n(x_1, \dots, x_n)$  are significant for the null hypothesis  $H: \theta \in \Theta_0$ . Let  $L_n$  be the level attained by  $T_n$  in the terminology of stochastic comparison of tests as formulated by R. R. Bahadur. For a class of test statistics it is shown that, when a  $\theta \in \Theta - \Theta_0$  obtains,  $L_n$  cannot tend to zero at a rate faster than  $[\tau(\theta)]^n$ ,  $0 < \tau(\theta) < 1$  where  $\tau(\theta) \cong \exp \{-J(\theta)\}$  with  $J(\theta) = \inf \{K(\theta, \theta_0): \theta_0 \in \Theta_0\}$  and  $K(\theta, \theta_0)$  the Kullback-Liebler Information number. Some of the results on the exponential rates of convergence to zero of the first and second kinds of errors when the other is held fixed are extended to tests based on a given statistic. These extensions yield a method to evaluate the "exact slope" defined by Bahadur and the "Index" defined by Hodges and Lehmann for some well-known test statistics not dealt with so far. These tests include among others the test of correlation coefficient and the multiple correlation coefficient from normal samples and also a new two-sample non-parametric test for location when the distributions are one-sided. (Received March 17, 1970.)

**70T-46. Two characterizations of the first passage time distribution of a standard Brownian Motion.** M. T. WASAN, Queen's University.

We assign a set of conditions to a strong Markov Process and arrive at a differential equation analogous to the Kolmogorov equation. However, in this case the duration variable is the net distance travelled and the state variable is a time, a situation precisely opposite to that of Brownian Motion. By solving this differential equation under suitable boundary conditions we obtain a function which is the density function of first passage time of Standard Brownian Motion Process. We characterize the first passage time process of the standard Brownian Motion as an infinitely divisible process by embedding Brownian Motion in the infinitely divisible process. (Received March 23, 1970.)