The Annals of Mathematical Statistics
1970, Vol. 41, No. 3, 876–887

CONSTRUCTION OF A SET OF 512-RUN DESIGNS OF RESOLUTION $\geqq 5$ AND A SET OF EVEN 1024-RUN DESIGNS OF RESOLUTION $\geqq 6$

BY NORMAN R. DRAPER[1] AND TOBY J. MITCHELL

University of Wisconsin and Oak Ridge National Laboratory

1. Introduction. In a previous paper [2], the authors constructed and blocked the complete set of even 512-run two-level fractional factorial designs of resolution $\geqq 6$ and the complete set of 256-run two-level fractional factorial designs of resolution $\geqq 5$. (An even design is one whose defining relation is composed entirely of words of even length.) The motivation for undertaking this previous work was to discover the maximum number of variables which can be accommodated in a 256-run resolution V design. It was in fact shown [2] that the maximum number of variables which can be accommodated in a 512-run design of resolution VI is 18, and hence that the maximum number of variables which can be accommodated in a 256-run design of resolution V is 17. In each case, the saturated design (i.e., the design which contains the largest number of variables, subject to the restrictions imposed by the specified resolution and run length) was found to be unique. (Note that, as in our previous papers, we are using a broadened definition of the word "saturated," which has often been restricted to designs having no degrees of freedom for experimental error.)

The purpose of the present paper is to tackle a similar question at the next stage of difficulty, namely, how many variables can be accommodated in a 512-run design of resolution V? The answer, as we shall see, appears to be 23.

We first briefly recapitulate the previous work. The construction algorithm for the earlier designs was described in [1]. In general, for specified (odd) resolution R and run length 2^q, $q = k-p$, the method involves:

(i) the stage-by-stage construction of the set of distinct even $2^{(k+1)-p}$ designs of resolution $\geqq R+1$, followed by

(ii) the erasure of variables from these designs (i.e., the removal of a variable from each word in the defining relation of each design) to obtain the set of distinct odd 2^{k-p} designs of resolution $\geqq R$.

At each stage of (i), the exhaustive enumeration of all designs at that stage is coupled with the discarding of all designs which are equivalent to a design previously constructed at the same stage. (Two designs are equivalent if the defining relation of one can be obtained from the defining relation of the other by relabeling the variables.) As a result, only one representative of each equivalence class of designs is saved at each stage.

In the present paper, we extend the procedure, with certain modifications

Received March 19, 1969; revised October 6, 1969.

[1] Supported by the National Science Foundation (grant GP–7842).

described below, to the case $q = 9$. In other words, we attempt to construct the complete set of 1024-run even designs of resolution $\geqq 6$ and, from these, obtain the set of 512-run designs of resolution $\geqq 5$.

The modifications of the procedure will now be described:

First, it became necessary to alter the algorithm for testing the equivalence of two designs because this algorithm used an excessive amount of computer time when applied in the present investigation. Unfortunately, the result of this modification (which is discussed more fully in Section 2) is that we cannot now be sure, as we were for $q < 9$, that we obtain *all* possible designs which meet our requirements.

Our modified procedure leads to designs which we believe to be saturated. We shall use the notation "saturated" (with quotation marks) to indicate such designs, since it is possible that there exists a design which accommodates more variables; such a design, if it existed, would be developed from a design which our modified procedure discarded at an earlier stage.

The second modification (discussed in Section 3) was to refine the blocking program in such a way that each of the designs we present here is indeed blocked into the maximum number of blocks allowed by the confounding requirements. (For the resolution VI designs, these requirements are that no interaction involving three or fewer variables may be confounded with blocks. For the resolution V designs, no main effect or two-factor interaction may be confounded with blocks.)

In summary, therefore, the equivalence procedure is somewhat weaker, while the blocking procedure is considerably improved.

In Section 4, we present tables of the constructed designs, together with a discussion of their use. Because of the large number of different designs which were originally constructed, we have given in the tables a set of "key designs," from which the whole set can easily be derived.

2. Modification of the test for equivalence. In the case $q = k-p = 8$, the authors [2] constructed the set of even $2_{\mathrm{VI}}^{(k+1)-p}$ designs, using a "sequential conjecture" algorithm [1] to test the equivalence of two designs. In the present case ($q = 9$), the doubling of the number of words in the defining relation and the tremendous increase in the number of designs which arise at each stage made the use of this algorithm impractical because of the amount of computer time required.

An alternative method, based on distinct "word length patterns" was proposed by the authors [1] and used by them [2] to block the designs constructed in the case $q = 8$. This procedure, which is relatively fast on the computer, involves counting the number, a_j, of words of length j in the defining relation, where $j = 1, 2, \cdots, k$. Two designs are called "equivalent" if the word length pattern vectors $(a_1, a_2, \cdots, a_k)$ associated with each of them are identical. It is possible, however, for distinct designs to have identical word length patterns, though this event is very rare if $q \leqq 8$. (See [2].)

To generate the set of distinct designs presented here, we apply a more sensitive test for equivalence, using a "letter pattern comparison." This first examines the defining relation of a design and counts the number, a_{ij}, of words of length j in which letter i appears. We then form the $k \times k$ letter pattern matrix $A = \{a_{ij}\}$ for each design, and declare two designs D and D' to be equivalent if and only if $A = P(A')$, where $P(A')$ is some permutation of the rows of A' and where A and A' are the letter pattern matrices corresponding to D and D', respectively. We note that the word length pattern of D is just $(\sum_{i=1}^{k} a_{i1}, \cdots, j^{-1} \sum_{i=1}^{k} a_{ij}, \cdots, k^{-1} \sum_{i=1}^{k} a_{ik})$, so that two designs having equivalent letter pattern matrices necessarily have equivalent word length pattern vectors. That this letter pattern comparison procedure (which *may* be exhaustive in this case) is distinctly superior to the word length pattern procedure (which is certainly ***not*** exhaustive) can be seen from Table 2.1. The table shows that the letter pattern procedure makes available to us considerably more designs than its rival. We should remark that the set of designs arising from the word length pattern procedure is not necessarily a subset of the designs arising from the letter pattern procedure, unless the latter method is, in fact, exhaustive.

TABLE 2.1
Number of distinct even 1024-run $2_R^{(k+1)-p}$ designs ($R \geqq 6$) generated using two different tests of equivalence

Test for Equivalence	Stage (p) 1	2	3	4	5	6	7	8	9	10	11	12	13	14
Word Length Pattern	3	7	8	15	24	35	39	47	48	49	39	19	6	1
Letter Pattern	3	7	11	23	50	119	267	635	1012	1218	568	121	8	1

We have conjectured in this work that two designs with equivalent letter pattern matrices are indeed equivalent designs. It may be noted that in the previous case ($q = 8$), this conjecture is in fact true [2]. In the present case, however, we have been unable either to prove this assertion or to find a counterexample to it. At present, therefore, we regard it simply as a useful conjecture which has enabled us to construct a large set of distinct designs. Clearly, further examination of this point would be desirable.

It is interesting to note, in Table 2.1, that although use of the letter pattern comparison (rather than the word length pattern comparison) permitted the construction of a much larger number of designs, both methods result in a single "saturated" 2_{VI}^{24-14} design. We thus tentatively conclude (*subject to the reservations expressed earlier*) that a 1024-run design of resolution VI can accommodate no more than 24 variables, while a 512-run design of resolution V can accommodate no more than 23 variables.

3. Modification of the blocking program. In the case $q = 8$, Draper and Mitchell [2] used the word length pattern comparison to block the designs which had been constructed. For several of those designs, therefore, it was not certain that the maximum possible number of blocking generators had been added. In many cases, however (e.g., for the saturated 2_{VI}^{18-9} design), it was possible to show from other considerations that the number of blocking generators added was indeed maximal.

For the set of designs presented here (1024-run, even, resolution VI) the blocking program has been improved to guarantee a maximal set of blocking generators. As before, we require that the combined defining relation, which is generated by the p generators of the resolution VI "base design" and the t blocking generators, must be such that all words have length $\geqq 4$. (Our notational description of such a design is $2_{\text{VI; IV}}^{(k+1)-p-t}$, indicating a two-level fractional factorial of resolution VI, accommodating $(k+1)$ variables, in 2^t blocks of $2^{(k+1)-p-t}$ runs each, where each block is itself a fractional factorial of resolution IV.) This resolution requirement ensures that the blocked resolution V designs, which are derived from the $2_{\text{VI; IV}}^{(k+1)-p-t}$ designs, consist of blocks of resolution III, so that no main effect or two-factor interaction is confounded with blocks.

First, it may be noted that no 1024-run even design of resolution $\geqq 6$ can be blocked into more than 32 blocks of resolution IV. This is seen if we recognize that the set of blocking generators defines an even design of resolution IV in ten variables. We know, however, that a 2_{IV}^{10-p} design does not exist if $p > 5$. Therefore, the number of even blocking generators which can be added to a 1024-run even design of resolution $\geqq 6$ must be $\leqq 5$.

We now show that, if this upper bound is attained, i.e., if the number of blocking generators (t) is equal to 5, the maximum number of variables which can be accommodated in the blocked design is 16. If there existed a $2_{\text{VI; IV}}^{17-7-5}$ design, each block would be a 2_{IV}^{17-12} design, but such a design does not exist, since 16 is the maximum number of variables which can be accommodated in a 32-run resolution IV design. There do exist $2_{\text{VI; IV}}^{16-6-5}$ designs, however; one has only to combine any two members of the family of blocked designs associated with the defining relation of "Design 7.3" given by Draper and Mitchell [2]. Hence, 16 is the maximum number of variables which can be accommodated in a 32-block, 1024-run design of resolution VI. Since all the designs listed in Section 4 accommodate *more* than 16 variables, it is clear from the above that none of them can be blocked into as many as 32 blocks of resolution IV.

We now consider the possibilities $t = 4$ and $t = 3$.

Given a resolution VI base design, the new blocking program first constructs the set of all even blocking generators which are compatible with the defining relation of the base design, i.e., whose product with each word in the defining relation of the base design is $\geqq 4$. In this phase of the program, time is saved by restricting attention (with no loss of generality) to blocking generators which are composed of letters from the set $1, 2, \cdots, q+1$, where $q = k-p$. (See [1].) It is necessary to test compatibility only against that subset of words in the defining relation of the base design which are members of the given set of generators or which can be expressed as the product of two such generators. This is because compatibility with respect to this subset implies compatibility with the whole defining relation, as can easily be shown.

Once the set of compatible candidates is constructed, "compatible pairs" are formed by checking to see if the product of each pair of compatible candidates is itself a compatible candidate. If so, the pair is defined as compatible with the base design, in the sense that both words in the pair could be added to the base design as blocking generators without violating the resolution conditions.

The blocking procedure for adding four blocking generators to a design consists of testing various pairs of compatible pairs and stopping as soon as a set of four blocking generators, compatible with the base design, is found. If necessary, all possible combinations of pairs are tested.

If four blocking generators compatible with the base design cannot be found, three blocking generators are added instead. In this case, however, all combinations of one compatible candidate with one compatible pair are tested, stopping as soon as a set of three blocking generators, compatible with the base design, is found. Again, all combinations are tested if necessary.

The addition of at least three compatible blocking generators was found to be possible for every design constructed in the present investigation.

4. Tables of designs. We present, in Table 4.1, 74 even 1024–run "dead-end" designs of resolution 6. (A dead-end design is one with which all candidates in the stage-by-stage construction are incompatible. This definition implies, for example, that the words of the defining relation of an even 2^{k-p} dead-end design cannot be a subset of the words of an even $2^{(k+1)-(p+1)}$ design of the same resolution.) All 4043 1024-run designs of resolution $\geqq 6$ which were constructed in the course of this investigation can be derived from these 74 dead-end designs through the deletion of variables. (The deletion of a set Ω of variables is merely the removal, from the defining relation, of all words which contain variables in Ω; the resulting words form the defining relation of a new design.)

Each of the designs in Table 4.1 is presented in "best-blocked" form, where "best-blocked" means that the maximum number of blocking generators has been added, subject to the resolution requirements. It should be noted that the deletion of variables from a maximally blocked dead-end design will result in a new blocked design, accommodating fewer variables, in the *same* number of blocks. This new design is not necessarily maximally blocked, however. For example, we found forty $2_{\text{VI; IV}}^{22-12-4}$ designs, not shown in Table 4.1, which are not dead-end designs, nor can they be obtained through the deletion of a variable from a $2_{\text{VI; IV}}^{23-13-4}$ design. Their 2_{VI}^{22-12} base designs can be obtained only through the deletion of a variable from one of the designs: 13.2, 13.3, 13.5, 13.6, 13.7, none of which can be blocked into 16 blocks.

We now show how Tables 4.2a and 4.2b can be used to construct all the designs in Table 4.1. Following the number of variables (V) and the word length pattern of each design in Table 4.1 is a letter (in the column headed K_1 to K_6) and a set of numbers (in the columns headed K_7, K_8, $\cdots$), which denote an appropriate set of "reference words" to be used in the construction of the generators of that design. These reference words, which are composed of letters from the set $(1, 2, \cdots, 10)$ are listed in Table 4.2a. Each of them can be converted into a generator by attaching an indicator variable i, where $i \in (11, 12, \cdots, \mathrm{V})$. For example, to construct the "saturated" design 14.1, we find in Table 4.1 that we need the reference words corresponding to A, 58, 43, 6, 9, 32, 52, 70, and 79. These words are, from Table 4.2a: 12345, 12367, 12468, 13469, 15789, 1256(10), 2379(10), 234678(10), 25678, 34579, 1368(10), 1359(10), 123489(10), 245789(10). If we now attach one indicator variable (from the set (11, 12, $\cdots$, 24)) to each of these words, we obtain the generators of the base design of Design 14.1:

$$\begin{array}{ll} W_1 = 12345(11) & W_8 = 234678(10)(18) \\ W_2 = 12367(12) & W_9 = 25678(19) \\ W_3 = 12468(13) & W_{10} = 34579(20) \\ W_4 = 13469(14) & W_{11} = 1368(10)(21) \\ W_5 = 15789(15) & W_{12} = 1359(10)(22) \\ W_6 = 1256(10)(16) & W_{13} = 123489(10)(23) \\ W_7 = 2379(10)(17) & W_{14} = 245789(10)(24). \end{array} \tag{4.1}$$

To block any of the designs constructed in this way, it is necessary first to refer to the columns B_1, B_2, B_3, and B_4 in Table 4.1, which indicate the appropriate blocking generators to be taken from Table 4.2b. No indicator variable should be attached to these blocking generators; they are to be written as they stand. For example, to block Design 14.1, the generators of whose base design we have already found (4.1), we find in Table 4.1 that the appropriate blocking generators are numbered 2, 13, and 45 in Table 4.2b, i.e.,

$$B_1 = 2346, \qquad B_2 = 2457, \qquad B_3 = 3489. \tag{4.2}$$

Because of the extremely large number of resolution V designs which can be derived from the resolution VI designs in Table 4.1, we have not attempted to list them, or even to derive them all. However, the information provided in the tables is sufficient to construct these $2_{\mathrm{V;\,III}}^{k-p-t}$ designs, if desired.

For example, suppose we wish to construct a resolution V design which accommodates 22 variables in 16 blocks of 32 runs each. We first select either of the two $2_{\mathrm{VI;\,IV}}^{23-13-4}$ designs 13.1 or 13.4, and use the tables to write down its generators as described above. We then erase any variable of our choosing wherever it appears in a generator. Taking, for example, Design 13.1 and erasing variable (23) we have, for the required $2_{\mathrm{V;\,III}}^{22-13-4}$ design, the generators:

$$\begin{array}{ll} W_1 = 12345(11) & W_8 = 234678(10)(18) \\ W_2 = 12367(12) & W_9 = 23578(19) \\ W_3 = 12468(13) & W_{10} = 34579(20) \\ W_4 = 13469(14) & W_{11} = 124578(10)(21) \\ W_5 = 15789(15) & W_{12} = 3489(10)(22) \\ W_6 = 1256(10)(16) & W_{13} = 456789(10), \\ W_7 = 2379(10)(17) & \end{array}$$

with blocking generators

$$B_1 = 2467, \qquad B_2 = 1458, \qquad B_3 = 2349, \qquad B_4 = 356(10).$$

TABLE 4.1a
Even 1024-*run "dead-end" designs of resolution* 6

No.	V	Word length pattern									
		6	8	10	12	14	16	18	20	22	24
10.1	20	90	255	332	255	90	0	0	1	0	0
10.2	20	96	234	352	264	64	13	0	0	0	0
11.1	21	132	386	668	600	220	37	4	0	0	0
11.2	21	132	386	668	600	220	37	4	0	0	0
11.3	21	133	382	673	600	215	41	3	0	0	0
11.4	21	130	398	638	640	190	49	2	0	0	0
11.5	21	134	378	678	600	210	45	2	0	0	0
11.6	21	133	382	673	600	215	41	3	0	0	0
11.7	21	134	378	678	600	210	45	2	0	0	0
11.8	21	132	386	668	600	220	37	4	0	0	0
11.9	21	132	386	668	600	220	37	4	0	0	0
11.10	21	131	392	653	620	205	43	3	0	0	0
11.11	21	133	382	673	600	215	41	3	0	0	0
11.12	21	131	392	653	620	205	43	3	0	0	0
11.13	21	132	388	658	620	200	47	2	0	0	0
11.14	21	130	396	648	620	210	39	4	0	0	0
11.15	21	131	392	653	620	205	43	3	0	0	0
11.16	21	132	386	668	600	220	37	4	0	0	0
11.17	21	134	378	678	600	210	45	2	0	0	0
11.18	21	132	385	672	595	220	42	0	1	0	0
11.19	21	131	392	653	620	205	43	3	0	0	0
11.20	21	133	382	673	600	215	41	3	0	0	0
11.21	21	133	380	683	580	235	31	5	0	0	0
11.22	21	133	382	673	600	215	41	3	0	0	0
11.23	21	133	382	673	600	215	41	3	0	0	0
11.24	21	134	378	678	600	210	45	2	0	0	0
12.1	22	184	594	1248	1304	600	149	16	0	0	0
12.2	22	186	582	1278	1264	630	137	18	0	0	0
12.3	22	186	582	1278	1264	630	137	18	0	0	0
12.4	22	185	588	1263	1284	615	143	17	0	0	0
12.5	22	184	594	1248	1304	600	149	16	0	0	0
12.6	22	187	577	1287	1259	625	146	13	1	0	0
12.7	22	186	582	1278	1264	630	137	18	0	0	0
12.8	22	185	589	1257	1299	595	158	11	1	0	0
12.9	22	185	589	1257	1299	595	158	11	1	0	0
12.10	22	184	594	1248	1304	600	149	16	0	0	0
12.11	22	184	594	1248	1304	600	149	16	0	0	0

TABLE 4.1a—*continued*
Even 1024-*run "dead-end" designs of resolution* 6

No.	V	Word length pattern									
		6	8	10	12	14	16	18	20	22	24
12.12	22	188	574	1288	1264	620	145	16	0	0	0
12.13	22	185	589	1257	1299	595	158	11	1	0	0
12.14	22	187	581	1267	1299	585	166	9	1	0	0
12.15	22	186	586	1258	1304	590	157	14	0	0	0
12.16	22	186	582	1278	1264	630	137	18	0	0	0
12.17	22	186	586	1258	1304	590	157	14	0	0	0
12.18	22	185	588	1263	1284	615	143	17	0	0	0
12.19	22	186	582	1278	1264	630	137	18	0	0	0
12.20	22	184	594	1248	1304	600	149	16	0	0	0
12.21	22	187	580	1273	1284	605	151	15	0	0	0
12.22	22	186	586	1258	1304	590	157	14	0	0	0
12.23	22	185	588	1263	1284	615	143	17	0	0	0
12.24	22	185	588	1263	1284	615	143	17	0	0	0
12.25	22	184	594	1248	1304	600	149	16	0	0	0
12.26	22	185	589	1257	1299	595	158	11	1	0	0
12.27	22	186	586	1258	1304	590	157	14	0	0	0
12.28	22	186	586	1258	1304	590	157	14	0	0	0
12.29	22	184	594	1248	1304	600	149	16	0	0	0
12.30	22	192	570	1264	1320	576	157	16	0	0	0
12.31	22	183	600	1233	1324	585	155	15	0	0	0
12.32	22	184	594	1248	1304	600	149	16	0	0	0
12.33	22	184	595	1242	1319	580	164	10	1	0	0
12.34	22	185	589	1257	1299	595	158	11	1	0	0
12.35	22	189	570	1288	1288	570	189	0	0	1	0
12.36	22	186	583	1272	1279	610	152	12	1	0	0
12.37	22	189	570	1288	1288	570	189	0	0	1	0
12.38	22	189	570	1288	1288	570	189	0	0	1	0
12.39	22	189	570	1288	1288	570	189	0	0	1	0
12.40	22	189	570	1288	1288	570	189	0	0	1	0
13.1	23	252	894	2244	2692	1540	505	60	4	0	0
13.2	23	253	887	2265	2657	1575	484	67	3	0	0
13.3	23	253	887	2265	2657	1575	484	67	3	0	0
13.4	23	251	899	2235	2697	1545	496	65	3	0	0
13.5	23	256	878	2264	2692	1520	521	56	4	0	0
13.6	23	256	870	2304	2612	1600	481	64	4	0	0
13.7	23	252	892	2256	2662	1580	475	72	2	0	0
14.1	24	336	1335	3888	5264	3888	1335	336	0	0	1

TABLE 4.1b
Even 1024-*run "dead-end" designs of resolution* 6

	Reference words												
	K_1 to K_6	K_7	K_8	K_9	K_{10}	K_{11}	K_{12}	K_{13}	K_{14}	B_1	B_2	B_3	B_4
10.1	*A*	58	43	88	84	*	*	*	*	2	9	19	33
10.2	*E*	89	90	52	91	*	*	*	*	2	14	34	53
11.1	*A*	58	43	4	12	75	*	*	*	2	10	44	70
11.2	*A*	58	43	4	64	71	*	*	*	2	10	34	73
11.3	*A*	58	43	4	76	83	*	*	*	2	10	51	67
11.4	*A*	58	43	17	65	56	*	*	*	2	13	49	55
11.5	*A*	58	69	9	12	39	*	*	*	2	9	33	76
11.6	*A*	58	69	9	54	81	*	*	*	2	13	21	55
11.7	*A*	58	69	12	33	64	*	*	*	2	10	34	67
11.8	*A*	58	69	17	46	72	*	*	*	2	10	47	69
11.9	*A*	58	69	26	28	81	*	*	*	2	22	43	55
11.10	*A*	58	69	26	34	81	*	*	*	2	13	33	76
11.11	*A*	58	69	34	47	81	*	*	*	2	13	49	70
11.12	*A*	58	69	37	62	74	*	*	*	2	10	18	34
11.13	*A*	58	69	47	62	74	*	*	*	2	10	18	34
11.14	*A*	58	69	47	74	81	*	*	*	2	10	18	34
11.15	*A*	58	78	4	15	44	*	*	*	2	10	50	52
11.16	*A*	58	78	35	39	18	*	*	*	2	9	21	79
11.17	*A*	58	78	35	62	74	*	*	*	2	10	18	34
11.18	*A*	58	78	37	46	70	*	*	*	2	10	17	52
11.19	*A*	59	72	4	62	25	*	*	*	2	10	44	55
11.20	*A*	59	72	40	55	48	*	*	*	3	7	45	58
11.21	*A*	59	72	45	69	87	*	*	*	2	10	51	72
11.22	*A*	59	72	4	11	66	*	*	*	2	14	51	57
11.23	*A*	59	72	24	12	44	*	*	*	2	9	21	79
11.24	*A*	59	72	24	44	31	*	*	*	2	9	21	79
12.1	*A*	58	43	4	9	50	75	*	*	3	12	48	66
12.2	*A*	58	43	4	9	54	71	*	*	4	20	32	61
12.3	*A*	58	43	4	9	54	79	*	*	2	28	37	63
12.4	*A*	58	43	4	9	79	83	*	*	3	12	48	66
12.5	*A*	58	43	4	12	41	73	*	*	4	22	36	63
12.6	*A*	58	43	4	12	50	73	*	*	2	14	34	75
12.7	*A*	58	43	4	12	54	65	*	*	3	16	42	60
12.8	*A*	58	43	4	12	54	73	*	*	2	22	39	63
12.9	*A*	58	43	4	12	71	73	*	*	2	10	49	73
12.10	*A*	58	43	4	41	64	65	*	*	2	14	51	53
12.11	*A*	58	43	4	50	65	76	*	*	2	10	50	52

TABLE 4.1b—*continued*
Even 1024-*run "dead-end" designs of resolution* 6

No.	Reference words											
	K_1 to K_6	K_7	K_8	K_9	K_{10}	K_{11}	K_{12}	K_{13}	K_{14}	B_1	B_2	B_3	B_4
12.12	*A*	58	43	4	54	65	76	*	*	4	20	41	53
12.13	*A*	58	43	4	54	71	73	*	*	3	8	30	71
12.14	*A*	58	43	6	9	17	76	*	*	2	28	37	81
12.15	*A*	58	43	6	9	32	41	*	*	2	9	19	79
12.16	*A*	58	43	6	9	32	54	*	*	6	8	44	54
12.17	*A*	58	43	6	12	73	84	*	*	2	10	49	71
12.18	*A*	58	43	6	17	57	71	*	*	5	10	29	70
12.19	*A*	58	43	9	12	50	84	*	*	2	27	37	62
12.20	*A*	58	43	9	35	56	65	*	*	3	26	37	64
12.21	*A*	58	69	9	12	57	74	*	*	4	13	17	56
12.22	*A*	58	69	9	17	29	42	*	*	2	13	19	77
12.23	*A*	58	69	9	27	29	39	*	*	3	12	17	78
12.24	*A*	58	69	9	34	57	79	*	*	2	10	51	75
2.25	*A*	58	69	9	42	39	79	*	*	2	10	49	73
12.26	*A*	58	69	12	26	49	57	*	*	5	24	29	61
12.27	*A*	58	69	12	27	44	54	*	*	3	8	30	68
12.28	*A*	58	69	12	27	62	54	*	*	2	27	39	57
12.29	*A*	58	69	12	28	46	77	*	*	2	18	40	63
12.30	*A*	58	69	12	28	57	74	*	*	2	25	41	62
12.31	*A*	58	69	17	27	28	34	*	*	4	14	32	67
12.32	*A*	58	69	17	33	37	72	*	*	4	17	41	62
12.33	*A*	58	69	26	33	37	81	*	*	2	23	38	52
12.34	*A*	58	69	37	62	54	87	*	*	2	28	37	55
12.35	*A*	59	72	11	14	38	55	*	*	2	9	49	*
12.36	*A*	59	75	53	36	67	80	*	*	5	24	29	65
12.37	*B*	34	60	11	18	63	68	*	*	2	9	74	*
12.38	*C*	10	12	22	30	51	85	*	*	2	14	46	*
12.39	*C*	10	12	30	44	51	61	*	*	2	9	46	*
12.40	*D*	13	21	23	28	53	82	*	*	1	11	80	*
13.1	*A*	58	43	4	9	41	65	86	*	15	19	31	59
13.2	*A*	58	43	4	9	50	73	83	*	2	10	49	*
13.3	*A*	58	43	4	9	54	65	81	*	2	10	35	*
13.4	*A*	58	43	4	9	71	73	83	*	2	10	49	73
13.5	*A*	58	43	6	9	12	17	65	*	2	9	53	*
13.6	*A*	58	43	6	9	12	41	65	*	2	9	33	*
13.7	*A*	58	43	6	9	32	57	70	*	2	9	44	*
14.1	*A*	58	43	6	9	32	52	70	79	2	13	45	*

TABLE 4.2a

Reference words for base designs in table 4.1

	$(K_1$ to $K_6)$
A	12345, 12367, 12468, 13469, 15789, 1256(10)
B	12345, 12367, 12468, 13469, 15789, 2356(10)
C	12345, 12367, 12468, 13469, 13578, 2356(10)
D	12345, 12367, 12468, 13578, 23569, 34579
E	12345, 12367, 12468, 13469, 13578, 1256(10)

$(K_7, K_8, \cdots)$

1	12345	24	1457(10)	47	135678(10)	70	123489(10)
2	12367	25	2467(10)	48	5678(10)	71	134589(10)
3	12468	26	134567(10)	49	245678(10)	72	4689(10)
4	23578	27	234567(10)	50	1249(10)	73	123689(10)
5	13578	28	1348(10)	51	2349(10)	74	234689(10)
6	25678	29	2348(10)	52	1359(10)	75	345689(10)
7	13469	30	1258(10)	53	1459(10)	76	145689(10)
8	23569	31	2368(10)	54	2469(10)	77	4789(10)
9	34579	32	1368(10)	55	2369(10)	78	345789(10)
10	14579	33	234568(10)	56	3569(10)	79	245789(10)
11	35679	34	1278(10)	57	4569(10)	80	235789(10)
12	24589	35	134568(10)	58	2379(10)	81	12345789(10)
13	12589	36	1378(10)	59	3479(10)	82	6789(10)
14	34589	37	2478(10)	60	123479(10)	83	356789(10)
15	45689	38	1478(10)	61	3579(10)	84	12456789(10)
16	15789	39	3678(10)	62	5679(10)	85	12356789(10)
17	36789	40	234578(10)	63	125679(10)	86	456789(10)
18	46789	41	124578(10)	64	345679(10)	87	23456789(10)
19	1256(10)	42	134578(10)	65	3489(10)	88	5689(10)
20	2356(10)	43	234678(10)	66	1489(10)	89	23579
21	2456(10)	44	4678(10)	67	2489(10)	90	2457(10)
22	1247(10)	45	134678(10)	68	1389(10)	91	346789(10)
23	2347(10)	46	235678(10)	69	3589(10)		

The resolution V design of most direct interest to us is the "saturated" $2^{23-14}_{V;\,III}$ design obtained by erasing any one of the variables of Design 14.1. If variable (24) is erased, for example, the generators of this design are exactly as given in (4.1) and (4.2), with the single exception that W_{14} is now 245789(10). It is interesting to note that *no matter which variable is erased,* the word length pattern of the resulting resolution V base design is:

(4.3)

Word Length	5	6	7	8	9	10	11	12	13	14
No. of Words	84	252	445	890	1620	2268	2632	2632	2268	1620

Word Length	15	16	17	18	19	20	21	22	23
No. of Words	890	445	252	84	0	0	0	0	1

TABLE 4.2b
Blocking generators for designs in table 4.1

			$(B_1–B_4)$				
1	1346	21	3458	41	4579	61	12347(10)
2	2346	22	1278	42	124579	62	257(10)
3	1356	23	2378	43	134579	63	12357(10)
4	2356	24	2478	44	1489	64	457(10)
5	1456	25	3578	45	3489	65	13457(10)
6	2456	26	123578	46	1589	66	138(10)
7	1247	27	4578	47	2589	67	238(10)
8	2347	28	124578	48	3589	68	148(10)
9	1257	29	1239	49	123589	69	248(10)
10	1357	30	1249	50	134589	70	348(10)
11	2357	31	2349	51	234589	71	12348(10)
12	1457	32	1359	52	123(10)	72	158(10)
13	2457	33	2359	53	124(10)	73	258(10)
14	3457	34	1459	54	135(10)	74	12358(10)
15	2467	35	2459	55	235(10)	75	12458(10)
16	1238	36	3459	56	145(10)	76	23458(10)
17	1258	37	1379	57	245(10)	77	249(10)
18	2358	38	3479	58	345(10)	78	159(10)
19	1458	39	123479	59	356(10)	79	459(10)
20	2458	40	2579	60	247(10)	80	189(10)
						81	157(10)

It is tempting to conjecture that these 2_V^{23-14} designs (all of which have word length pattern (4.3)) are all equivalent to a unique saturated design, parallel to the 256-run and 128-run cases. As we have already noted, however, our tests for the equivalence of two designs are not yet sufficiently well developed to allow us to confirm or deny such conjectures.

Acknowledgments. The authors would like to express their appreciation to Mrs. Kay Kannan and Mrs. Diana Smith of the Mathematics Division, Oak Ridge National Laboratory, for their valuable assistance in computer programming, and to Mrs. Charlotte Montgomery and Mrs. Betty Mashburn for their help in preparation of the manuscript. Thanks are due to Professor D. R. Cox of Imperial College, London, for kindly providing research facilities in the fall semester of 1968.

REFERENCES

[1] Draper, N. R. and Mitchell, T. J. (1967). The construction of saturated 2_R^{k-p} designs. *Ann. Math. Statist.* **38** 1110–1126.

[2] Draper, N. R. and Mitchell, T. J. (1968). Construction of the set of 256-run designs of resolution ≧ 5 and the set of even 512-run designs of resolution ≧ 6 with special reference to the unique saturated designs. *Ann. Math. Statist.* **39** 246–255.