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A CLASS OF ORTHOGONAL SERIES RELATED TO MARTINGALES[1]

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1. Introduction. In this paper we study convergence problems for sums of dependent random variables. The particular type of series considered here includes all discrete parameter martingales, but is more restricted than the class of all orthogonal series.

Let $\{X_k\}_{k=1}^{\infty}$ be a sequence of centered (mean zero) random variables on a probability space (Ω, F, P) and $\{Y_n = \sum_{k=1}^{n} X_k\}_{n=1}^{\infty}$ their sequence of partial sums. Consider the following hierarchy of dependence.

(1.1) Mutual independence:

$$\int_\Omega \Theta(X_1, X_2, \cdots, X_n)\Phi(X_{n+1})\,dP = \int_\Omega \Theta(X_1, X_2, \cdots, X_n)\,dP \int_\Omega \Phi(X_{n+1})\,dP$$

for all pairs $(\Theta(\cdot), \Phi(\cdot))$ of integrable functions of the indicated variables, $n = 1, 2, \cdots$.

(1.2) The *martingale* property: $\int_\Omega \Theta(X_1, X_2, \cdots, X_n)X_{n+1}\,dP = 0$

for all bounded measurable functions $\Theta(\cdot)$ of the indicated variables, $n = 1, 2, \cdots$.

(1.3) The *weak martingale* property: $\int_\Omega \Phi(Y_m)X_n\,dP = 0$

for all bounded measurable functions $\Phi(Y_m)$, $1 \leqq m < n = 2, 3, \cdots$.

(1.4) Orthogonality: $\int_\Omega X_n X_m\,dP = 0, \qquad n \neq m.$

If $X_m X_n \in L^1$ for $m \neq n$, the weak martingale property implies orthogonality (Proposition 2.2). Otherwise, the increasing dependence of the hierarchy is clear. If $\{Y_n\}_{n=1}^{\infty}$ satisfies (1.2), it is called a *martingale*. If $\{Y_n\}_{n=1}^{\infty}$ satisfies (1.3), we call it a *weak martingale*. Weak submartingales are defined analogously. Clearly, every martingale is a weak martingale and Gaussian weak martingales are martingales (Section 3).

Martingales which converge in L^2, converge a.e.; but, there are orthogonal series which converge in L^2 and diverge a.e. ([1] Theorem 2.4.1, page 88). Since weak martingales lie between orthogonal series and martingales in the hierarchy of dependence, it is natural to investigate the pointwise convergence of L^2—convergent weak martingales. We show that on totally finite signed measure spaces martingales whose L^2 norms are bounded converge a.e. (Theorem 4.1). But, we construct an example of an L^2-bounded weak martingale on a totally finite signed measure space whose paths oscillate between plus and minus infinity.

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Frank Knight and Donald L. Burkholder have given independent proofs (both unpublished) that if the L^1 norms of a weak martingale are bounded, the weak martingale converges in probability (Theorem 2.1).

2. Basic properties. Given a sequence of random variables $Y = \{Y_n\}_{n=1}^{\infty}$ on a probability space (Ω, F, P), let $F_n = \sigma(Y_1, Y_2, \cdots, Y_n)$ be the σ-field generated by $\{Y_k\}_{k=1}^{n}$. We use $\|Y_n\|_p$ to denote the L^p norm of Y_n and let $\|Y\|_p = \sup_n \|Y_n\|_p$. The sequence $Y = \{Y_n\}_{n=1}^{\infty}$ is said to be *L^p-bounded* if $\|Y\|_p < \infty$. Let $I(\cdot)$ be the indicator function of the set within the parentheses and write $\Omega - A$ as A'. We use $a \wedge b$ to denote the minimum of a and b. Unless otherwise noted, all integrals are taken over Ω.

Weak martingales are most conveniently defined in terms of conditional expectations. The weak martingale property, (1.3), may be expressed in the following two equivalent forms.

DEFINITION 2.1. A sequence $\{Y_n = \sum_{k=1}^{n} X_k\}_{n=1}^{\infty}$ of integrable random variables is a weak martingale if and only if

$$E(X_n \mid Y_m) = 0 \text{ a.e.}, \qquad 1 \leqq m < n = 2, 3, \cdots. \tag{2.1}$$

DEFINITION 2.2. A sequence $\{Y_n\}_{n=1}^{\infty}$ of integrable random variables is a weak martingale if and only if

$$E(Y_n \mid Y_m) = Y_m \text{ a.e.}, \qquad 1 \leqq m < n = 2, 3, \cdots. \tag{2.2}$$

Clearly, if $\{Y_n = \sum_{k=1}^{n} X_k\}_{n=1}^{\infty}$ is a weak martingale, then (2.1) and (2.2) imply that

$$E(X_n) = 0, \qquad E(Y_n) = E(Y_1) \qquad n = 2, 3, \cdots. \tag{2.3}$$

Future references to the weak martingale property will be to Condition (2.2).

The weak martingale property should be compared to the martingale property, $E(Y_{n+1} \mid Y_1, Y_2, \cdots Y_n) = Y_n$ a.e., $n = 1, 2, \cdots$. In the language of gambling, a martingale is a sequence of fortunes resulting from a game which is fair in the sense that, given the gambler's entire history up to time n, his average gain in the future is zero. A weak martingale need only be fair given the gambler's fortune at any particular time; the possibility is allowed that some past histories are favorable, on the average, to the future fortunes of the gambler.

Martingales have the property that $\{Y_{n \wedge t}\}_{n=1}^{\infty}$ is also a martingale for all stopping times t; this is not true for weak martingales.

PROPOSITION 2.1. *Let $\{Y_n = \sum_{k=1}^{n} X_k\}_{n=1}^{3}$ be a weak martingale which is not a martingale. Then, there is a bounded stopping time $t(\cdot)$ such that:*

(i) $E(Y_t) > E(Y_1)$

(ii) $\{Y_{t \wedge n} = \sum_{k=1}^{n} X_k I(t \geqq k)\}_{n=1}^{3}$ *is not a weak martingale.*

PROOF. Since $\{Y_n\}_{n=1}^3$ is not a martingale and $E(X_3) = 0$, there is a set A of positive probability in F_2 such that $E(X_3 \mid F_2) < 0$ a.e. on A. Let

$$\begin{aligned} t(\omega) &= 2 \quad \text{if} \quad \omega \in A; \\ &= 3 \quad \text{if} \quad \omega \in A'. \end{aligned}$$

The random variable $Y_t = \sum_{n=1}^3 Y_n I(t=n)$ is clearly integrable and

$$\begin{aligned} \int Y_t\, dP &= \int_A Y_2\, dP + \int_{A'} Y_3\, dP > \int_A (Y_2 + E(X_3 \mid F_2))\, dP + \int_{A'} Y_3\, dP \\ &= \int_A Y_3\, dP + \int_{A'} Y_3\, dP = \int Y_1\, dP. \end{aligned}$$

To verify (ii), observe that:

$$E(Y_{t \wedge 2}) = E(Y_2) = E(Y_1) < E(Y_t) = E(Y_{t \wedge 3}).$$

Therefore, (2.3) does not hold and $\{Y_{t \wedge n}\}_{n=1}^3$ is not a weak martingale.

The following proposition states that a weak martingale whose difference sequence has the property that $X_n X_m \in L^1$ for $n \neq m$ is a sequence of the partial sums of an orthogonal series. This is true, in particular, if $Y_n \in L^2$, $n = 1, 2, \cdots$.

PROPOSITION 2.2. *Let $\{Y_n = \sum_{k=1}^n X_k\}_{n=1}^\infty$ be a weak martingale with $X_m X_n \in L^1$ for $n \neq m$ and $Y_0 = 0$. Then,*

$$\int X_m X_n\, dP = 0, \qquad n \neq m.$$

PROOF. Note that $E(X_n \mid X_m) \neq 0$; however, for $n > m$

$$\int X_n X_m\, dP = \int X_n (Y_m - Y_{m-1})\, dP = \int E(X_n \mid Y_m) Y_m\, dP - \int E(X_n \mid Y_{m-1}) Y_{m-1}\, dP = 0.$$

An L^2-bounded weak martingale, being an orthogonal series, converges in probability. It is also true that L^1-bounded weak martingales converge in probability. We present D. L. Burkholder's proof, modified to cover the case of weak submartingales.

THEOREM 2.1. (Burkholder) *Let $\{Y_n\}_{n=1}^\infty$ be an L^1-bounded weak submartingale. Then, $\{Y_n\}_{n=1}^\infty$ converges in probability.*

PROOF. With the aid of Jensen's inequality it is easily seen that $\{(Y_n - a)^+\}_{n=1}^\infty$ is an L^1-bounded weak submartingale for any constant a. Given $\delta > 0$, choose $\lambda > 0$ such that $2 \sup \|Y_n\|_1 / \lambda < \delta$. For $\varepsilon > 0$, the set $\{(x, y); |x| + |y| \leqq \lambda, y - x \geqq \varepsilon\}$ is closed, bounded and hence compact. Therefore, there is a sequence of constants $\{a_i\}_{i=1}^q$ such that for $n > m$,

$$\begin{aligned} &P\{Y_n \geqq Y_m + \varepsilon\} \\ &\quad \leqq P\{|Y_n| + |Y_m| > \lambda\} + P\{|Y_m| + |Y_n| \leqq \lambda,\ Y_n \geqq Y_m + \varepsilon\} \\ &\quad \leqq 2 \sup_k \|Y_k\|_1 / \lambda + \textstyle\sum_{i=1}^q P\{Y_n > a_i + \varepsilon/2,\ Y_m < a_i\} \\ &\quad \leqq \delta + (2/\varepsilon) \textstyle\sum_{i=1}^q \int (Y_n - a_i)^+ I\{(Y_m - a_i)^+ = 0\}\, dP \\ &\quad = \delta + (2/\varepsilon) \textstyle\sum_{i=1}^q \left(\int (Y_n - a_i)^+\, dP - \int (Y_n - a_i)^+ I\{(Y_m - a_i)^+ > 0\}\, dP\right) \\ &\quad = \delta + (2/\varepsilon) \textstyle\sum_{i=1}^q \left(\int (Y_n - a_i)^+\, dP - \int E((Y_n - a_i)^+ \mid Y_m) I\{(Y_m - a_i)^+ > 0\}\, dP\right) \\ &\quad \leqq \delta + (2/\varepsilon) \textstyle\sum_{i=1}^q \left(\int (Y_n - a_i)^+\, dP - \int (Y_m - a_i)^+\, dP\right) \\ &\quad = \delta + o(1) \text{ as } m, n \to \infty. \end{aligned}$$

Since δ is arbitrary, $P(Y_n - Y_m \geqq \varepsilon) = o(1)$ as $n > m \to \infty$ for all $\varepsilon > 0$.

To prove the theorem, it is sufficient to show that there is a subsequence of the positive integers $\{n_k\}_{k=1}^{\infty}$ such that if $\{r_k\}_{k=1}^{\infty}$ is any subsequence of the integers with $r_k \geqq n_k$ for all k, then $\{Y_{r_k}\}_{k=1}^{\infty}$ converges a.e. We can choose, by using the above, n_k such that

$$P\{Y_n - Y_m > 2^{-k}\} < 2^{-k}, \qquad \text{for all} \quad n > m \geqq n_k.$$

For any subsequence of the integers $\{r_k\}_{k=1}^{\infty}$ with $r_k \geqq n_k$, $k = 1, 2, \cdots$, we obtain

$$\sum_{k=1}^{\infty} P\{Y_{r_{k+1}} - Y_{r_k} > 2^{-k}\} < \sum_{k=1}^{\infty} 2^{-k} < \infty. \tag{2.4}$$

The Borel–Cantelli Lemma and (2.4) yield

$$\sum_{k=1}^{\infty} (Y_{r_{k+1}} - Y_{r_k})^+ < \infty \quad \text{a.e.} \tag{2.5}$$

From Fatou's lemma we obtain

$$\int \liminf |Y_{r_k}|\, dP \leqq \liminf \int |Y_{r_k}|\, dP < \infty. \tag{2.6}$$

Hence, $\liminf |Y_{r_k}| < \infty$ a.e. Together with (2.5) this implies the a.e. convergence of $\{Y_{r_k}\}_{k=1}^{\infty}$ to a finite limit. The theorem is proved.

Uniformly integrable weak martingales can be characterized in much the same way as uniformly integrable martingales. A chief difference is that a uniformly integrable weak martingale is the restriction of a measure to σ-fields which are not necessarily increasing.

THEOREM 2.2. *Let $\{Y_n\}_{n=1}^{\infty}$ be a weak martingale. The following are equivalent.*

(i) *$\{Y_n\}_1^{\infty}$ converges in the L^1 norm to an integrable random variable.*
(ii) *There is an integrable random variable Y such that*

$$E(Y \mid Y_n) = Y_n \quad \text{a.e.} \qquad n = 1, 2, \cdots.$$

(iii) *$\{Y_n\}_1^{\infty}$ is uniformly integrable.*

PROOF. We use lim* to denote convergence in the L^1 norm. First we show that (i) implies (ii). Let $Y = \lim^* Y_n$. From the continuity of conditional expectations with respect to L^1 convergence and the weak martingale property we obtain:

$$E(Y \mid Y_m) = E(\lim^* Y_n \mid Y_m) = \lim^* E(Y_n \mid Y_m) = Y_m \quad \text{a.e.}$$

The integrability of Y implies the uniform integrability of $\{E(Y \mid \beta_\alpha)\}_{\alpha \in \Gamma}$ for any family $\{\beta_\alpha\}_{\alpha \in \Gamma}$ of sub-σ-fields. Thus, (ii) implies (iii).

Finally, (iii) implies (i) since Theorem 2.1 shows that $\{Y_n\}_1^{\infty}$ converges in probability. Recall that a uniformly integrable sequence of random variables which converges in probability, converges in the L^1 norm.

3. Examples of weak martingales. We construct examples of weak martingales which are not martingales by working directly with distributions. The general procedure is to mix multivariate probabilities in such a way that their bivariate distributions behave in the manner required by the weak martingale property.

Recall that if X and Y are centered random variables with a *joint* normal distribution, then

$$E(Y \mid X) = XE(XY)/E(X^2) = X\sigma_{xy}/\sigma_x^{\ 2} \quad \text{a.e.} \tag{3.1}$$

Since a centered Gaussian process is characterized by its covariance structure, (3.1) shows that every Gaussian weak martingale is a martingale.

EXAMPLE 3.1. We need the following lemma, which shows that for $\alpha > 1$ there is a centered Gaussian process $\{Y_n\}_{n=1}^{\infty}$ with

$$E(Y_n \mid Y_m) = \alpha Y_m \quad \text{a.e.}, \qquad 1 \leqq m < n = 2, 3, \cdots. \tag{3.2}$$

LEMMA 3.1. *Given $\alpha > 1$, a sequence of variances $\{\sigma_n^{\ 2}\}_{n=1}^{\infty}$ may be defined so that if $V_n = (v_{ij})i, j = 1, 2, \cdots$, is the symmetric matrix with $v_{ii} = \sigma_i^{\ 2}$, $i \leqq n$ and $v_{ij} = \alpha\sigma_i^{\ 2}$, $i < j$, then V_n is positive definite for $n = 1, 2, \cdots$.*

PROOF. It suffices to show that $|V_n| > 0$, $n = 1, 2, \cdots$ ([4] Theorem 1.23, page 3). The proof follows by induction. Choose $\sigma_1^{\ 2} > 0$ and we have

$$|V_1| = \sigma_1^{\ 2} > 0.$$

Assume $\sigma_2^{\ 2}, \sigma_3^{\ 2}, \cdots, \sigma_{n-1}^2$ have been chosen as required. Subtracting row $(n-1)$ from row (n) in V_n, we see that

$$|V_n| = (\sigma_n^{\ 2} - \alpha\sigma_{n-1}^2)|V_{n-1}| - (\alpha-1)\sigma_{n-1}^2|V_{n-1}^*|, \tag{3.3}$$

where $V_{n-1}^* = V_{n-1}$ except in the $(n-1) \times (n-1)$ place, in which $\alpha\sigma_{n-1}^2$ appears instead of σ_{n-1}^2. We have $|V_{n-1}| > 0$ by the inductive hypothesis. Choose $\sigma_n^{\ 2} > ((\alpha-1)\sigma_{n-1}^2|V_{n-1}^*|/|V_{n-1}|) + \alpha\sigma_{n-1}^2$. The lemma is proved.

Let $g(y_1, y_2, \cdots, y_n)$ be the Gaussian density with mean zero and covariance matrix V_n as given by Lemma 3.1. We use the convention that $g(y_{n_1}, y_{n_2}, \cdots, y_{n_k})$ represents the marginal density of $(Y_{n_1}, Y_{n_2}, \cdots, Y_{n_k})$. The following mixture is a weak martingale probability that is not a martingale probability.

$$f_n(y_1, y_2, \cdots, y_n) = (1/\alpha)g(y_1, y_2, \cdots, y_n) + (1-1/\alpha)g(y_1)g(y_2)\cdots g(y_n). \tag{3.4}$$

The sequence $\{f_n(\cdot)\}_{n=1}^{\infty}$ clearly constitutes a consistent family of probability densities. Let $Y = \{Y_n\}_{n=1}^{\infty}$ be the coordinate random variables with the distribution thus formed. For $1 \leqq m < n = 2, 3, \cdots$, using (3.1) we have

$$\begin{aligned}
E(Y_n \mid Y_m = y_m) &=_{\text{a.e.}} \textstyle\int_{-\infty}^{\infty} y_n f(y_m, y_n)\, dy_n / \int_{-\infty}^{\infty} f(y_m, y_n)\, dy_n \\
&= ((1/\alpha)\textstyle\int_{-\infty}^{\infty} y_n g(y_m, y_n)\, dy_n + (1-1/\alpha)g(y_m)\int_{-\infty}^{\infty} y_n g(y_n)\, dy_n) \\
&\qquad \cdot ((1/\alpha)g(y_m) + (1-1/\alpha)g(y_m))^{-1} \\
&= 1/\alpha \textstyle\int_{-\infty}^{\infty} y_n g(y_m, y_n)\, dy_n / g(y_m) \\
&= (1/\alpha)\alpha y_m = y_m.
\end{aligned}$$

Hence Y is a weak martingale. A similar computation shows that Y is not a martingale.

EXAMPLE 3.2. Let P_1 and P_2 be two different probabilities with the same bivariate distributions. If there is a martingale probability P_3 and a positive constant c such that

$$P = (1/c)(cP_3 + P_2 - P_1) \geqq 0, \tag{3.5}$$

then P is a weak martingale probability. To verify this, let $\{Y_n\}_{n=1}^{\infty}$ be the coordinate random variables with joint distribution given by P. For $1 \leqq m < n$, $A \in \sigma(Y_m)$ we have:

$$\begin{aligned} \int_A Y_n\,dP &= \int_A Y_n\,dP_3 + (1/c)\int_A Y_n\,d(P_2 - P_1) \\ &= \int_A Y_n\,dP_3 = \int_A Y_m\,dP_3 = \int_A Y_m\,dP. \end{aligned} \tag{3.6}$$

Let us investigate a specific instance of this procedure. Consider $n \geqq 3$ balls placed independently and at random into one of two urns. Let

$$\begin{aligned} X_{ij} = X_{ji} &= 1 \quad \text{if ball } i \text{ and ball } j \text{ are in the same urn;} \\ &= -1 \quad \text{otherwise.} \end{aligned}$$

We take $\{Y_k\}_{k=1}^{\binom{n}{2}}$ to be some ordering of the X_{ij}. The sequence $\{Y_k\}_{k=1}^{\binom{n}{2}}$ is then a familiar example of a process which is pairwise independent but *not* mutually independent. Let P_1 be its joint distribution.

Let P_2 be the product measure of $\binom{n}{2}$ copies of the distribution of Y_1. Then P_1 and P_2 are two different probabilities with the same bivariate distributions. Observe that the support of both P_1 and P_2 consists of sequences of ± 1 of length $\binom{n}{2}$, where both probabilities are now considered as defined on $R^{\binom{n}{2}}$.

The probability P_3 is taken to be a Markov Chain, constructed as follows:

$$\begin{aligned} &P(Y_1 = 1) = P(Y_1 = -1) = \tfrac{1}{2}. \\ &P\{Y_{j+1} = k \mid Y_j = i\} = \tfrac{1}{4}; \\ &\qquad j = 1, 2, \cdots \tbinom{n}{2} - 1; \quad k = 1, -1, 2i+1, 2i-1. \end{aligned}$$

The probability P_3 is a martingale probability since, with respect to it:

$$E(Y_{j+1} \mid Y_1, \cdots, Y_j) = E(Y_{j+1} \mid Y_j) = Y_j \quad \text{a.e.,} \qquad j = 1, 2, \cdots, \tbinom{n}{2} - 1.$$

Furthermore, P_3 places strictly positive probability on the support of P_2. Hence, a $c > 0$ exists such that P as defined in (3.5) is a weak martingale probability. It is easy to verify that P is not a martingale probability.

EXAMPLE 3.3. Let $g(y_1, y_2, \cdots, y_n)$ be the density of a centered Gaussian martingale ($n \geqq 3$). Note that $g(\cdot)$, being positive and continuous everywhere, has a strictly positive infimum on every compact set in its domain.

Let $u(\cdot)$ be a continuous odd function vanishing outside of $[-1, 1]$ and such that:

$$\int_{-1}^{1} y_n u(y_n)\,dy_n \neq 0 \tag{3.7}$$

$$\sup_y |u(y)|^n < \inf\{g(y_1, \cdots, y_n); |y_i| \leqq 1, i \leqq n\}. \tag{3.8}$$

We now define a mixture of probabilities which has the same bivariate distributions as $g(y_1, y_2, \cdots, y_n)$ but a different n-fold distribution. The procedure used is suggested by an example of E. Nelson ([3] Example 2, page 99).

$$f(y_1, y_2, \cdots, y_n) = g(y_1, y_2, \cdots, y_n) + u(y_1)u(y_2)\cdots u(y_n). \tag{3.9}$$

Clearly, $f(\cdot)$ is a probability density. Take $Y^{(n)} = \{Y_i\}_{i=1}^n$ to be the coordinate random variables with this density. Observe that every sub-collection of random variables except $Y^{(n)}$ itself is a Gaussian martingale. Therefore, $Y^{(n)}$ is a weak martingale with the property that if one function is omitted, those remaining form a martingale. To show that $Y^{(n)}$ is not a martingale we compute:

$$\begin{aligned}
&E(Y_n \mid Y_1 = y_1, Y_2 = y_2, \cdots, Y_{n-1} = y_{n-1}) \\
&\quad =_{\text{a.e.}} \textstyle\int_{-\infty}^{\infty} y_n f(y_1, y_2, \cdots, y_n)\, dy_n / \int_{-\infty}^{\infty} f(y_1, y_2, \cdots, y_n)\, dy_n \\
&\quad = (\textstyle\int_{-\infty}^{\infty} y_n g(y_1, \cdots, y_n)\, dy_n + u(y_1)u(y_2)\cdots u(y_{n-1})\int_{-1}^{1} y_n u(y_n)\, dy_n)/ \\
&\qquad\qquad g(y_1, \cdots, y_{n-1}) \\
&\quad = (y_{n-1} g(y_1, \cdots y_{n-1}) + u(y_1)u(y_2)\cdots u(y_{n-1})\textstyle\int_{-1}^{1} y_n u(y_n)\, dy_n)/g(y_1, \cdots y_{n-1}) \\
&\quad = y_{n+1} + (u(y_1)\cdots u(y_{n-1})/g(y_1, \cdots y_{n-1}))\textstyle\int_{-1}^{1} y_n u(y_n)\, dy_n \\
&\quad \neq y_{n-1} \qquad \text{on a set of positive probability because of (3.7).}
\end{aligned}$$

Hence, $Y^{(n)}$ is not a martingale.

The sequence $Y^{(n)}$ can be extended to an infinite weak martingale by considering a difference sequence $\{X_j\}_{j=n+1}^{\infty}$ of mutually independent, centered random variables which are jointly independent of $Y^{(n)}$. Taking $Y_{n+m} = Y_n + \sum_{j=n+1}^{n+m} X_j$, $Y = \{Y_i\}_{i=1}^{\infty}$ is a weak martingale which is not a martingale. If the X_j are taken to be normally distributed, Y is a non-Gaussian process all of whose bivariate distributions are those of a Gaussian martingale.

EXAMPLE 3.4. A sequence $\{Y_n = \sum_{k=1}^n X_k\}_{n=1}^{\infty}$ of integrable random variables with $E(X_n) = 0$ and X_n independent of each Y_m, $1 \leqq m < n = 2, 3 \cdots$, is a weak martingale since $E(X_n \mid Y_m) = E(X_n) = 0$ a.e., $m < n$. We can construct such a sequence of random variables with the additional properties that $\{X_n\}_{n=1}^{\infty}$ does not consist of mutually independent random variables and $\{Y_n\}_{n=1}^{\infty}$ is not a martingale. The construction is omitted.

4. Martingales and weak martingales on signed measure spaces. Let (Ω, F, μ) be a totally finite signed measure space and $\Omega = J \cup K$ a Hahn decomposition. For $A \in F$, let $\mu^+(A) = \mu(AJ)$, $\mu^-(A) = -\mu(AK)$ and $|\mu| = \mu^+ + \mu^-$. Given a σ-field $F_n = \sigma(Y_1, Y_2, \cdots, Y_n) \subset F$, μ_n is μ restricted to F_n. Note that $|\mu_n|$ is not necessarily the same as $|\mu|$ restricted to F_n.

In considering the a.e. convergence of a sequence $\{Y_n\}_{n=1}^{\infty}$ of random variables, the important measure is $|\mu_\infty|$, where $F_\infty = \sigma(\bigcup_{n=1}^{\infty} F_n)$. To clarify this point, consider the example with $\Omega = (0, 1]$, F equal to the Borel sets and μ equal to Lebesgue measure on $(0, \frac{1}{2}]$ and negative Lebesgue measure on $(\frac{1}{2}, 1]$. Let X_k equal one if k is odd and minus one if k is even. The sequence $\{Y_n = \sum_{k=1}^{n} X_k\}_{n=1}^{\infty}$ diverges everywhere. But, F_∞ is the trivial σ-field and $|\mu_\infty|(\Omega) = 0$. Observe that

$$\int_A Y_{n+1}\, d\mu = \int_A Y_n\, d\mu \quad \text{for} \quad A \in F_n, \qquad n = 1, 2, \cdots. \tag{4.1}$$

This example shows that the magnitude of $\sup_n \int |Y_n|\, d|\mu|$ does not limit the average number of upcrossings of bounded intervals by martingales on signed measure spaces. Moreover, this observation implies that signed martingale measures cannot be written as finite linear combinations of martingales probabilities. The obvious extension of martingales to signed measure spaces indicated by (4.1) is too restrictive. It is only necessary that (4.1) hold for sets $A \in F_n$ where $|\mu_n|(A) > 0$.

DEFINITION 4.1. A set $E \in F_\alpha \subset F$ such that $|\mu_\alpha|(E) = 0$ but $|\mu|(E) > 0$ is called α-nil.

The fact that a countable union of α-nil sets is α-nil justifies the following definition.

DEFINITION 4.2. Let N_α be the a.e. maximal α-nil set if N_α is an α-nil set and $|\mu|(N_\alpha) = \sup\{|\mu|(E);\ E \text{ is } \alpha\text{-nil}\}$.

DEFINITION 4.3. A sequence $\{Y_n\}_{n=1}^{\infty}$ of $|\mu|$ integrable random variables is a (sub) martingale if

$$\int_{AN_n'} Y_{n+1}\, d\mu (\geqq) = \int_A Y_n\, d\mu \quad \text{for all} \quad A \in F_n, \qquad n = 1, 2, \cdots. \tag{4.2}$$

The following example of a martingale on a totally finite signed measure space will help clarify the preceding definitions. Let $\Omega = (0, 1]$ and F_n be the σ-field generated by the partition

$$\left\{\left(0, \frac{1}{2^n}\right], \left(\frac{1}{2^n}, \frac{2}{2^n}\right], \cdots \left(\frac{2^{n-1}-2}{2^n}, \frac{2^{n-1}-1}{2^n}\right], \right. \tag{4.3}$$
$$\left. \left(\frac{2^{n-1}-1}{2^n}, \frac{2^{n-1}+1}{2^n}\right], \left(\frac{2^{n-1}+1}{2^n}, \frac{2^{n-1}+2}{2^n}\right], \cdots, \left(\frac{2^n-1}{2^n}, 1\right]\right\}.$$

Observe that (4.3) is the nth dyadic partition of (0, 1] with the middle two atoms joined together. Let $F = F_\infty = \sigma(\bigcup_{n=1}^{\infty} F_n)$.

$$\mu = \text{Lebesgue Measure on } (0, \tfrac{1}{2}]$$
$$- \text{Lebesgue Measure on } (\tfrac{1}{2}, 1].$$

We have $J = (0, \frac{1}{2}]$, $K = (\frac{1}{2}, 1]$ as a Hahn decomposition of (Ω, F, μ) and $N_n = \left(\frac{2^{n-1}-1}{2^n}, \frac{2^{n-1}+1}{2^n}\right]$ a.e. Therefore, in this case

$$N_n \supset N_{n+1} \quad \text{a.e.} \tag{4.4}$$ (Note that (4.4) does not hold in general.)

A martingale $\{Y_n\}_1^{\infty}$ may now be constructed as *conditional expectations* of

$Y(x) = x$ with respect to $\{F_n\}_{n=1}^{\infty}$. Let

$$(4.5)\qquad Y_n(x) = 2^n \int_{k2^{-n}}^{(k+1)2^{-n}} x\, d\mu \quad \text{if} \quad x \in \left(\frac{k}{2^n}, \frac{k+1}{2^n}\right], \qquad k \leqq 2^{n-1}-2,\ k \geqq 2^{n-1}+1$$

$$= 0 \qquad\qquad \text{if} \quad x \in N_n.$$

For any atom $A \in F_n$, we have:

$$(4.6)\qquad \int_{AN_n'} Y_{n+1}\, d\mu = \int_A Y_n\, d\mu.$$

But

$$(4.7)\qquad \int_{N_n} Y_{n+1}\, d\mu = \int_{N_n - N_{n+1}} Y_{n+1}\, d\mu = \int_{N_n - N_{n+1}} x\, d\mu \neq 0 = \int_{N_n} Y_n\, d\mu.$$

Thus, (4.2) holds but (4.1) does not.

We are now able to prove a submartingale convergence theorem.

THEOREM 4.1. *Let* $\{Y_n\}_{n=1}^{\infty}$ *be a submartingale on a totally finite signed measure space* (Ω, F, μ). *Then,* $\sup \int |Y_n| d|\mu| < \infty$ *and* $F = \sigma(\bigcup_{n=1}^{\infty} F_n)$ *imply that* $\{Y_n\}_{n=1}^{\infty}$ *converges* a.e. $[|\mu|]$ *to a finite limit.*

PROOF. We may assume that $|\mu|(\Omega) = 1$. Let

$$(4.8)\qquad \delta_n(\omega) = I(\bigcup_{i=1}^{n} N_i)'(\omega) = \prod_{i=1}^{n} I(N_i')(\omega)$$
$$\delta_0(\omega) \equiv 1.$$

From (4.2) it follows that $\{Y_n \delta_{n-1}\}_{n=1}^{\infty}$ is a submartingale with the property that

$$(4.9)\qquad \int_A Y_{n+1}\, \delta_n\, d\mu \geqq \int_A Y_n\, \delta_{n-1}\, d\mu \quad \text{for all} \quad A \in F_n, \qquad n = 1, 2, \cdots.$$

Since $F = \sigma(\bigcup_{n=1}^{\infty} F_n)$, given any $\varepsilon \geqq 0$ there is an integer j and a set $A \in F_j$ such that $|\mu|(A \,\Delta\, J) \leqq \varepsilon$($\Delta$ denotes symmetric difference and J is the positive set of a Hahn decomposition of (Ω, F, μ)). Therefore,

$$(4.10)\qquad |\mu|(\bigcup_{i=j}^{\infty} N_i) \leqq 2|\mu|(A \,\Delta\, J) \leqq 2\varepsilon.$$

Re-index the martingale $\{Y_n\}_{n=j}^{\infty}$ and write it as $\{Y_n\}_{n=1}^{\infty}$. Since ε is arbitrary, it suffices to show that $\{Y_n \delta_{n-1}\}_{n=1}^{\infty}$ converges a.e. $[|\mu|]$ to a finite limit.

Take $q(\omega) = I(J)(\omega) - I(K)(\omega)$. Then from (4.9) we obtain:

$$(4.11)\qquad \int_A Y_{n+1}\, \delta_n q\, d|\mu| = \int_A Y_{n+1}\, \delta_n\, d\mu$$
$$\geqq \int_A Y_n\, \delta_{n-1}\, d\mu = \int_A Y_n\, \delta_{n-1} q\, d|\mu| \quad \text{for all} \quad A \in F_n, \qquad n = 1, 2, \cdots.$$

It is now easy to verify that $\{Y_n \delta_{n-1} E(q \mid F_n)\}_{n=1}^{\infty}$ is a submartingale on $(\Omega, F, |\mu|)$, where conditional expectations are with respect to $|\mu|$. That is, for any $A \in F_n$, $n = 1, 2, \cdots$, using (4.11):

$$\int_A Y_{n+1}\, \delta_n E(q \mid F_{n+1})\, d|\mu| = \int_A Y_{n+1}\, \delta_n q\, d|\mu|$$
$$\geqq \int_A Y_n\, \delta_{n-1} q\, d|\mu| = \int_A Y_n\, \delta_{n-1} E(q \mid F_n)\, d|\mu|.$$

Furthermore, $\sup_n \int |Y_n \delta_{n-1} E(q \mid F_n)| d|\mu| \leqq \sup_n \int |Y_n| d|\mu| < \infty$. By the standard submartingale convergence theorem ([2] page 324), $\{Y_n \delta_{n-1} E(q \mid F_n)\}_{n=1}^{\infty}$ converges a.e. to a finite limit. The sequence $\{E(q \mid F_n)\}_{n=1}^{\infty}$ is a uniformly integrable martingale which converges to q a.e. since $F = \sigma(\bigcup_{n=1}^{\infty} F_n)$. Hence, $\{Y_n \delta_{n-1}\}_{n=1}^{\infty}$ converges a.e. to a finite limit. The proof of the theorem is complete.

In view of the preceding, we give the following definition of weak martingales on totally finite signed measure spaces.

DEFINITION 4.4. A sequence of $|\mu|$ integrable random variables $\{Y_n\}_1^{\infty}$ is a weak martingale if

$$\int_{A\hat{N}_m'} Y_n \, d\mu = \int_A Y_m \, d\mu \quad \text{for all} \quad A \in \sigma(Y_m), \qquad n > m = 1, 2, \cdots, \tag{4.12}$$

where $\hat{N}_n$ is the maximal nil set in $\sigma(Y_n)$ (see Definition 4.2).

To construct an example of an L^2-bounded a.e. divergent weak martingale on a totally finite signed measure space, we first show that there is a centered Gaussian process $\{Y_n\}_{n=1}^{\infty}$ such that $E(Y_n \mid Y_m) = \frac{1}{2} Y_m$ a.e. $n > m \geqq 1$.

LEMMA 4.1. *Let $V_n = (v_{ij}) i, j = 1, 2, \cdots, n$ be the symmetric $n \times n$ matrix with all diagonal elements equal to one and all other elements equal to $\frac{1}{2}$. Then, V_n is positive definite, $n = 1, 2, \cdots$.*

PROOF. As pointed out in Lemma 3.1, it suffices to show that $|V_n| > 0$, $n = 1, 2, \cdots$.

$$|V_1| = 1, \qquad |V_2| = \begin{vmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{vmatrix} = \tfrac{3}{4}.$$

Subtracting row $(n-1)$ from row (n), for $n \geqq 3$ we obtain:

$$\begin{aligned} |V_n| &= \tfrac{1}{2}|V_{n-1}| + \tfrac{1}{2}(|V_{n-1}| - \tfrac{1}{2}|V_{n-2}|) \\ &= |V_{n-1}| - \tfrac{1}{4}|V_{n-2}|. \end{aligned}$$

Solution of this second-degree difference equation with boundary conditions $|V_3| = \frac{1}{2}$ and $|V_4| = 5/16$ yields:

$$|V_n| = (n+1)(\tfrac{1}{2})^n > 0, \qquad n \geqq 3.$$

The lemma is proved.

The L^2-bounded a.e. divergent weak martingale is constructed by mixing two measures. Let $g(y_1, y_2, \cdots, y_n)$ be the density of the centered Gaussian process with covariance matrix V_n, as given by Lemma 4.1. We take $g(y)$ to be the standard normal density, which is the marginal of $g(y_1, \cdots, y_n)$. If P_1 is the probability generated by $\{g(y_1, y_2, \cdots y_n)\}_{n=1}^{\infty}$ and P_2 is the probability generated by $\{g(y_1) g(y_2) \cdots g(y_n)\}_{n=1}^{\infty}$, then

$$\mu = 2P_1 - P_2 \tag{4.13}$$

is a signed weak martingale measure on the coordinate random variables $Y = \{Y_n\}_{n=1}^{\infty}$. Using Lemma 4.1 and (3.1), we have

$$\begin{aligned} \int_A Y_n \, d\mu &= 2\int_A Y_n \, dP_1 - \int_A Y_n \, dP_2 \\ &= 2\int_{-\infty}^{\infty} \int_A y_n g(y_m, y_n) \, dy_m \, dy_n - \int_{-\infty}^{\infty} y_n g(y_n) \, dy_n \int_A g(y_m) \, dy_m \\ &= 2\int_A \tfrac{1}{2} y_m g(y_m) \, dy_m = \int_A Y_m \, d\mu, \; A \in \sigma(Y_m). \end{aligned} \tag{4.14}$$

Note that because of the stationarity of P_1 and P_2, (4.14) holds as long as $m \neq n$. Thus, Y is a weak martingale in both orders. (This can only happen on probability spaces if $Y_m = Y_n$ a.e., $m \neq n$ ([2] page 314).)

The sequence Y is $L^2(|\mu|)$-bounded since:

$$\int Y_n^2 \, d|\mu| \leqq 2\int Y_n^2 \, dP_1 + \int Y_n^2 \, dP_2 = 3, \qquad n = 1, 2, \cdots.$$

However, P_1 and P_2 are both measures whose paths diverge. Under P_1, the odd numbered differences $(X_1, X_3, \cdots)$ consist of mutually independent, standard normal random variables.

$$\int X_n^2 \, dP_1 = \int (Y_n^2 + Y_{n-1}^2 - 2Y_n Y_{n-1}) \, dP_1 = 1 + 1 - 2(\tfrac{1}{2}) = 1.$$

$$\int X_{2k+1} X_{2j+1} \, dP_1 = \int (Y_{2k+1} - Y_{2k})(Y_{2j+1} - Y_j) \, dP_1 = 0, \qquad j \neq k.$$

The same is true of the even numbered ones. Under P_2, the difference sequence $\{X_k\}_{k=1}^{\infty}$ consists of mutually independent, normal random variables with mean zero and variance two. By symmetry and the Borel–Cantelli Lemma we obtain:

$$\{\limsup Y_n = \infty, \liminf Y_n = -\infty\} = \Omega \text{ a.e. } [2P_1 + P_2].$$

$$\{\limsup Y_n = \infty, \liminf Y_n = -\infty\} = \Omega \text{ a.e. } [|\mu|].$$

We have therefore constructed an $L^2(|\mu|)$-bounded weak martingale on a totally finite signed measure space whose paths oscillate between $+\infty$ and $-\infty$. As shown by Theorem 4.1, martingales behave quite differently. Clearly, Y does not converge in μ measure. Therefore, there is *no* martingale with the same bivariate distributions as Y. We conjecture that this cannot happen on probability spaces. We do not know whether or not L^2-bounded weak martingales converge a.e. on probability spaces.

Additional examples of weak martingales on totally finite signed measure spaces may be constructed by taking the difference of two different probabilities with the same bivariate distributions. Two such pairs of probabilities were constructed in Example 3.2 and Example 3.3.

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