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ON A CLASS OF UNIFORMLY ADMISSIBLE ESTIMATORS OF A FINITE POPULATION TOTAL[1]

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1. Introduction. In a recent paper Godambe (1969) established the uniform admissibility of a class of estimators of a finite population total. In the present note we extend this class. The notation and definitions of this section follow that of Godambe (1969).

Any subset s of the integers, $1, \cdots, N$, which label the N distinguishable population units, is called a sample. A sampling design is defined by a probability mass function, p, on S, the set of all possible samples. Let x_i be the real (unknown) value associated with the ith population unit and let $\mathbf{x} = (x_1, \cdots, x_N) \in R^N$. Any real-valued function $e(\mathbf{x}, s)$ which depends on $\mathbf{x}$ only through those values x_i for $i \in s$ will be termed an estimator. We will be concerned with estimation of the population total, $T(\mathbf{x}) = \sum_1^N x_i$, under quadratic losses.

Definition 1.1. For any given sampling design p, an estimator e' is said to *dominate* the estimator e if for all $\mathbf{x} \in R^N$

$$\sum_S p(s)[e'(s,\mathbf{x}) - T(\mathbf{x})]^2 \leqq \sum_S p(s)[e(s,\mathbf{x}) - T(\mathbf{x})]^2$$

with strict inequality for at least one $\mathbf{x}$.

Definition 1.2. A pair (e', p') of an estimator e' and a sampling design p' is said to *uniformly dominate* another pair (e, p) if for all $\mathbf{x} \in R^N$

$$\sum_S p'(s)[e'(s,\mathbf{x}) - T(\mathbf{x})]^2 \leqq \sum_S p(s)[e(s,\mathbf{x}) - T(\mathbf{x})]^2$$

with strict inequality for at least one $\mathbf{x}$.

The notions of *admissibility* of an estimator for a given sampling design and that of *uniform admissibility* of a pair (e, p) for p in a class, C, of designs are then defined in the standard manner.

If $C_n = \{p \mid \sum_S p(s)n(s) = n\}$ where $n(s)$ is the cardinality of s then the main result of Godambe (1969) is that with respect to the class C_n the pair (e^*, p^*) is uniformly admissible where $e^*(s, \mathbf{x}) = \sum_{i \in s} x_i + \sum_{i \notin s} \lambda_i$, ($\lambda_1, \cdots, \lambda_N$ being *any* fixed values) and where p^* is *any* member of C_n.

Definition 1.3. For $0 < n < N$, $D_n \equiv \{p \mid p(s) = 0 \text{ if } n(s) \neq n\}$ i.e., D_n is the class of fixed size sample designs.

The main result of this note is then

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THEOREM 1.1. *With respect to the class of designs D_n the pair (e^*, p^*) is* uniformly admissible *where*

$$e^*(s, \mathbf{x}) = \alpha_n \textstyle\sum_{i \in s} x_i + \beta_n \tag{1.1}$$

α_n and β_n being any fixed arbitrary values such that $1 < \alpha_n < N/n$, and p^* is any *member of D_n.*

Note that for this class of estimators the coefficient of $\sum_{i \in s} x_i$ is realistically allowed to depend on the sample size, n. A proof of this theorem as well as some other results are given in the next section. A brief discussion of the estimator is given in Section 3.

2. Proof of main and supplementary results. It is clear that for any prior distribution, ω, on R^N

$$e(s, \mathbf{x}) = \textstyle\sum_{i \in s} x_i + \sum_{i \notin s} E_\omega(x_i \mid s, x_j : j \in s) \tag{2.1}$$

is a Bayes estimator of $T(\mathbf{x})$ provided that the conditional expectation above exists. The following is then immediate.

THEOREM 2.1. *For any specified fixed values β_n and $1 < \alpha_n < N/n$ the estimator, $e^*(s, \mathbf{x})$ in (1.1), is a Bayes estimator under any prior distribution, ω, for which*

$$\textstyle\sum_{i \notin s} E_\omega(x_i \mid s, x_j : j \in s) = (\alpha_n - 1) \sum_{i \in s} x_i + \beta_n$$

for all $\mathbf{x} \in R^N$ and all s for which $n(s) = n$.

To prove admissibility properties of e^* we will utilize the following definitions:

DEFINITION 2.1. Let Ω denote any class of discrete prior distributions on R^N such that for any point $\mathbf{x}_0 \in R^N$ there exists an $\omega \in \Omega$ such that $\omega(\mathbf{x}_0) > 0$.

DEFINITION 2.2. Let Ω^* denote the class of discrete prior distributions, ω, on R^N such that

(i) $x_1, \cdots, x_N$ when distributed as ω are *exchangeable* and possess a variance, σ_ω^2 and

(ii) For all $\mathbf{x} \in R^N$, all s for which $n(s) = n$, and all $i \notin s$ $E_\omega(x_i \mid s, x_i : i \in s) = \alpha_n' \sum_{i \in s} x_i + \beta_n'$, α_n' and β_n' being any fixed values satisfying $0 < \alpha_n' < 1/n$.

In order to prove the main results we use the following lemmas.

LEMMA 2.1. (Godambe) *Let Ω satisfy Definition 2.1. If e, given in (2.1), is a Bayes estimator for all $\omega \in \Omega$ and if p is any arbitrary sampling design, then e is admissible.*

LEMMA 2.2. *For any point $\mathbf{x}_0 \in R^N$ there exists an $\omega_0 \in \Omega^*$ such that $\omega_0(\mathbf{x}_0) > 0$.*

PROOF. For the given $\mathbf{x}_0 = (x_{10}, \cdots, x_{N0})$ let $y_1, \cdots, y_{r-1}$ be the set of distinct values of the x_{i0}'s. Let $Y = \{y_1, \cdots, y_{r-1}, y_r\}$ where y_r is determined by (2.3) below. For any $\mathbf{x} \in R^N$ let n_i be the number of the N coordinate x_j's equal to y_i

$i = 1, \cdots, r$. We then assign a discrete probability distribution to R^N as follows. Let

$$(2.1)\quad \omega_0(\mathbf{x}) = 0 \quad \text{if} \quad \sum_1^r n_i < N,$$

$$= P_r(n_1, \cdots, n_r) = \frac{\Gamma(N+1)\prod_1^r \Gamma(n_i+\varepsilon_i)\Gamma(\varepsilon)}{\prod_1^r \Gamma(n_i+1)\Gamma(N+\varepsilon)\prod_1^r \Gamma(\varepsilon_i)} \quad \text{if} \quad \sum_1^r n_i = N;$$

where $\varepsilon_i > 0, n_i = 0, 1, \cdots, N, \varepsilon = \sum_1^r \varepsilon_i$.

This distribution, variously called the Dirichlet-Multinomial (Ericson (1969)), the compound multinomial (Mosimann (1962)) etc. clearly satisfies the discreteness, exchangeability, and $\omega_0(\mathbf{x}_0) > 0$ requirements. It also is easily seen that under this prior $\sigma_{\omega_0}^2 = \sum_1^r y_i^2\varepsilon_i/\varepsilon - (\sum_1^r y_i\varepsilon_i/\varepsilon)^2$. In addition, it can be shown (Ericson (1969)), that for $i \notin s$

$$(2.2)\quad E_{\omega_0}(x_i \mid s, x_j : j \in s) = \sum_{i\in s} x_i/(n+\varepsilon) + \sum_1^r y_i\varepsilon_i/(n+\varepsilon)$$

and thus it is clear that one can choose ε, y_r, and the ε_i's such that for *any* specified $0 < \alpha_n' < 1/n$ and $-\infty < \beta_n' < \infty$

$$(2.3)\quad \alpha_n' = 1/(n+\varepsilon) \quad \text{and} \quad \beta_n' = \sum_1^r y_i\varepsilon_i/(n+\varepsilon).$$

From these two lemmas we then have

THEOREM 2.2. *For any fixed sample size, p, the estimator e^* in* (1.1) *for any fixed β_n and* $1 < \alpha_n < N/n$ *is admissible.*

To prove Theorem 1.1 we utilize the following result.

THEOREM 2.3. (Godambe) *Let Ω satisfy Definition* 2.1. *If e, given in* (2.1) *is a Bayes estimator for all $\omega \in \Omega$ and if C is a class of sampling designs such that for all $p \in C$ and all $\omega \in \Omega$ the Bayes risk is independent of $p \in C$, i.e.,*

$$(2.4)\quad \sum_{R^N} \omega(x)\{\sum_S p(s)[e(s, x) - T(x)]^2\} = c_\omega,$$

where c_ω does not depend upon p, then with respect to the class C the pair (e, p), where p is any element of C, is uniformly admissible.

The proof of Theorem 1.1 then follows from Theorem 2.3, Theorem 2.1 and Lemma 2.2 by taking $\Omega = \Omega^*$ in Theorem 2.3, taking $C = D_n$, taking e as in (2.1), and noting that for any $\omega \in \Omega^*$ the Bayes risk of e is constant over $p \in D_n$. The latter observation follows since for $\omega \in \Omega^*$, by (2.1) and Definition 2.2, $e(s, \mathbf{x}) = [(N-n)\alpha_n' + 1]\sum_{i\in s} x_i + (N-n)\beta_n'$.

Hence, on interchanging the order of summation and summing first on $\mathbf{x} \in R^N$ such that x_i for $i \in s$ are fixed, the Bayes risk, (2.4), becomes

$$(2.5)\quad \sum_S p(s) E_{x_i:\, i\in s} \operatorname{Var}[T(\mathbf{x}) \mid s, x_i : i \in s]$$

$$= \sum_S p(s)[\operatorname{Var}(T(\mathbf{x})) - \operatorname{Var}_{x_i:\, i\in s}\{E(T(\mathbf{x}) \mid s, x_i : i \in s\}]$$

$$= N\sigma_\omega^2 + N(N-1)\rho_\omega\sigma_\omega^2 - [(N-n)\alpha_n' + 1]^2(n\sigma_\omega^2 + n(n-1)\rho_\omega\sigma_\omega^2) \equiv c_\omega$$

for all $p \in D_n$ and where ρ_ω is the common correlation coefficient under $\omega \in \Omega^*$ of x_i and x_j. Finally we make the identification $\alpha_n = [(N-n)\alpha_n' + 1]$ and $\beta_n = (N-n)\beta_n'$.

3. Discussion. It follows as a simple corollary of Theorem 1.1 that with respect to D_n uniformly admissible estimators of $\mu(\mathbf{x}) = T(\mathbf{x})/N$ are given by

$$e_\mu(s, \mathbf{x}) = n\alpha_n \bar{x}/N + \beta_n/N \equiv a_n{}^* \bar{x} + \beta_n{}^* \tag{3.1}$$

where $n/N < \alpha_n{}^* < 1$.

Note that the uniform admissibility of the estimators in (1.1) and (3.1) for $\alpha_n = 1$ and $\alpha_n{}^* = n/N$ respectively was shown by Godambe (1969). The restriction $\alpha_n{}^* \geqq n/N$ seems intuitively reasonable and under the class Ω^* of priors it corresponds to the restriction that $\mathrm{Cov}(x_i, x_j) \geqq 0$.

The class of estimators, (3.1), of $\mu(\mathbf{x})$ also has intuitive appeal from a Bayesian viewpoint for it has been shown (Ericson (1969)) that if $e_\mu(s, \mathbf{x})$, (3.1), is a Bayes estimator of μ for some $\omega \in \Omega^*$ then $\beta_n{}^* = (1-\alpha_n{}^*)E(\mu(\mathbf{x}))$ and $\alpha_n{}^* = \mathrm{Var}(\mu(\mathbf{x}))/\mathrm{Var}(\bar{x})$ or $\alpha_n{}^* = \mathrm{Var}(\mu(\mathbf{x}))/[\mathrm{Var}(\mu(\mathbf{x})) + E_{\bar{x}}\mathrm{Var}(\mu(\mathbf{x}) \mid \bar{x})]$. It then follows that e_μ has the interpretation of being a weighted average of $\bar{x}$, the sample mean, and $E(\mu(\mathbf{x}))$, the prior mean, with weights inversely proportional to the prior expectation of the sampling variance of $\bar{x}$ and the prior variance of $\mu(\mathbf{x})$ respectively.

REFERENCES

[1] Ericson, W. A. (1969). Subjective Bayesian models in sampling finite populations (with discussion). *J. Roy. Statist. Soc. Ser. B.* **31** 195–233.

[2] Godambe, V. P. (1969). Admissibility and Bayes estimation in sampling finite populations V. *Ann. Math. Statist.* **40** 672–676.

[3] Mosimann, J. E. (1962). On the compound multinomial distribution, the multivariate β-distribution, and correlations among proportions. *Biometrika* **49** 65–82.