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SOME OBSERVATIONS ON THE WHITE–HULTQUIST PROCEDURE FOR THE CONSTRUCTION OF CONFOUNDING PLANS FOR MIXED FACTORIAL DESIGNS[1]

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1. Summary. In a recent paper [3], White and Hultquist extended the use of finite fields for the construction of confounding plans to include "asymmetrical" or "mixed" factorials. The technique, in their own words, was to define addition and multiplication of elements from distinct finite fields by mapping these elements into a finite commutative ring containing subrings isomorphic to each of the fields in question. The standard techniques were then applied to the asymmetrical case illustrating the procedure with a numerical example for $3^2 \times 5$ factorial experiment. Later, Raktoe [1] also provided an equivalent theoretical basis for the results obtained by White and Hultquist [3], worked out a generalization of the technique and illustrated his procedure with an example of $2^2 \times 3 \times 5$ factorial experiment.

It appears, however, that to provide a basis of the required calculus covering confounding plans of "mixed factorials" of the types discussed by them, it may not be necessary to invoke the properties of finite fields and to combine them. Instead, properties of finite multiplicative groups may be sufficient to construct such confounding plans.

The aim of the present note is to indicate that this alternative approach, when it exists, is structurally identical with the procedure as outlined in [3], and that this methodology is simple, taking as it does only the properties of multiplicative groups. The procedure has been illustrated with reference to the same example of $3^2 \times 5$ factorial design as discussed in full in [3]. The correspondence relationships between the levels of the factors and the elements of the group may be so worked out that the complete model for the $3^2 \times 5$ experiment as provided in Table 4.1 of [3] would come out exactly the same by this alternative approach. The procedure of White and Hultquist would require that the number of levels of a factor be prime. But by the procedure presented here, it would be possible to cover mixed factorials of other types where the number of levels of a factor may not be a prime number.

Analysis of variance is not attempted here, as such analysis can be carried out following the procedures as given by White and Hultquist [3].

2. Introduction. White and Hultquist [3] combines residue classes of integers (mod p_1) with the residue classes of integers (mod p_2), all elements of both sets of residue classes being considered as members of the set of residue classes of integers (mod $p_1 \cdot p_2$) where

(i) p_1, p_2 are prime numbers;

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(ii) the correspondences between $i(p_1)$ or $j(p_2)$ and the integers (mod $p_1 \cdot p_2$) [x (mod a) being written for short as $x(a)$] are determined by the following rules: $i(p_1)$ and $j(p_2)$ are made to correspond respectively to $\phi[i(p_1)]$ and $\phi[j(p_2)]$ as elements (mod $p_1 \cdot p_2$), where

(a) $\phi[1(p_u)] = k_u(p_1p_2) \cdot p_v(p_1p_2)$, and $k_up_v(p_1p_2) \equiv 1(p_u)$

(b) $\phi[1(p_u)] = 1(p_1p_2) \cdot k_u(p_1p_2) \cdot p_v(p_1p_2)$; $u, v = 1, 2, u \neq v$.

Some basic results are then indicated on the "mapping" and it is eventually proved that in the residue class ring (mod $p_1 \cdot p_2$), every element $x(p_1 \cdot p_2)$ has a unique decomposition, $x(p_1 \cdot p_2) \equiv x(p_1) + x(p_2)$, where $x(p_1)$ and $x(p_2)$ are not necessarily the same integers.

As a numerical illustration combining residue classes of integers (mod 2) and (mod 3) (i.e. $p_1 = 2, p_2 = 3$), the following mappings are indicated in [3] along with the decompositions of the residue classes (mod 6).

Mappings

$$\begin{aligned} 0(3) &\to 0(6) \cdot 4(6) \equiv 0(6) \\ 1(3) &\to 1(6) \cdot 4(6) \equiv 4(6) \\ 2(3) &\to 2(6) \cdot 4(6) \equiv 2(6) \\ \cdots & \quad \cdots \quad \cdots \\ 0(2) &\to 0(6) \cdot 3(6) \equiv 0(6) \\ 1(2) &\to 1(6) \cdot 3(6) \equiv 3(6) \end{aligned}$$

Decompositions

$$\begin{aligned} &0(2)+0(3) \equiv 0(6) \qquad 1(2)+0(3) \equiv 3(6) \\ &0(2)+1(3) \equiv 4(6) \qquad 1(2)+1(3) \equiv 1(6) \\ &0(2)+2(3) \equiv 2(6) \qquad 1(2)+2(3) \equiv 5(6). \end{aligned} \tag{2.1}$$

3. Alternative procedure. As a first step, we recognize that for $p_1 = 2$ and $p_2 = 3$, we may utilize the group properties of the multiplicative group of the 6 (2×3) non-zero residue classes (mod 7), the elements being given by the set of residue classes $G = \{(1), (2), (3), (4), (5), (6)\}$. G has two subgroups (normal divisors) $S_1 = \{(1), (6)\}$ and $S_2 = \{(1), (2), (4)\}$ of orders 2 and 3 respectively such that G is the direct product of S_1 and S_2, being given by $G = S_1 \otimes S_2$ with the properties [see 2] that

(i) $S_1 \cap S_2 = (1)$,

(ii) every element of G is expressible as a product, $g = ab$, $a \in S_1$, $b \in S_1$, $g \in G$, and that a and b are uniquely determined by g.

Since 3 is a primitive element of $GF(7)$, we may rewrite G in terms of the power cycle of 3, as $G = \{3^0 = 1, 3^1 = 3, 3^2 = 2, 3^3 = 6, 3^4 = 4, 3^5 = 5\}$. The subgroups (normal divisors) S_1 and S_2 may be shown as $S_1 = \{x^0 = 1, x^3\}$ and $S_2 = \{x^0 = 1, x^2, x^4\}$, where $x = 3$.

If a 2×3 factorial experiment is attempted, 0(2) and 1(2) will correspond to the levels 0 and 1 of the first factor A, and 0(3), 1(3) and 2(3) to the levels 0, 1 and 2 of the second factor $\boldsymbol{B}$.

The correspondences of the levels of the factors with the elements of the subgroups (the powers of x) may be made in any manner. But to bring out the analogy with the procedure of White and Hultquist [3] and to point out that the present procedure leads to the same structure we introduce the following correspondences.

Factor A	Factor B
$0(2) \to x^0 = 1$	$0(3) \to x^0 = 1$
	$1(3) \to x^4$
$1(2) \to x^3$	$2(3) \to x^2.$

With the above correspondences, we rewrite the decompositions (2.1) with "multiplication" substituted for "addition" as given below:

$$\begin{aligned} &x^0 \cdot x^0 = x^0 = 1 \qquad && x^3 \cdot x^0 = x^3 \\ &x^0 \cdot x^4 = x^4 && x^3 \cdot x^4 = x^1 \\ &x^0 \cdot x^2 = x^2 && x^3 \cdot x^2 = x^5. \end{aligned}$$

The above gives the table of unique decompositions of the elements of G as products of the elements of S_1 and S_2. It will be noticed that the powers of x on the right-hand sides of the above decompositions are exactly the same as the residue classes (mod 6) as given in (2.1).

We now indicate how we would obtain the treatment combinations corresponding to an effect (a main effect or an interaction) according to this procedure. Let us, for this purpose, consider the same example of the mixed factorial $3^2 \times 5$ with three factors A, B, C where A, B have three levels each, and C, five levels. The effect AB^2C is represented in [3] by the equations (mod numbers omitted),

$$i+2j+k = (0), (1), \cdots, (14), \tag{3.1}$$

where i, j and k represent the running variables corresponding to the three factors, and (0), (1), $\cdots$, (14) represent the residue classes (mod 15). In the present procedure, the corresponding equation will be denoted in the product form by

$$i(j)^2k = x^0, x^1, \cdots, x^{14}, \tag{3.2}$$

where i, j and k will represent the running variables, and $x^0, x^1, \cdots, x^{14}$, the 15 elements of a multiplicative group which is easily available as the direct product of two subgroups (normal divisors) of orders 3 and 5 respectively. It will be noticed that in this representation of the effects, the running variables have been combined as a "product" instead of a sum.

We would thus get an analog of the procedure of White and Hultquist, where the operation "addition" is substituted throughout by the operation "multiplication."

We indicate below the mappings and the decompositions in respect of the above factorial experiment on the lines as indicated by White and Hultquist.

Mappings

$0(3) \to 0(15)$	$0(5) \to 0(15)$
$1(3) \to 10(15)$	$1(5) \to 6(15)$
$2(3) \to 5(15)$	$2(5) \to 12(15)$
	$3(5) \to 3(15)$
	$4(5) \to 9(15)$

Decompositions

(3.3)

$0(3)+0(5) = 0(15)$	$1(3)+0(5) = 10(15)$
$0(3)+1(5) = 6(15)$	$1(3)+1(5) = 1(15)$
$0(3)+2(5) = 12(15)$	$1(3)+2(5) = 7(15)$
$0(3)+3(5) = 3(15)$	$1(3)+3(5) = 13(15)$
$0(3)+4(5) = 9(15)$	$1(3)+4(5) = 4(15)$

$$
\begin{aligned}
2(3)+0(5) &= 5(15)\\
2(3)+1(5) &= 11(15)\\
2(3)+2(5) &= 2(15)\\
2(3)+3(5) &= 8(15)\\
2(3)+4(5) &= 14(15).
\end{aligned}
$$

It is well known that it is possible to have a Galois Field $GF(2^4)$ with 16 elements, and that the 15 non-zero elements giving the multiplicative group G will be available as a power cycle of the primitive element x being given by

$$x^0 = 1, \quad x = x, \quad x^2 = x^2, \quad x^3 = x^3, \quad x^4 = x^3+1, \quad x^5 = x^3+x+1,$$

$$x^6 = x^3+x^2+x+1, \quad x^7 = x^2+x+1, \quad x^8 = x^3+x^2+x,$$

$$x^9 = x^2+1, \quad x^{10} = x^3+x, \quad x^{11} = x^3+x^2+1, \quad x^{12} = x+1,$$

$$x^{13} = x^2+x, \quad x^{14} = x^3+x^2,$$

where the minimum function is x^4+x^3+1. [We note here the fact that it will be enough for purposes of the calculus if it is granted that the 15 non-zero elements are available as a power cycle of the primitive element. We do not need to know the different forms of the polynomials which are congruent to the different powers of x.] The above group G is the direct product of the subgroups (normal divisors) S_1 and S_2 such that G is given by

$$G = S_1 \otimes S_2 = \{x^0 = 1, x^5, x^{10}\} \otimes \{x^0 = 1, x^3, x^6, x^9, x^{12}\}.$$

The above gives us the required calculus with 15 elements. An exact replica of the decompositions in (3.3) is obtained, if the levels of the factors are made to correspond as given below, it being understood that "addition" is substituted by "multiplication" in the present procedure.

Correspondences

Between the levels and the powers of the primitive element

For factors A and B		For factor C	
$x^0 = 1$	$\to 0$	$x^0 = 1$	$\to 0$
x^{10}	$\to 1$	x^6	$\to 1$
x^5	$\to 2$	x^{12}	$\to 2$
		x^3	$\to 3$
		x^9	$\to 4$

The decompositions may be similarly indicated. It will be noticed that residue classes $0, 1, 2, \cdots, 14$ (mod 15) of (3.3) are the same as the powers of x in the decompositions. The structures of the two decompositions are identical.

By the present procedure, the table of the complete model for the $3^2 \times 5$ factorial experiment as given in Table 4.1 of [3] will be exactly the same. The table is not reproduced here.

The principles for confounding are the same as outlined in [3]. For instance, when AB^2C is confounded, AB^2 and C are also confounded. If AB^2C is confounded a block will be assigned to each of the 15 levels which are shown in the outermost column of the table under AB^2C of [3]. The corresponding treatment combinations are available in the first column of the same table. These 15 blocks will confound, 14 degrees of freedom of which 2 degrees of freedom will belong to AB^2, 4 to C, and 8 to AB^2C [3].

Raktoe [1] has illustrated his procedure with reference to the mixed factorial $2^2 \times 3 \times 5$. In this case, we need a calculus with $2 \times 3 \times 5 = 30$ elements, and these 30 elements may be obtained from the non-zero residue classes (mod 31), 31 being a prime number.

As in each $GF(p)$, where p is a prime, there exists a primitive element, the 30 non-zero elements can be expressed as a power cycle of the primitive element. For the required calculus, all that we need to know is to recognize the fact that the multiplicative group G of 30 elements is the direct product of 3 subgroups (normal divisors) of orders 2, 3 and 5 respectively as given by

$$G = S_1 \otimes S_2 \otimes S_3 = \{x^0 = 1, x^{15}\} \otimes \{x^0 = 1, x^{10}, x^{20}\} \otimes \{x^0 = 1, x^6, x^{12}, x^{18}, x^{24}\}.$$

In the solution of equations of the type (3.2) we need to know only the powers of x with the condition that $x^{30} = 1$.

As it is always possible to have a cyclic group of order $p^n - 1$ from $GF(p^n)$, it will be clear that this methodology will work for any mixed factorial of the type,

$$p_1^{m_1} \times p_2^{m_2} \times \cdots \times p_1^{m_1},$$

where $m_1, m_2, \cdots, m_1$ are any positive integers and the product of the primes, $p_1 p_2 \cdots p_1 = p^n - 1$ for any prime p and positive integer n.

It is well known (p. 148 of Van Der Waerden [2], Vol. I) that a cyclic group $\{a\}$ of order $n = r \cdot s$ is the direct product of its subgroups $\{a^r\} \otimes \{a^s\}$, where $(r, s) = 1$. Hence, the multiplicative group of the 30 non-zero elements of $GF(31)$ will be available as the direct product of two subgroups of orders 5 and 6 which are relatively prime. From this representation it would be possible to provide the calculus for mixed factorials of the types $5^{m_1} \times 6^{m_2}$, where m_1 and m_2 are any two positive integers. It is noticed that 6 is not a prime number, and in order to cover this case by the methodology of White and Hultquist [3] and Raktoe [1], it would first be necessary to decompose 6 into 2×3, and then to combine them all. On the other hand, in case of 5×7, for example, the present method will fail while White–Hultquist–Raktoe would apply.

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