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ABSTRACTS OF PAPERS

(*Abstracts of papers presented at the Eastern Regional meeting, Chapel Hill, North Carolina, May 5–7, 1970. Additional abstracts have appeared in earlier issues.*)

125-52. Nonparametric tests in the presence of nuisance parameters (preliminary report). C. B. Bell and V. Kurotschka, University of Michigan.

Let Ω be a family of distributions on the sample space $\mathscr{X}$ and $\Omega_\tau := \{H \circ \tau;\ H \in \Omega\}$ where τ is a nuisance parameter with values in a "nuisance parameter group" T. For the family $S(\Omega(H_0))$ of all similar sets with respect to (wrt) $\Omega(H_0) := \bigcup_{\tau \in T} \Omega_\tau$ one has Theorem A. (i) $S(\Omega(H_0)) = \bigcap_{\tau \in T} S(\Omega_\tau)$ (ii) *A is similar of size* α wrt $\Omega(H_0)$ iff *for every* $\tau \in T$ *the set* $\tau(A)$ *is similar of size* α wrt Ω. (iii) *Let A be invariant* wrt T. *Then A is similar* wrt $\Omega(H_0)$ iff *A is similar* wrt Ω. Theorem B. *Let* $\mathscr{X} = R_N$ *and* Ω *be symmetrically complete.* ϕ *is similar of size* α wrt. $\Omega(H_0)$ iff $\sum_{\gamma \in S_N} \phi(\tau(\gamma(Z))) = N!\alpha$ *for all* $\tau \in T$ *and* $\Omega(H_0)$*—almost all* $z \in \mathscr{X}$. Corollary C. *The above results hold in particular, for the two-sample problem when* $\Omega = \Omega_2(N)$ *and when T is the scale or the location group applied to the coordinates of the second sample only.* Work is in progress to apply and extend to problems arising in analysis of variance settings like testing for interactions in a two-way layout. (Received March 17, 1970.)

125-53. Asymptotic properties of branching renewal processes. Peter A. W. Lewis, Imperial College and IBM Watson Research Center. (Invited)

Branching renewal processes (BRP) and generalized branching renewal processes (GBRP) are defined and their properties investigated. Analogs of the elementary renewal theorem and Blackwell's theorem are proved for generalized branching renewal processes, and convergence of the intensity function is discussed. Structural theorems on boundedness of the number of events in finite intervals and existence of moments of counts of events are given. Finally, two limit theorems on the asymptotic distribution of the numbers of events in the branching renewal process and in the generalized branching renewal process are proven. (Received May 18, 1970.)

(*Abstracts of papers to be presented at the Annual meeting, Laramie, August 25–28, 1970. Additional abstracts have appeared in earlier issues.*)

126-5. The asymptotic configuration of Wishart eigenvalues. C. L. Mallows and K. W. Wachter, Bell Telephone Laboratories.

Suppose, A, B are independent $p \times p$ Wishart matrices on m, n degrees of freedom respectively. We consider the empiric distribution of the eigenvalues of A and of $B^{-1}A$, as $p \to \infty$ with $m/p, n/p$ fixed; these converge in probability to remarkably simple forms, making feasible some methods of graphical analysis of observed matrices. Normality is not an essential assumption in the one-matrix case. There are some similarities between these results and those of Wigner and his colleagues (see e.g., Wigner, *SIAM Rev.* (1967)) for symmetric random matrices, though our problems are much more difficult combinatorially; in one special case our result can be related to Wigner's semi-circle law. (Received March 18, 1970.)

126-6. Small sample properties of estimators for the gamma distribution. L. R. Shenton and K. O. Bowman, University of Georgia and Union Carbide.

The exact distribution of the maximum likelihood estimators $\hat{\rho}$, $\hat{a}$ of the parameters ρ, a of the gamma density $f(x) = e^{-(x/a)}(x/a)^{\rho-1}/a\Gamma(\rho)$ is only known theoretically and in intractable form,

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$\hat{p}$ being a solution of the equation $\log \hat{p} - \psi(\hat{p}) = \ln(A/G)$ $(=y)$ where ψ is the logarithmic derivative of the gamma function, A and G are the sample (size n) arithmetic and geometric means respectively. In previous studies (CTC Report 1, Union Carbide, Oak Ridge, and elsewhere) we have developed series in descending powers of n for the means covariances, skewness parameters $\beta_1^{\frac{1}{2}}, \beta_2$ for $\hat{p}, \hat{a}$ up to and including the term in n^{-6}. The accuracy of these results (extensive Monte Carlo simulations notwithstanding) presents serious problems and the present study develops expansions for these moments in terms of the random variate y, which surprisingly enough, in expectation, produces series in descending powers of ρ. Usable expressions are now given for the moment parameters mentioned previously in $\hat{p}$ and $\hat{a}$, for $\rho > 2$ approximately, these give what we believe is sufficient accuracy for most practical purposes. The agreement (or otherwise) between the new assessments and the old (n^{-1} series) is discussed at length and is entirely satisfactory as long as n and ρ are not small. Several applications are mentioned, and a listing is given of moment coefficients of order n^{-1}, n^{-2}, and many limiting values as $\rho \to \infty$, $\rho \to 0^+$, and $n \to \infty$. (Received April 3, 1970.)

126-7. A sequential confidence interval for the mean of a U-statistic. RAYMOND N. SPROULE, University of California, Davis.

Let $f(x_1, x_2, \cdots, x_r)$ be the symmetrical kernel of a U-statistic U_n whose expectation is θ. We develop a sequential confidence interval for θ of fixed-width $2d$, where $d > 0$, and such that the coverage probability approaches a specified α as $d \to 0$, where $0 < \alpha < 1$. The problem was solved by Chow and Robbins (*Ann. Math. Statist.* **36** (1965) 457–462) for a special U-statistic, the sample mean. The sequential procedure may be simply described as follows: at each stage of sampling U_n and an estimate of its variance are calculated, and sampling is terminated as soon as the approximate coverage probability for the interval $[U_n - d, U_n + d]$, based on a normal approximation, is at least α. It is shown that the expected sample size of the sequential procedure is asymptotically equal to the sample size of the corresponding non-sequential scheme used when the variance of U_n is known; that is, the sequential procedures are efficient. (Received April 21, 1970.)

126-8. Some multiple comparison selection procedures based on ranks. GARY C. MCDONALD, General Motors Corporation. (Invited)

The selection procedures investigated in this paper arise from consideration of the following model. Each of n independent judges orders k populations (categories, objects, people, etc.) according to some suitably defined criterion of desirability. That is, each judge assigns a rank of 1 to the population least desirable in his opinion, $\cdots$, and a rank of k to that which is most desirable. The "best" population is defined to be that (unknown) population which is in fact most desirable according to this given criterion. Based on these ranks, several classes of selection procedures for choosing a subset of the k populations so as to guarantee that the best is included with probability no less than P^*, a given constant between k^{-1} and 1, are constructed and investigated. (Received April 30, 1970.)

126-9. Some results on estimation and testing hypotheses for the general linear model using generalized inverses. GEORGE A. MILLIKEN, Kansas State University.

The general linear model with fixed effects is defined as $\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{e}$ where $\mathbf{y}$ is an $n \times 1$ random observation vector, $\mathbf{X}$ is an $n \times p$ matrix of known constants of rank q with $q \leqq p < n$, $\boldsymbol{\beta}$ is a $p \times 1$ vector of unknown parameters defined in E_p, and $\mathbf{e}$ is an unobserved random vector with $E(\mathbf{e}) = \mathbf{0}$ and $E(\mathbf{e}'\mathbf{e}) = \sigma^2\mathbf{I}$. The set of linear combinations $\mathbf{A}\boldsymbol{\beta}$, where $\mathbf{A}$ is a $k \times p$ matrix of rank k, is estimable if and only if there is a $k \times n$ matrix $\mathbf{C}$ such that $\mathbf{A} = \mathbf{CX}$. The hypothesis $H_0: \mathbf{A}\boldsymbol{\beta} = \mathbf{0}$ vs. $H_a: \mathbf{A}\boldsymbol{\beta} \neq \mathbf{0}$ is defined to be testable if and only if $\mathbf{A}\boldsymbol{\beta}$ is a set of estimable functions. In this paper it is proved that the linear combinations $\mathbf{A}\boldsymbol{\beta}$ are estimable if and only if the rank of the matrix

X(I − A⁻A) is $q-k$. The matrix $\mathbf{A}^-$ is the generalized inverse of the matrix A (Penrose, R. (1955) *Proc. Cambridge Philos. Soc.* **51** 406–413). This result is then expressed as the trace of the matrix $\mathbf{X}(\mathbf{I}-\mathbf{A}^-\mathbf{A})[\mathbf{X}(\mathbf{I}-\mathbf{A}^-\mathbf{A})]^-$. The sum of squares due to the null hypothesis $\mathbf{A}\boldsymbol{\beta} = \mathbf{0}$ is obtained using Bose's principle of conditional error (Bose, R. C., *Least Squares Aspects of Analysis of Variance*, Institute of Statistics Mimeo, Series 9, Chapel Hill, North Carolina). The reduced model, i.e., the model restricted under the null hypothesis, is shown to be $\mathbf{y} = \mathbf{X}(\mathbf{I}-\mathbf{A}^-\mathbf{A})\boldsymbol{\beta}+\mathbf{e}$. The final results are numerical techniques for computing the matrix products $\mathbf{HH}^-$ and $\mathbf{H}^-\mathbf{H}$ and the matrix $\mathbf{H}^-$. Using the theory and computational techniques, a computer program can be developed to provide a complete analysis of the linear model using generalized inverses. (Received May 5, 1970.)

126-10. The distribution of linear combinations of frequency counts of adjusted ranks: two samples from continuous distributions. C. J. Park and K. M. Lal Saxena, University of Nebraska.

Let $X_1, X_2, \cdots, X_m$ and $Y_1, Y_2, \cdots, Y_n$ be random samples from continuous distribution $F(\cdot)$ and $G(\cdot)$ respectively. Let R_i denote the rank of Y_j's in the combined ordered sample, and define $T_i = R_i - i, i = 1, 2, \cdots, n$, T_i's denoting the adjusted ranks. Let S_j denote the number of T_i's equal to $j, j = 0, 1, \cdots, m$. Then S_j's are the frequency counts of adjusted ranks. In this paper we prove that, under $F(x) \equiv G(x)$, the asymptotic distribution of $W = \sum_{j=0}^m w_j S_j$ is normal, when m and n tend to infinity and (i) $n/m \to \alpha$, $0 < \alpha < \infty$, and (ii) $\sum_{j=0}^m |w_j|^3/(\sum_{j=0}^m w_j^2)^{3/2} > 0$. In the proof we use a generating function. When $w_j = j+(n+1)/2$, W is the Wolcoxon test statistic. (Received May 6, 1970.)

126-11. Asymptotically optimal subset selection procedures. Vijay S. Bawa, Bell Telephone Laboratories, Inc.

In this paper, we develop single-stage (fixed sample size) asymptotically optimal procedure for the problem of selecting a subset of populations containing the best population. The optimality criterion used is as in Studden (*Ann. Math. Statist.* **38** (1967) 1072–1078). Studden obtained a subset selection procedure which is optimal among the class of invariant decision rules for translation or scale invariant exponential families of distributions. We have developed a procedure which is an asymptotically optimal procedure among the class of all decision rules for a much larger class of distribution functions satisfying certain mild regularity conditions. The results obtained in this paper are applicable to the problem of selecting an optimal subset from several univariate or multivariate populations, when the populations are ranked according to the values of a scalar valued parameter. The results are also applicable to the subset selection problem associated with a single multivariate population when certain symmetry conditions hold. (For example, the results apply to ranking means of a single multivariate normal population with common correlation coefficient and common variance of the components.) Relation to existing subset selection procedures is noted. (Received May 6, 1970.)

126-12. On the generalization of binomial distribution. P. C. Consul and G. C. Jain, University of Calgary.

The numerical data collected on sampling of attributes over an extended area or over a period of time does not usually conform with the binomial distribution. To meet such difficulties different workers like Dandeker (1955), Shumway and Gurland (1960), Katti and Gurland (1962), Kendall and Stuart (1963) have suggested various modifications and generalizations to the binomial distribution. Considering the probability p of the occurrence of an event to be a linear functional form of the number of successes observed in a set of n independent observations, we give another generalization, containing two parameters, α and β, of the binomial distribution as $b_\beta(x, n, \alpha) = \binom{n}{x}\alpha(\alpha+\beta x)^{x-1}(1-\alpha-\beta x)^{n-x}$; $x = 1, 2, 3, \cdots, n$ for $1 < \alpha < 0$ and $0 < \alpha+\beta n < 1$, β being small. We also give a method to obtain its moments, a number of its important properties like convolution and its approximation with a Poisson type distribution given by $p_x(\lambda_1, \lambda_2) = \lambda_1(\lambda_1+x\lambda_2)^{x-1} e^{-(\lambda_1+x\lambda_2)}/x!$ which has been named by us as the generalized Poisson distribution. (Received May 19, 1970.)

126-13. On the generalization of Poisson distribution.

P. C. CONSUL AND G. C. JAIN, University of Calgary.

Though a large variety of problems are Poissonian in nature, it has been practically observed that the probability of occurrence of an event does not remain constant and is often affected by the previous occurrence. Many workers like Greenwood and Yule (1920), Neyman (1939), Anscombe (1950), Evans (1953), Skellam (1952), Dandeker (1955), Shumway and Gurland (1960), Cassie (1962), Bhattacharya and Hola (1965) have discussed this problem and have suggested various mixtures of distributions. We define a generalized Poisson distribution, based on two parameters, λ_1 and λ_2, such that $\lambda_1 > 0$, $|\lambda_2| < 1$ and $0 < \lambda_1 + m\lambda_2 < 1$, as $p_x(\lambda_1, \lambda_2) = \lambda_1(\lambda_1 + x\lambda_2)^{x-1} e^{-(\lambda_1+x\lambda_2)}/x!$, $x = 0, 1, 2, \cdots$ We further prove that it is a probability density function, study its important properties and show that it gives a much better fit than others to numerical data in cases where mean $m = \sigma^2$, $m < \sigma^2$ and $m > \sigma^2$. (Received May 19, 1970.)

(Abstracts of papers to be presented at the European regional meeting, Hanover, Germany, August 19–20, 1970.)

127-1. A Bayesian approach to a specific outlier problem in small samples.

IRWIN GUTTMAN, University of Wisconsin and University of Massachusetts.

The problem of detecting whether one observation in a small sample of size $n \leqq 10$ is spurious of the shifted mean type is approached using a Bayesian analysis. Specifically, suppose a sample of n independent observations is such that $(n-1)$ of the observations are generated from the m-variate normal $N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ distribution, while one observation is spurious, and generated from the m-variate $N(\boldsymbol{\mu}+\mathbf{a}, \boldsymbol{\Sigma})$ distribution, where of course, we do not know which observation is spurious. Using non-informative priors, the posterior of $\mathbf{a}$ is shown to be a weighted combination of generalized t-distributions, mean value $n(\mathbf{y}_j - \bar{\mathbf{y}})/(n-1)$, degrees of freedom $(n-m-1)$, variance-covariance matrix $nA^{(j)}/[(n-1)(n-m-3)]$, where $A^{(j)} = \sum_{i \neq j}(\mathbf{y}_i - \bar{\mathbf{y}}^{(j)})(\mathbf{y}_i - \bar{\mathbf{y}}^{(j)})'$, $\bar{\mathbf{y}}^{(j)} = (n-1)^{-1}\sum_{i \neq j}\mathbf{y}_i$ and $\bar{\mathbf{y}} = n^{-1}\sum_{j=1}^{n}\mathbf{y}_j$. The weights, say c_j, are given by $c_j = |V^{(j)}|^{-(n-2)/2} / \sum_{j=1}^{n}|V^{(j)}|^{-(n-2)/2}$ where $V^{(j)} = (n-2)^{-1}A^{(j)}$. These results may be used to examine the statement "$\mathbf{a} = \mathbf{0}$". Numerical examples are given to illustrate the cases $m = 1$ and $m = 2$. (Received May 18, 1970.)

127-2. An item analysis model for pairwise comparisons with an application.

P. ALLERUP, S. JOHANSEN, P. LEWY AND P. PEDERSON, University of Copenhagen.

We consider a situation where n persons are asked to compare k objects by a pairwise comparison of all objects. The outcome of comparing i and j are $\{i > j\}$, $\{i \sim j\}$ or $\{i < j\}$. The response of person No. ν to comparison (i, j) is the random variable $X_{ij\nu} = \{X_{ij\nu}^{(1)}, X_{ij\nu}^{(0)}, X_{ij\nu}^{(-1)}\}$ having a multinomial distribution with probabilities proportional to $(\theta_i/\theta_j, \alpha_\nu, \theta_j/\theta_i)$. By means of this model it is possible to distinguish the persons by the number of times the response $\{i \sim j\}$ was used and the objects by the number of times it was judged better minus the number of times it was judged worse than its opponent. We describe the conditional estimation and test procedures and give various approximations to these. (Received May 19, 1970.)

127-3. Asymptotic properties of Bayesian decision rules for two terminal decisions and multiple sampling II. A. HALD AND N. KEIDING, University of Copenhagen.

The paper is a continuation of a paper by the same authors (*J. Roy. Statist. Soc. Ser. B* **31** (1969) 455–471. The model is based on a differentiable prior distribution, a linear loss function, an asymptotically normal sampling distribution, and sampling costs proportional to the sample size. Asymptotic expressions are derived for sample sizes, acceptance and rejection criteria and the minimum regret for some k-stage sampling and decision procedures. In the present paper three cases not considered in the previous paper are treated. Some comments on the relation to sequential sampling, the relation to models for clinical trials, and efficiency problems are also given. (Received May 19, 1970.)

(*Abstracts of papers contributed by title.*)

70T-47. On ties in triple comparisons (preliminary report). R. J. BEAVER AND P. V. RAO, University of Florida.

The triple comparison model of Bradley and Pendergrass [Contributions to Probability and Statistics, Stanford University Press, Stanford, California, 1960] is generalized to account for the occurrence of ties in the data. Maximum likelihood estimates of the parameters involved in the model are given, together with their large sample properties. Likelihood ratio tests of the parameters of the model are developed. In testing the equality of treatment ratings using the triple comparison model, the asymptotic relative efficiency for normal alternatives using untied triple comparisons and analysis of variance for complete and balanced incomplete block designs are given. (Received March 26, 1970.)

70T-48. Sequential designs for estimating the ED50. P. R. FREEMAN, University College, London.

In sequentially estimating the ED50 of a logistic, quantal response model, a stopping rule and a rule for deciding at what dose level to perform the next trial have to be specified. Using a Bayesian approach with quadratic loss and constant cost of observation, optimal policies are derived using a dynamic programming analysis, for the cases when one, two or three dose levels are available. The expected losses of these policies are compared, under certain assumptions, with those using an up-and-down method. The latter is shown, in most cases, to be surprisingly close to optimality. Possible extensions to tailored testing and sequential estimation of the size of a population are discussed. (Received March 18, 1970.)

70T-49. Entropy characterizations of clustering and randomness for channels with memory. J-P. A. ADOUL, B. D. FRITCHMAN AND L. N. KANAL, Bethlehem University and LNK Corporation.

We propose a quantitative measure for the memory of ergodic, binary discrete-time stochastic processes, and in particular transmission channels. Labelling the states error and non error, we characterize the memory μ of a process of error density Pe by its relative deviation in average conditional entropy from the discrete memoryless channel (D.M.C.), which we prove has maximum entropy for the class of (finite and infinite memory length) processes of density Pe. We obtain an upper bound for μ for real channels, derive μ for the general discrete renewal process from the error gap probability mass function (EGPMF) and prove that it is a lower bound for any processes having the same EGPMF. We demonstrate some limitations of finite error-free state models by showing that their EGPMF is bounded from above by a geometric series. To estimate the counting distribution with flexibility we introduce conditional gap distributions and multigap statistics; we use these in implementing a denumerable Markov Chain model which, free from finite state model limitations and more general than renewal processes, allows the derivation of all classical statistics including entropy. In addition we show that a qualitative characterization of memory involves the description of dual phenomena: cyclical and clustering trends. (Received April 1, 1970.)

70T-50. Performance of compound sequential detector for intersymbol interference channels. B. D. Fritchman, L. N. Kanal and J. D. Womer, Lehigh University and LNK Corporation.

We consider the transmission of m-ary digital information over a channel which induces both intersymbol interference and additive, Gaussian noise. We show a way of computing the performance of the optimum compound sequential decision procedure (Chang and Hancock, *IEEE Trans. Information Theory* (1966), Abend *et al.*, *IEEE Trans. Information Theory* (1968) etc.) which, by definition bases its decision only on all past and present samples. We are able to reduce the decision problem to the classical m-class pattern classification problem in which the kth symbol is corrupted by noise of variance σ_k. This formulation allows us to calculate the probability of error of this detector. This had not previously been done. The calculated performance is shown to be in good agreement with the performance obtained by simulation of an optimum compound detector. We present the relationship between the channel impulse response and the value of σ_k. This relationship involves a difference equation whose coefficients depend directly on the impulse response of the channel. The $n+1$ coefficients of the difference equation must lie in a certain region of n-space in order for the solution of the difference equation to converge. These regions are determined. Also, the sequential compound detector performs satisfactorily only for impulse responses which meet certain other criteria. We present these criteria. (Received April 1, 1970.)

70T-51. On the best linear estimates of the location and scale parameters based on k selected order statistics from distributions truncated at the middle (preliminary report). A. K. Md. Ehsanes Saleh and M. Ahsanullah, Carleton University and Food and Drug Directorate.

In this paper double exponential, extreme value distributions of Type I and Type II, normal, lognormal and logistic distributions are considered. From a sample of size n let $x_{(1)} < x_{(2)} \cdots < x_{(n)}$ be the order statistics, where the integers $1, 2, \cdots n$ denote the ranks of the order statistics. The interval $[1, n]$ is subdivided into 3 disjoint interval $[1, r_1)$, $[r_1, r_2]$ and (r_2, n) where r_1 and r_2 are determined by prefixed proportion of censoring $\beta - \alpha$ with $r_1 = [n\alpha]+1$ and $r_2 = [n\beta]+1$. Assuming all observations with ranks in the interval $[1, r_1)$ and $(r_2, n]$ are available for estimation purposes, we have obtained the BLUES' of the scale parameter σ and location parameter μ for the distributions mentioned above based on k (prefixed) selected order statistics allowing k_1 $(< r_1)$ order statistics with ranks in the interval $[1, r_1)$ and k_2 $(\leqq n - r_2)$ order statistics with ranks in the interval $(r_2, n]$ such that $k_1 + k_2 = k$. The resulting BLUES' possess minimum variance property among all other choices of the k order statistics. The problem has been solved numerically for $n = 3(1)10$, $k = 2(1)4$, $r_1 = 1(1)n-1$ and $r_2 = r_1(1)n-1$. (Received April 6, 1970.)

70T-52. A sequential procedure for the fixed-width interval estimation of the mean of a uniform distribution. NICO F. LAUBSCHER AND M. J. A. ANDREW, National Research Institute for Mathematical Sciences.

Let $X_1, X_2, \cdots$ be independent rv's from the uniform distribution $R(\mu, \omega)$ where both the mean μ and range ω are unknown. The distribution of Carlton's statistic, $C_n = (m_n - \mu)/r_n$, where m_n is the midrange and r_n the range from a sample of size n is used to construct a $(1-\alpha)$ confidence interval for μ. Following Chow and Robbins a sequential stopping rule may be formulated thus: observe the sequence $X_1, X_2, \cdots$, term by term, stopping with X_N where N is the first integer $(\geqq 2)$ satisfying $r_N(\alpha^c - 1) \leqq 2d$, where $c = 1/(1-N)$; then form the interval $I_N = (m_N - d, m_N + d)$. It is easy to see that $P(N < \infty) = 1$. A simulation study has been done to compare this stopping rule with one based on the distribution-free sign test (Geertsema) and a study is at present being undertaken of the coverage probability of I_N and the expected stopping value of N. (Received April 9, 1970.)

70T-53. Asymptotic tests for serial correlation with exponentially distributed random variables. G. L. YANG, University of Maryland.

Let $X_1, X_2, \cdots$ be a sequence of random variables which are identically distributed over $(0, \infty)$ and serially dependent in a Markovian manner. The conditional density of X_{j+1} given X_j is $p(x_{j+1}/x_j) = (\lambda + \beta x_j) \exp(-(\lambda + \beta x_j) x_{j+1})$ for $x_{j+1} \geqq 0$ and is zero otherwise, where $\lambda > 0$ and $\beta \geqq 0$. The problem of testing the hypothesis H_0 that the serial correlation parameter $\beta = 0$ against a sequence of alternatives $H_1: \beta_n = v/n^{\frac{1}{2}} > 0$ is considered in this paper. The following asymptotic normal distributions of the test statistics both under H_0 and H_1 are obtained. If the parameter λ is assumed to be known, the test statistic $\Delta_n = \sum_{j=0}^{n-1} X_j(1 - X_{j+1})/n^{\frac{1}{2}} \rightarrow_{\mathscr{L}} \mathscr{N}(0, 2/\lambda^4)$ under H_0 and $\Delta_n \rightarrow_{\mathscr{L}} \mathscr{N}(2\beta/\lambda^4, 2/\lambda^4)$ under H_1. If λ is a nuisance parameter, the test statistic $\hat{\Delta}_n = (n - \sum_{j=0}^{n-1} X_j X_{j+1})/n^{\frac{1}{2}}$ is used. $\hat{\Delta}_n$ converges in law to $\mathscr{N}(0,1)$ under H_0 and to $\mathscr{N}(\lambda/\beta^2, 1)$ under H_1. The critical region defined by $\hat{\Delta}_n > b$, where b is determined by $(2\pi)^{-\frac{1}{2}} \int_b^\infty e^{-u^2/2}\, du = \alpha$ and α is the significance level, is asymptotically uniformly most powerful on $0 \leqq \beta/\lambda \leqq c < \infty$ among all uniformly asymptotically similar tests of the same size. To obtain the above result, the essential part is to establish the contiguity of the sequences of probability measures specified under H_0 and H_1 (cf. L. LeCam, *Univ. California Publ. Statist.* **3** (1960)). The method of proof is analogous to the one given in *Ann. Inst. Statist. Math.* **39** (1968) 1863–89. However, the regularity conditions used here such as the assumptions imposed on $(\lambda + \beta x_j)$ are different from that of "the rate of occurrence" $(\lambda + \beta \sum_{i=0}^{j-1} f(t_i, t_j))$ used in the above mentioned paper. (Received April 13, 1970.)

70T-55. A note on round off errors and the uniform distribution over a compact group. M. T. CHAO AND T. F. HOU, Bell Telephone Laboratories.

Let $X_1, X_2, \cdots$ be independent random variables and let $S_n = X_1 + \cdots + X_n - [(X_1 + \cdots + X_n)/d]d$, where $d > 0$ and $[x]$ denotes the largest integer less than or equal to x. It is known that under general conditions the limiting distribution of S_n converges weakly to the uniform distribution over $[0, d)$. The discrete case was treated by Dvoretzky and Wolfowitz (*Duke Math. J.* **18** (1951) 501–507) and the continuous case by Nagaev and Muhin (Limit theorems statist. inference (Russian) 113–117 *Izdat.* "Fan," Tashkent, 1966). In this paper, it is noticed that the set $[0, d)$ may be considered as a group if the group multiplication is defined as addition modulo d. By restricting the values of X_i in the group $[0, d)$, S_n may be written as $S_n = X_1 \cdot X_2 \cdot \cdots \cdot X_n$ in terms of the group multiplication. The following general result is obtained: Let $X_1, X_2, \cdots$ be independent random variables which take values in a compact second countable Abelian group G; let $S_n = X_1 \cdot X_2 \cdot \cdots \cdot X_n$. It is shown that, as $n \rightarrow \infty$, the probability measure P_n induced by S_n converges weakly to the normalized Haar measure over G. The only condition we impose on the random variables X_i is that they cannot essentially assume values in a proper compact subgroup of G; namely, $\limsup \Pr[X_k \cdot X_{k+1} \cdot \cdots \cdot X_n \in G_1] < 1$ for all proper compact subgroups G_1 of G. (Received April 23, 1970.)

70T-56. Exact tests in quantal bioassay for the logistic model (preliminary report). JOHN J. GART, National Cancer Institute.

Consider two independent series of mutually independent binomial variates $\{x_i, y_i\}$ with respective sample sizes, $\{n_{1i}, n_{2i}\}$, and parameters, $\{p_{1i}, p_{2i}\}$, for $i = 1, 2, \cdots, k$. We assume the logistic model, $p_{ji} = \exp(\alpha_j + \beta_j d_i)/\{1 + \exp(\alpha_j + \beta_j d_i)\}$, for all i and $j = 1, 2$. Let $\alpha_1 = \alpha + \lambda$ and $\alpha_2 = \alpha - \lambda$. An hypothesis of interest in bioassay is $H_0: \lambda = 0$; $\alpha, \beta_1 = \beta_2 = \beta$ (unspecified) versus $H_1: \lambda > 0; \alpha, \beta$. Following the approach of Cox [*J. Roy. Statist. Soc. Ser. B* **20** (1958) 215–232], we find the jointly sufficient set of statistics to be $\sum(x_i - y_i)$, $\sum(x_i + y_i)$, and $\sum(x_i + y_i) d_i$ which relate to λ, α, and β respectively. The uniformly most powerful unbiased test of H_0 is based on the distribution of $\sum(x_i - y_i)$ given $\sum(x_i + y_i)$ and $\sum(x_i + y_i) d_i$ fixed [cf. Lehmann, *Testing Statistical Hypotheses*, Wiley, New York (1959) chapter 4]. The conditional distribution is independent of α and β [cf. *ibid.* 52] and is proportional to $f(x_i, y_i) = \prod_{i=1}^{k} \binom{n_{1i}}{x_i}\binom{n_{2i}}{y_i}$. The appropriate exact probability in the tail is $P = \sum_C f(x_i, y_i)/\sum_R f(x_i, y_i)$ where C is the subset of R such that $\sum x_i$ is greater than or equal to its observed value and R is the set of all (x_i, y_i) such that $\sum(x_i + y_i)$ and $\sum(x_i + y_i) d_i$ equal their observed values. Exact tests for the difference in β's may also be constructed. Exact confidence limits for λ may also be found. Exact confidence limits for $2\lambda/\beta$ (the relative potency) may be found using the approach of Cox [*Biometrika* **54** (1967) 567–572]. Comparison of the exact methods with approximate normal deviate tests and methods based on the usual estimation procedures are under study. (Received April 23, 1970.)

70T-57. Estimation and design for multiresponse nonlinear models, nonhomogenous variance structure. M. J. BOX AND N. R. DRAPER, I.C.I., Ltd. and University of Wisconsin.

The subjects of parameter estimation and choice of experimental design for multiresponse nonlinear models have been discussed for a variety of situations in a series of papers (see, for example, Box and Draper, *Biometrika* **52** 355–365; Draper and Hunter, *Biometrika* **53** 525–533, and **54** 662–665). A situation which has not previously been examined is that in which the variance-covariance structure is not constant and the data can be split up into independent blocks, each block having a different structure, some of which may be known, some unknown. For such a case it can be shown that the expected extension of previous work applies. The marginal posterior distribution for the parameters $\boldsymbol{\theta}$ (which can be maximized to produce the required estimators $\hat{\boldsymbol{\theta}}$) is the product of the marginal posteriors appropriate for each of the blocks separately. Moreover, the criterion for selection of additional runs becomes "maximize $|\mathbf{A}|$" where $\mathbf{A} = \Sigma \mathbf{A}_i$ and each $\mathbf{A}_i (i \geqq 1)$ is the matrix of the quadratic form in the exponential portion of the expanded form of the marginal posterior for $\boldsymbol{\theta}$, while $\mathbf{A}_0$ is a similar matrix derived from the prior density. Each $\mathbf{A}_i$ for blocks with unknown structure is of the same form as those A_i for blocks with known structure but all variance-covariance elements are replaced by estimates derived from the empirical scatter matrices. (Details are given in University of Wisconsin Statistics Department Technical Report No. 144.) (Received May 4, 1970.)

70T-58. Functional limit theorems for the queue $GI/G/1$ in light traffic. DONALD L. IGLEHART, Stanford University.

Consider a $GI/G/1$ queue with traffic intensity less than one. Let W_n be the waiting time of the nth customer, $Q(t)$ the number of customers in the system at time t, $W(t)$ the virtual waiting time at time t, $D(t)$ the number of customers departing the system in $[0, t]$, $B(t)$ the amount of time the server is busy in $[0, t]$, and $I(t)$ the amount of time the server is idle in $[0, t]$. Functional strong laws, central limit theorems, and laws of the iterated logarithm are obtained for the processes $\sum_{j=0}^{[nt]} W_j$, $\int_0^{ut} Q(s)\,ds$, $\int_0^{ut} W(s)\,ds$, $D(ut)$, $B(ut)$, and $I(ut)$, as either n or u tends to infinity. (Received May 4, 1970.)

70T-59. On distribution-free confidence bounds for PR($Y < X$). HANS URY, California State Department of Public Health.

For the case in which X and Y are both unknown, distribution-free confidence intervals for $\Pr(Y < X)$ are obtained by means of the Chebyshev inequality and van Dantzig's (*Koninklijke Nederlandse Akademie van Wetenschappen Proc. Ser. A* **54** (1951) 1–8) upper bound for the variance of the Mann–Whitney statistic U. The (two-sided) intervals are shorter than the upper bounds obtained by Birnbaum and McCarty (*Ann. Math. Statist.* **29** (1958) 558–562), provided the confidence coefficient does not exceed .925 and the sample sizes are not too unequal; and they are more reliable for small sample sizes than the shorter intervals given by Govindarajulu (*Ann. Inst. Statist. Math.* **20** (1968) 229–238), especially when $\Pr(Y < X)$ is close to 0 or 1. For equal sample sizes and for several confidence levels and interval widths, the required sample size is compared with the corresponding Birnbaum–McCarty entry. For unequal sizes, lower bounds are given for the ratio of smaller to larger size which will ensure that the intervals remain shorter than the Birnbaum–McCarty bounds. (Received May 6, 1970.)

70T-60. On the distribution of the sum of independent discrete random variables. B. K. SHAH, Rockland State Hospital.

Let $X_1, X_2, \cdots, X_k$ be k independent, but not necessarily identical, discrete positive integer valued random variables. Let S_k denote their sum i.e. $S_k = \sum_{i=1}^k X_i$. Using probability generating functions we obtain $\Pr\{S_k = x\}$ recursively. The relation $x \Pr\{S_k = x\} = \sum_{j=1}^x \{(j-1)!\}^{-1} \Pr\{S_k = x-j\} \sum_{i=1}^k \{\partial^j \ln A_i(s)/(\partial s^j)\}_{s=0}$, where $A_i(s)$ is the probability generating function for the ith random variable X_i and $\Pr\{S_k = 0\} = \prod_{i=1}^k \Pr\{X_i = 0\}$, is established in Theorem 1. This result is applied to the sums of binomial random variables with varying probabilities (p_i) of successes and we obtain the distribution in the following form: $\Pr\{S_k = x\} = \sum_{i=0}^{nk} g_i\, b(x, nk-i, \bar{p})$, $\bar{p} > 0$, $x = 0, 1, \cdots, nk$, where $b(x, n, p) = \binom{n}{x} p^x (1-p)^{n-x}$ and g_i satisfy the recursion $rg_r = -n\sum_{i=1}^r g_{r-i}\{\sum_{j=1}^k [1-\bar{p}/p_j]^i\}$, $g_0 = \prod_{i=1}^k (p_i/\bar{p})^n$. A similar result holds for the sums of the negative binomial random variables. (Received May 6, 1970.)

70T-61. Jackknifing maximum likelihood estimates in the multiparameter case (preliminary report). J. G. FRYER, University of Exeter.

The object of this paper is to generalize all of the results in Brillinger (*Rev. Inst. Internat. Statist.* **32** (1964), 202) to cover multiparameter distributions. Formulae are obtained both for fixed r (the number of groups used) and for fixed s (the number of elements in any group). There is a noticeable increase in the complexity of the results. Let I^{ab} denote the (a, b)th element of the inverse of the matrix $[-E(\partial^2 \log f/\partial\theta_\alpha \partial\theta_\beta)]$, where f is the basic density function, and denote $E(\partial^2 \log f/\partial\theta_\alpha \partial\theta_\beta \cdot \partial \log f/\partial\theta_\gamma)$ by $(\alpha\beta, \gamma)$ for example. Then denoting the jackknifed estimates of θ_a and θ_b by $\theta_a{}^*$ and $\theta_b{}^*$ respectively, we find for fixed r for instance that to $0(n^{-2})$ $E(\theta_a{}^* - \theta_a)(\theta_b{}^* - \theta_b) = I^{ab}/n - rI^{ab}/n^2(r-1) + rI^{a\beta}I^{b\alpha}I^{\gamma\delta}(\beta\gamma, \alpha\delta)/n^2(r-1) + rI^{a\beta}I^{b\alpha}I^{\gamma\delta}I^{\varepsilon\zeta}\{(\beta\varepsilon, \delta)(\alpha\gamma, \zeta) + (\beta\gamma\varepsilon)(\alpha\delta\zeta)/2 + (\beta\gamma\varepsilon)(\alpha\delta, \zeta) + (\alpha\gamma\varepsilon)(\beta\delta, \zeta)\}/\{n^2(r-1)\}$. In this expression, we use the convention of summation over repeated indices. Despite the fact that this formula reduces to Brillinger's in the one-parameter case, other generalized formulae do not. It appears that Brillinger did not take his basic expansion quite far enough to gather all of the terms in n^{-2} in general, but that the extra terms in n^{-2} for $E(\theta_a{}^* - \theta_a)^2$ are self-cancelling. (Received May 7, 1970.)

70T-62. A characterization of the geometric distribution. ANDRE G. LAURENT AND RAMESH GUPTA, Wayne State University.

Let the random variable X be distributed on the set of nonnegative integers and Y_i be the ith order statistic of a random sample of n observations of X. X has a geometric distribution iff for one triplet $1 \leqq i \leqq j < k \leqq n$, Y_i and $Y_k - Y_j$ are independent. (Received May 8, 1970.)

70T-63. A characterization of rotatable arrangements (preliminary report). LAWRENCE L. KUPPER, University of North Carolina.

Working in the framework of the *approximate* theory of the optimal design of experiments, Hoel (*Ann. Math. Statist.* **36** (1965) 1097–1106) characterized the G-optimal spacing for Fourier series (F.S.) and spherical harmonics (S.H.) regression of order d in terms of certain normalizing coefficients which are difficult to interpret. In this paper the author proves a theorem stating that the "best" designs ("best" in the sense that they are shown to possess many highly desirable properties) for F.S. and S.H. regression of order d are expressible as rotatable arrangements of points (Box and Hunter, *Ann. Math. Statist.* **28** (1957) 195–241) of order d in two and three dimensions, respectively. This theorem is used to construct optimal *exact* designs for F.S. regression of all orders and for S.H. regression of orders one and two. Sequential designs and infinite classes of designs depending only on one parameter are given. (Received May 13, 1970.)