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SOME REMARKS ON THE FELLER PROPERTY[1]

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1. Introduction. Let X_t be a temporally homogeneous Markov process on a locally compact metric space E. Denote its transition probabilities by $P_t(x, B)$, $t \geqq 0$, $x \in E$, $B \in \mathscr{B}$, where $\mathscr{B}$ is the topological Borel field on E. Let C_0 and $\mathscr{B}_b$ be the set of continuous functions with limit zero at infinity, and the bounded Borel measurable functions respectively. The transition probabilities can be thought of as operators on $\mathscr{B}_b$ via the formula $P_t f(x) = \int_E P_t(x, dy) f(y)$. The process X is said to be a Feller process if

(i) $P_t : C_0 \to C_0$,

(ii) for each $x \in E$ and $f \in C_0$, $P_t f(x) \to f(x)$ as $t \downarrow 0$.

If X is a Feller process, we can always find a standard modification $\hat{X}$ of X which is a Hunt process; that is, $\hat{X}$ is a right-continuous strong Markov process having left limits in E, and furthermore is quasi-left continuous: if $\{T_n\}$ is an increasing sequence of stopping times with limit T, then $\hat{X}(T_n) \to \hat{X}(T)$ w.p. 1. Most of the Markov processes presently admitted to the select circle of "well-behaved processes" —Brownian motion, for instance, or more generally most diffusion and birth-and-death processes—are Feller processes. On the other hand the Feller property is far from being a necessary condition that X be a Hunt process, and its consequences are often consequences of the continuity properties of $s \to P_t f \circ X_s$ rather than of $x \to P_t f(x)$. For instance, it is an easy exercise to show that if X is a right-continuous Markov process and $s \to P_t f \circ X_s$ is right continuous for each $f \in C_0$, then X must be strongly Markov. In fact, this condition turns out to be both necessary and sufficient for X to be strongly Markov (Theorem 2.1).

In this paper, we will show that every strong Markov process satisfies Feller properties of the second type—here we use the term "Feller property" very loosely to mean any relation of the type "P_t takes a class of 'well-behaved' functions into another class of 'well-behaved' functions." Such properties can often be described topologically, though this is not always the most convenient way. If X_t is a right-continuous strong Markov process, there is a topology on E, called the fine topology, which is particularly well adapted to the process. A set $B \subset E$ is open in the fine topology if and only if for each $x \in B$, $P^x\{X_t \in B$ for some interval $(0, \delta)\} = 1$. Girsanov [5] and Šur [10] proved that if X_t is a Hunt process, P_t takes the class of bounded fine continuous functions into itself. This is equivalent to the statement that $s \to P_t f \circ X_s$ is right continuous a.e. (P^x) for each x whenever f is bounded and fine continuous.

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We will generalize this in several ways, first to arbitrary right-continuous strong Markov processes, then to general Markov processes, and finally, in both cases, to continuity properties uniform in t. We find, not too surprisingly in view of the compactification results of Ray [8], that the results for Markov processes are quite similar to those for strong Markov processes except that "limit" is replaced by "essential limit," that is, a limit over all but a subset of the parameter set having Lebesgue measure zero.

2. The main theorems. Let $(E, \mathscr{B})$ be a locally compact space with its Borel field and let $\{X_t, t > 0\}$ be a Markov process on E with stationary transition probabilities $P_t(x, B)$. By this we mean that X_t is adapted to an increasing family $\{\mathscr{F}_t, t > 0\}$ of σ-fields (always assumed to be complete) and for each $x \in E$ and $t \geqq 0$, $P_t(x, \cdot)$ is a probability measure on E such that for each $s, t > 0$ and $B \in \mathscr{B}$:

$$P\{X_{t+s} \in B \mid \mathscr{F}_t\} = P_s(X_t, B) \qquad \text{w.p. 1.} \tag{2.1}$$

We also assume:

$$t > 0, \qquad f \in \mathscr{B}_b \Rightarrow P_t f \in \mathscr{B}_b. \tag{2.2}$$

A positive random variable T is a stopping time in the wide sense if for each $t > 0$, $\{T < t\} \in \mathscr{F}_t$; and is a strict stopping time if one can replace "$<$" by "$\leqq$" above. We shall usually just say "stopping time" instead of "stopping time in the wide sense." The σ-field $\mathscr{F}_{T+}$ is defined as usual to be $\{\Lambda \in \mathscr{F} : \Lambda \cap \{T < t\} \in \mathscr{F}_t, \forall t > 0\}$. Then X is a strong Markov process if, in addition to the above, for each stopping time T, $t > 0$, $B \in \mathscr{B}$:

$$P\{X_{T+t} \in B \mid \mathscr{F}_{T+}\} = P_t(X_T, B) \qquad \text{w.p. 1.} \tag{2.3}$$

An additional condition, not implied by the above, which is frequently useful is

(2.4) for each $x \in E$ there exists a strong Markov process $\{X_t^x, t > 0\}$ with transition functions $P_t(\cdot, \cdot)$ and absolute distributions $P_t(x, \cdot)$, $t > 0$.

This condition turns out to be more of a condition on the transition probabilities than on the process itself, for, although it will not be proved here, if X_t is a right-continuous strong Markov process there is a modification of the transition functions for which both (2.3) and (2.4) hold. We will assume that all strong Markov processes in this section satisfy (2.4).

Let $\mathscr{D}$ be the class of functions $f \in \mathscr{B}_b$ with the property that $t \to P_t f(x)$ is continuous on $(0, \infty)$ for each fixed $x \in E$. This class is non-empty; for instance, if we define $R_\lambda f(x) = \int_0^\infty e^{-\lambda t} P_t f(x)\, dt$ for $f \in \mathscr{B}_b$, then $R_\lambda f \in \mathscr{D}$ for each $\lambda > 0$. Moreover, if X_t is a Hunt process, then $C_b \subset \mathscr{D}$, where C_b is the set of bounded continuous functions on E. Let $\mathscr{D}_R$ be the class of $f \in \mathscr{B}_b$ for which $t \to P_t f(x)$ is right continuous for every $x \in E$, and let $\mathscr{D}_{RL}$ be the class of $f \in \mathscr{B}_b$ for which $t \to P_t f(x)$ is right continuous with left limits for all $x \in E$. (When we say that a function $\{g(t), t > 0\}$ has right or left limits without further qualification we mean the designated limits exist for each $t > 0$.)

Our first theorems concern strong Markov processes. We will state the theorems in this section and prove them in Section 4.

THEOREM 2.1. *Let* $\{X_t, t > 0\}$ *be a right-continuous strong Markov process. Then if* $f \in \mathscr{D}_R$ *and* $t > 0$, $P_t f$ *is fine continuous. Moreover, if in addition* $f \in \mathscr{D}_{RL}$, *then* $s \to P_t f \circ X_s$ *has left limits* w.p. 1, *and if* X_t *is a Hunt process, then* $s \to P_t f \circ X_s$ *is quasi-left continuous as well.*

Since a bounded fine continuous function is of class $\mathscr{D}_R$, this includes the theorem of Girsanov [5] and Šur [10] mentioned in the introduction.

If we start with a function f in $\mathscr{D}$, rather than $\mathscr{D}_R$, we can get a stronger result. Let $\mathscr{C}$ be the space of continuous real-valued functions on $(0, \infty)$ with the topology of uniform convergence on compacts. If $f \in \mathscr{D}$, then for each x, $t \to P_t f(x)$ is an element of $\mathscr{C}$. We will often use the notation $P.f(x)$, or more generally, $P_{.-a}f(x)$, for the functions $t \to P_t f(x)$ and $t \to P_{t-a}f(x)$ respectively.

THEOREM 2.2. *Let* $\{X_t, t > 0\}$ *be a right-continuous strong Markov process and suppose* $f \in \mathscr{D}$. *Then with probability one, the process* $s \to P.f \circ X_s$ *is right continuous and has left limits in the topology of* $\mathscr{C}$. *If, further,* X_t *is a Hunt process, then* $s \to P.f \circ X_s$ *is also quasi-left continuous in the topology of* $\mathscr{C}$.

If f is merely in $\mathscr{B}_b$, one can say little about $s \to P_t f \circ X_s$ for fixed t, but it is possible to make some statements about convergence of the function $P.f \circ X_s$. If ν is a finite measure on $(0, \infty)$, absolutely continuous with respect to Lebesgue measure, then for each x the function $P.f(x)$ is in $L^p((0, \infty), \nu)$, where $1 \leqq p \leqq \infty$ and $f \in \mathscr{B}_b$.

THEOREM 2.3. *Let* $\{X_t, t > 0\}$ *be a right-continuous strong Markov process. Let* ν *be a finite measure on* $(0, \infty)$ *which is absolutely continuous with respect to Lebesgue measure. Then for each* $1 \leqq p < \infty$, $s \to P.f \circ X_s$ *is right continuous and has left limits in* $L^p((0, \infty), \nu)$. *If* X_t *is a Hunt process, then* $s \to P.f \circ X_s$ *is also quasi-left continuous.*

Notice that this theorem offers another proof of the well-known fact that $s \to R_\lambda f \circ X_s$ is right continuous and has left limits w.p. 1, $f \in \mathscr{B}_b$.

If we only require that X_t be Markov but not strongly so, it is clear that the theorems above will no longer be true in general. However, the process will still satisfy a Feller property, albeit a weaker one. If X_t is not strongly Markov the fine topology is no longer relevant, so it is not easy to see how to express these results topologically except in certain special cases, e.g., Markov chains, to be discussed later.

We again find that $P_t f$ has limits along the sample paths, but limits in a weaker sense than above. The proper notion of limit here turns out to be that of essential limit. If F is a metric space with metric d and $g : [0, \infty) \to F$ then g has the essential right (left) limit A at t_0 if for each $\varepsilon > 0$ there exists $\delta > 0$ such that the set of $t \leqq t_0$ ($t \geqq t_0$) for which $|t - t_0| < \delta$ and $d(g(t), A) > \varepsilon$ has Lebesgue measure zero. If g is real-valued, the essential supremum of g over an interval (a, b) is the smallest M for which the Lebesgue measure of the set $\{t \in (a, b) : g(t) > M\}$ is zero.

In the present situation the continuity properties of the sample paths of X_t are not vital. It is enough to assume the process is measurable, that is the function $(t, \omega) \to X_t(\omega)$, $(t, \omega) \in (0, \infty) \times \Omega$, is measurable with respect to $\mathscr{T} \times \mathscr{F}$, where $\mathscr{T}$ is the class of Borel sets on $(0, \infty]$. The following theorem, along with Lemma 4.4, was proved in a slightly different form in [3]. We state it here for completeness.

THEOREM 2.4. *Let* $\{X_t, t > 0\}$ *be a measurable Markov process on E and suppose* $f \in \mathscr{D}_R$. *Then for each* $t > 0$:

(i) $f \in \mathscr{D}_R \Rightarrow s \to P_t f \circ X_s$ *has essential right limits* w.p. 1.
(ii) $f \in \mathscr{D}_{RL} \Rightarrow s \to P_t f \circ X_s$ *has essential right and left limits* w.p. 1.

As before, if f is in the class $\mathscr{D}$, the same continuity properties of $s \to P_t f \circ X_s$ hold, but they hold uniformly for t in compact sets. We need, however, to assume the Chapman–Kolmogorov equations. (This assumption can be relaxed, but only slightly.)

THEOREM 2.5. *Let* $\{X_t, t > 0\}$ *be a measurable Markov process on E. Suppose the transition probabilities of* X_t *satisfy the Chapman–Kolmogorov equations, and suppose* $f \in \mathscr{D}$. *Then for each* $t > 0$, $s \to P.f \circ X_s$ *has essential left and right limits in the topology of* $\mathscr{C}$.

THEOREM 2.6. *Let* $\{X_t, t > 0\}$ *be a measurable Markov process, and let* $f \in \mathscr{B}_b$. *If* ν *is a finite measure on* $(0, \infty)$ *which is absolutely continuous with respect to Lebesgue measure, then* $s \to P.f \circ X_s$ *has essential right and left limits in* $L^p((0, \infty), \nu)$ w.p. 1.

3. Random variables in a Banach space. Let $\mathscr{X}$ be a Banach space with norm $\| \ \|$. Let X be a strong random variable taking values in $\mathscr{X}$ (for definitions and properties of strong random variables and strong conditional expectations, see [9]). If $\mathscr{X}$ is separable, a necessary and sufficient condition that X be a strong random variable is that $\mu(X)$ be a random variable for each bounded linear functional μ, or just for each μ in a countable determining set. (Recall that a countable determining set is a countable set A of linear functionals with the property that $\|x\| = \sup_{\mu \in A} |\mu(x)|$ for all $x \in \mathscr{X}$.)

If $E\{\|X\|\} < \infty$, and $\mathscr{F}_1 \subset \mathscr{F}$ then there exists a strong conditional expectation $E_{\mathscr{X}}\{X \mid \mathscr{F}_1\}$, which is itself a strong random variable. The strong conditional expectation operator commutes with bounded linear functionals. In particular, if X and Y are strong random variables and if for each μ in some countable determining set A we have $\mu(Y) = E\{\mu(X) \mid \mathscr{F}_1\}$ w.p. 1, then $Y = E_{\mathscr{X}}\{X \mid \mathscr{F}_1\}$; for then $\mu(Y) = E\{\mu(X) \mid \mathscr{F}_1\} = \mu(E_{\mathscr{X}}\{X \mid \mathscr{F}_1\})$ for all $\mu \in A$ w.p. 1.

The following theorem is due to Chatterji [1] and generalizes a result of Scalora [9]:

THEOREM. *Let X be a strong random variable with values in* $\mathscr{X}$ *and* $E\{\|X\|\} < \infty$. *Let* $\{\mathscr{F}_n\}$ *be an increasing (decreasing) sequence of σ-fields, and set* $X_n = E_{\mathscr{X}}\{X \mid \mathscr{F}_n\}$. *Then* $\lim_{n \to \infty} X_n$ *exists* w.p. 1, *and equals* $E_{\mathscr{X}}\{X \mid \bigwedge_n \mathscr{F}_n\}$ *if* $\mathscr{F}_n \downarrow$, *or* $E_{\mathscr{X}}\{X \mid \bigvee_n \mathscr{F}_n\}$, *if* $\mathscr{F}_n \uparrow$.

We will find a slight extension of this result useful later, a Banach space version of an observation of Hunt. According to Hunt's result ([6] page 47) if $\{\mathscr{F}_n\}$ is either an increasing or decreasing sequence of σ-fields with limit $\mathscr{F}_\infty$, and $\{X_n\}$ a sequence of real-valued random variables such that for some integrable random variable Y, $|X_n| \leqq Y$ a.e., and $X_n \to X_\infty$ w.p. 1, then $E\{X_n \mid \mathscr{F}_n\} \to E\{X_\infty \mid \mathscr{F}\}$ w.p. 1.

PROPOSITION 3.1. *Let $\mathscr{X}$ be a Banach space and let $\{X_n\}$ be a sequence of strong random variables with values in $\mathscr{X}$, such that $X_n \to X_\infty$ strongly* w.p. 1. *Suppose there exists a random variable Y such that $\|X_n\| \leqq Y$, all n and $E\{Y\} < \infty$. Then if $\{\mathscr{F}_n\}$ is an increasing or decreasing sequence of σ-fields with limit $\mathscr{F}_\infty$, $E_{\mathscr{X}}\{X_n \mid \mathscr{F}_n\} \to E_{\mathscr{X}}\{X_\infty \mid \mathscr{F}_\infty\}$ strongly* w.p. 1.

PROOF. $E_{\mathscr{X}}\{X_n \mid \mathscr{F}_n\} = E_{\mathscr{X}}\{X_\infty \mid \mathscr{F}_n\} + E_{\mathscr{X}}\{X_n - X_\infty \mid \mathscr{F}_n\}$. The first term converges to $E_{\mathscr{X}}\{X \mid \mathscr{F}_\infty\}$ by the Chatterji–Scalora theorem. The second term is dominated in norm by $E\{\|X_n - X_\infty\| \mid \mathscr{F}_n\}$. But this goes to zero w.p. 1. by Hunt's lemma quoted above. □

For our use we need only consider Banach spaces whose elements are (equivalence classes of) functions on $[0, \infty)$ or on some subset B of $[0, \infty)$, with Banach space addition and scalar multiplication corresponding to pointwise addition and scalar multiplication of functions. Such a Banach space will be called *type* α if

(i) For some $M > 0, f \in \mathscr{X} \Rightarrow \|f\| \leqq M \sup_{x \in B} |f(x)|$.

(ii) There exists a countable determining set $\{\hat{\mu}_i\}$ of bounded linear functionals such that for each i, there exists a finite measure μ_i on B for which $\hat{\mu}_i(f) = \int_B f(x)\mu_i(dx), f \in \mathscr{X}$.

It should be noted that condition (i) is empty unless f is bounded. However, the applications we have in mind concern bounded functions, and this condition is simply a convenient method of assuring us that if $\{X_t, t \in B\}$ is a bounded stochastic process such that $X. \in \mathscr{X}$, then $E_{\mathscr{X}}\{\|X\|\} < \infty$. In our applications, $\mathscr{X}$ will either be an L^p space, the real line, or a space of continuous functions on a compact set, which are clearly of type α.

PROPOSITION 3.2. *If $\mathscr{X}$ is a separable type α space and $\{Y_t, t \in B\}$ is a measurable stochastic process such that for* a.e. ω, *the function $Y.(\omega)$ is an element of $\mathscr{X}$, then $Y.$ is a strong random variable.*

PROOF. Let $\{\hat{\mu}_i\}$ be the determining set, and μ_i the corresponding measures. Then by Fubini's theorem, the functions $\hat{\mu}_i(Y.) = \int_B Y_t(\omega)\mu_i(dt)$ are random variables. □

PROPOSITION 3.3. *If $\mathscr{X}$ is a separable type α space and $\{Y_t, t \in B\}$ and $\{X_t, t \in B\}$ are measurable processes such that* w.p. 1 *$X.$ and $Y.$ are in $\mathscr{X}$, and $\mathscr{F}_1$ is a σ-field, then a sufficient condition that $Y. = E_{\mathscr{X}}\{X. \mid \mathscr{F}_1\}$ is that $Y_t = E\{X_t \mid \mathscr{F}_1\}$* w.p. 1 *for each $t \in B$.*

PROOF. Again by Fubini's theorem, for each $\hat{\mu}_i$ in the countable determining set:

$$\hat{\mu}_i(Y.) = \int_B Y_t \mu_i(dt) \tag{3.1}$$

and $\hat{\mu}_i(E_{\mathscr{X}}\{X.\,|\,\mathscr{F}_1\}) = E\{\hat{\mu}_i(X.)\,|\,\mathscr{F}_1\} = E\{\int_B X_t \mu_i(dt)\,|\,\mathscr{F}_1\}$. If we now take a measurable version of $E\{X_t\,|\,\mathscr{F}_1\}$ this is equal to $\int_B E\{X_t\,|\,\mathscr{F}_1\}\mu_i(dt) = \hat{\mu}_i(Y.)$ w.p. 1. □

4. Proofs of Theorems 2.1–2.6. Let $\{X_t, t > 0\}$ be a temporally homogeneous Markov process. The theorems in Section 2 can be reduced to questions concerning strongly Markov sub-processes $\{X_t, t \in \Gamma\}$ of X of two kinds:

Case A. Γ is a countable dense set for which $\{X_t, t \in \Gamma\}$ is strongly Markov.

Case B. $\Gamma = (0, \infty)$ and $\{X_t, t \in \Gamma\}$ is right continuous and strongly Markov.

We begin with a substitution lemma which is not new but which is given here in the form we need. Its only surprising feature is that it needs proof.

LEMMA 4.1. *Let T be a stopping time taking values in Γ and let $h \geqq 0$ be an $\mathscr{F}_{T+}$-measurable random variable which is countably-valued in Case A but arbitrary in Case B. Then for any $f \in \mathscr{B}_b$:*

$$E\{f \circ X_{T+h}\,|\,\mathscr{F}_T\} = P_{h(\omega)} f \circ X_{T(\omega)}(\omega) \quad \text{w.p. } 1, \tag{4.1}$$

where the right-hand side is $P_t f \circ X_T$ evaluated at $t = h(\omega)$.

PROOF. If h has its values in a countable set $\{t_j\}$, then (4.1) follows immediately from:

$$E\{f \circ X_{T+h}\,|\,\mathscr{F}_T\} = \textstyle\sum_{j=1}^{\infty} I_{\{h=t_j\}} E\{f \circ X_{T+t_j}\,|\,\mathscr{F}_T\}, \tag{4.2}$$

and the strong Markov property applied to T. This proves Case A, and Case B follows upon taking a sequence $\{h_n\}$ of countably-valued $\mathscr{F}_{T+}$-measurable random variables decreasing to h, and noting the right continuity of $t \to f \circ X_t$ and $t \to P_t f \circ X_T$ for each of a countable family of f dense in the space C_K of continuous functions of compact support. □

Let $\mathscr{X}$ be a separable type α Banach space and fix $x \in E$. For certain $f \in \mathscr{B}_b$, $P.f(x)$ may be an element of $\mathscr{X}$. This is true for all $f \in \mathscr{B}_b$, for instance, if $\mathscr{X}$ is an L^p space of a finite measure. If we apply P_s to $P.f$ we may again get an element of $\mathscr{X}$, which we denote by $P_s P.f(x)$. By the Chapman–Kolmogorov equations (when they are valid), $P_s P.f(x) = P_{s+}.f(x)$, so P_s is a restriction of the translation operator. In general, any continuity properties in s that this operator might have will depend on f. (In this section, the only Banach space topology to interest us is the norm topology, so that all statements about limits in $\mathscr{X}$ refer to limits in the Banach space norm.)

Accordingly, we define classes $\mathscr{D}(\mathscr{X})$, $\mathscr{D}_R(\mathscr{X})$ and $\mathscr{D}_{RL}(\mathscr{X})$ as follows: $\mathscr{D}_R(\mathscr{X})$ is the set of $f \in \mathscr{B}_b$ such that for each $x \in E$ and $t > 0$, $P.f(x)$ and $P_t P.f(x)$ are elements of $\mathscr{X}$ and for which $t \to P_t P.f(x)$ is right continuous. $\mathscr{D}_{RL}(\mathscr{X})$ and $\mathscr{D}(\mathscr{X})$ are the sets of $f \in \mathscr{D}_R(\mathscr{X})$ for which $t \to P_t P.f(x)$ has left limits and for which $t \to P_t P.f(x)$ is continuous, respectively, for each $x \in E$. Clearly $\mathscr{D}(\mathscr{X}) \subset \mathscr{D}_{RL}(\mathscr{X}) \subset \mathscr{D}_R(\mathscr{X})$.

The theorems of Section 2 can all be proved from the following theorem by choosing the Banach space $\mathscr{X}$ conveniently.

THEOREM 4.2. *Let $\varepsilon > 0$ and let $\mathscr{X}$ be a separable type α Banach space of functions on some subset B of* $[\varepsilon, \infty)$. *Let X be a Markov process and $\Gamma \subset (0, \infty)$ a set for which either Case A or Case B obtains. We have*:

(a) *if $f \in \mathscr{D}_R(\mathscr{X})$, then* w.p. 1, $\lim_{s \downarrow t, s \in \Gamma} P.f \circ X_s$ *exists for all $t > 0$; in Case B this limit is $P.f \circ X_t$.*

(b) *If $f \in \mathscr{D}_{RL}(\mathscr{X})$, then* w.p. 1, $\lim_{s \uparrow t, s \in \Gamma} P.f \circ X_s$ *exists for all $t > 0$.*

(c) *If $f \in \mathscr{D}(\mathscr{X})$ and X_t is a Hunt process, then $s \to P.f \circ X_s$ is quasi-left continuous.*

REMARK. The limits in the above theorem are all limits in the norm topology of $\mathscr{X}$.

Before proceeding to the proof, let us remark that for any $f \in \mathscr{D}_R(\mathscr{X})$ and $\xi \in \mathscr{X}$, the function $x \to \|P.f(x) - \xi\|$ is Borel measurable. This is easily seen since $\int_B \mu_i(dt)(P_t f(x) - \xi(t))$ is Borel measurable in x and $\|P.f(x) - \xi\|$ is just the supremum of these over all μ_i in the countable determining set of functionals on $\mathscr{X}$.

PROOF. The crucial step in the proof is to show $s \to P.f \circ X_s$ has a right limit at an arbitrary stopping time T. Once this is established, the existence of limits at all t follows by a modification of a transfinite induction argument of Doob.

Let $f \in \mathscr{D}_R(\mathscr{X})$ and let T be a stopping time. We claim that $\lim_{t \downarrow 0} P.f \circ X_{T+t}$ exists in $\mathscr{X}$ with probability one. To show this, it is sufficient to verify the existence of the limit on the set $\{t_0 \leqq T < t_0 + \frac{1}{3}\varepsilon\}$ for an arbitrary t_0. Thus, set

$$T' = T \quad \text{if} \quad t_0 \leqq T < t_0 + \tfrac{1}{3}\varepsilon,$$
$$= \infty \quad \text{otherwise.}$$

Fix $\delta > 0$ and choose a sequence $T_0, T_2, \cdots, T_{2k}, \cdots$ of Γ-valued stopping times decreasing to T' with $T_{2k} \leqq t_0 + \frac{1}{2}\varepsilon$ for all k. If X_t is right continuous or if Γ is countable, $\{X_t, t \in \Gamma\}$ is automatically well-measurable (see [7] page 156) for the definition and properties of well-measurable processes) so that for each n the set

$$A_n = \{(t, \omega) \in \Gamma \times \Omega : \| P.f \circ X_t - P.f \circ X_{T_{2n}} \| > \tfrac{1}{3}\delta, t \in (T_{2n}, T_{2n-2}]\}$$

is well-measurable. Let ΠA_n be the projection of A_n on Ω. By Theorem T21 of ([7] page 162), there is a stopping time S with values in $\Gamma \cap (T_{2n}, T_{2n-2}]$ for which $P\{(S, \omega) \in A_n\} \geqq P\{\Pi A_n\} - 2^{-n}$. Set $T_{2n-1} = S$.

Fix $t \in B$ (which implies $t \geqq \varepsilon$, where B and ε are given in the statement of the theorem) and choose a constant V in $\Gamma \cap (t_0 + \frac{1}{2}\varepsilon, t_0 + \varepsilon)$. In particular, T_n is less than V whenever T_n is finite, and, as T_n and V take values in Γ, by Lemma 4.1

$$P_t f \circ X_{T_n} = E\{f \circ X_{T_n + t} \mid \mathscr{F}_{T_n}\} = E\{P_{t+T_n-V} f \circ X_V \mid \mathscr{F}_{T_n}\}. \tag{4.3}$$

This being true for all $t \in B$ implies—by Proposition 3.3—

$$P.f \circ X_{T_n} = E_{\mathscr{X}}\{P._{+T_n-V} f \circ X_V \mid \mathscr{F}_{T_n+}\}. \tag{4.4}$$

Now $f \in \mathscr{D}_R(\mathscr{X})$ so as $n \to \infty$, $T_n \downarrow T'$, $\mathscr{F}_{T_n+} \downarrow \mathscr{F}_{T'+}$ and $P._{+T_n-V} f \circ X_V \to P._{+T'-V} \circ X_V$ boundedly. By Proposition 3.1 the right-hand side converges to $E_{\mathscr{X}}\{P._{+T'-V} f \circ X_V \mid \mathscr{F}_{T'+}\}$. If T takes values in Γ, we can apply Lemma 4.1 to see this is $P.f\circ X_{T'}$. But now, noting the choice of the T_n and applying the Borel–Cantelli lemma:

$$P\{\limsup_{r,s\downarrow T}\|P.f\circ X_r - P.f\circ X_s\| > \delta\}$$
$$\leqq P\{\limsup_{n,m\to\infty}\|P.f\circ X_{Tn} - P.f\circ X_{T_m}\| > \tfrac{1}{3}\delta\} = 0$$

since the sequence $P.f\circ X_{T_n}$ converges.

It follows upon letting δ go through a sequence tending to zero that $s\to P.f\circ X_s$ has a right limit at T' and hence T, and is right continuous there if T' takes values in Γ. From this, existence of right limits in $(0, \infty)$ may be shown as follows. Define $S_0 = 0$,

$$S_1 = \inf\{t > S_0 : \|P.f\circ X_t - \lim_{s\downarrow S_0} P.f\circ X_s\| > \delta\},$$

and by induction, $S_{n+1} = S_n + S_1\circ\theta_{S_n}$, where θ is the translation operator. If $\lim_{n\to\infty} S_n < \infty$, define $S_\omega = \lim_{n\to\infty} S_n$, $S_{\omega+1} = S_\omega + S_1\circ\theta_{S_\omega}$, and so on through the countable ordinals. As $P.f\circ X_s$ has a right limit at each S_β, $S_{\beta+1} > S_\beta$ a.s. on $\{S_\beta < \infty\}$. Thus, for some countable ordinal γ, $P\{S_\gamma = \infty\} = 1$. For ω not in some exceptional null set, for $t \geqq 0$ there is a $\beta \ni t\in [S_\beta(\omega), S_{\beta+1}(\omega))$, hence the right-hand oscillation of $P.f\circ X_s$ at t is less than δ. This being true for a sequence of δ decreasing to zero implies existence of right limits at all t and right continuity at all $t\in\Gamma$.

If now $f\in\mathscr{D}_{RL}(\mathscr{X})$, let $\{S_n\}$ be the sequence of stopping times defined above. Let $S = \lim_{n\to\infty} S_n$. Using Meyer's theorem again we can construct a sequence $\{S_n'\}$ of Γ-valued stopping times increasing to S so that a.e. on

$$\{S < \infty, \limsup_{n\to\infty}\|P.f\circ X_{S_{n'+1}'} - P.f\circ X_{S_n'}\| > \tfrac{1}{4}\delta.$$

As before, it is enough to show existence of the limit on the set $\{t_0 - \frac{1}{3}\varepsilon \leqq S < t_0\}$ for an arbitrary t_0 in Γ, so we may assume without loss of generality that $t_0 - \frac{1}{3}\varepsilon \leqq S < t_0$ everywhere, and that $S_n' > t_0 - \frac{1}{2}\varepsilon$. The proof then proceeds almost as above. We have, for $t\in B$ (hence $t \geqq \varepsilon$)

$$P_t f\circ X_{S_n'} = E\{P_{t+S_n'-t_0} f\circ X_{t_0} \mid \mathscr{F}_{S_n'+}\}. \tag{4.5}$$

This being true w.p. 1 for each $t\in B$ implies

$$P.f\circ X_{S_n'} = E_{\mathscr{X}}\{P._{+S_n'-t_0} f\circ X_{t_0} \mid \mathscr{F}_{S_n'+}\}. \tag{4.6}$$

But this converges almost surely by Proposition 3.1 above. By choice of the S_n', S must be infinite w.p. 1. But now this must hold simultaneously for a sequence of δ tending to zero, which implies that w.p. 1 $s\to P.f\circ X_s$ has no oscillatory discontinuities from the left.

Finally, if $f\in\mathscr{D}(\mathscr{X})$ and X is a Hunt process, the argument we have just given shows that if $S_n\uparrow S < \infty$ is a sequence of stopping times—again assume for the minute that $t_0 - \frac{1}{3}\varepsilon \leqq S < t_0$ everywhere—that

$$\lim_{n\to\infty} P.f\circ X_{S_n} = E_{\mathscr{X}}\{P._{+S-t_0} f\circ S_{t_0} \mid \bigvee_n \mathscr{F}_{S_n}\};$$

as $\bigvee_n \mathscr{F}_{S_n} = \mathscr{F}_{S+}$ for a Hunt process [7], this conditional expectation is equal to $P.f \circ X_S$ w.p. 1. □

Theorems 2.1, 2.2 and 2.3 are now readily established. Theorem 2.1 follows directly from Theorem 4.2 upon taking $\mathscr{X} = (-\infty, \infty)$ and noting

$$\mathscr{D} = \mathscr{D}(-\infty, \infty), \qquad \mathscr{D}_R = \mathscr{D}_R(-\infty, \infty), \qquad \text{and} \quad \mathscr{D}_{RL} = \mathscr{D}_{RL}(-\infty, \infty).$$

PROOF OF THEOREM 2.2. If $f \in \mathscr{D}$, then $t \to P_t f(x)$ is continuous for all $x \in E$, so for each $n \geqq 1$, $P.f(x) \in C[1/n, n]$, which is the space of continuous functions on $[1/n, n]$ with sup-norm. As $t \to P_t f(x)$ is uniformly continuous on $[\varepsilon, 1/\varepsilon]$ and $P_s P.f(x) = P._{+s} f(x)$, the mapping $s \to P_s P.f(x)$ is continuous for $t > 0$. Thus $f \in \mathscr{D}(C[1/n, n])$. If we apply Theorem 4.2, we see that w.p. 1 $s \to P.f \circ X_s$ is right continuous and has left limits in $C[1/n, n]$, $n = 1, 2, \cdots$ which implies that $s \to P.f \circ X_s$ is right continuous and has left limits in $\mathscr{C}$. If X_t is a Hunt process, then Theorem 4.2 implies quasi-left continuity of $s \to P.f \circ X_s$ in all $C[1/n, n]$, and hence in $\mathscr{C}$. □

PROOF OF THEOREM 2.3. Let ν be a finite measure on $(0, \infty)$ which is dominated by Lebesgue measure, and consider $L^p((0, \infty), \nu)$. We cannot apply Theorem 4.2 directly to this space, since the theorem applies only to spaces of functions on some subset of $[\varepsilon, \infty)$. However, we can apply it to the spaces $\mathscr{X}_n = L^p\{[1/(n+1), 1/n), \nu\}$, $n = 0, 1, 2, \cdots$ (where $\mathscr{X}_0 = L^p\{[1, \infty), \nu\}$). If $g_1, g_2, \cdots$ is a sequence of uniformly bounded functions, it is easily verified that $\{g_n\}$ converges in L^p iff it converges in each $\mathscr{X}_n$.

Observe that if f is a bounded Borel measurable function on E, $P.f \in L^p((0, \infty), \nu)$, $p \geqq 1$, and the map $t \to P_t P.f(x) = P._{+t} f(x)$ is just the translation map. As ν is dominated by Lebesgue measure, this map is continuous, so that

$$\mathscr{B}_b \subset \mathscr{D}(L^p((0, \infty), \nu)),$$

and therefore $\mathscr{B}_b \subset \mathscr{D}(\mathscr{X}_n)$, $n = 0, 1, 2, \cdots$. By Theorem 4.2, $s \to P.f \circ X_s$ is right continuous and has left limits in each $\mathscr{X}_n$, and if X_t is a Hunt process, is even quasi-left continuous in each $\mathscr{X}_n$. As $|P_t f(x)| \leqq \sup_{y \in E} |f(y)|$ for all $x \in E$, the same conclusions must hold in $L^p((0, \infty), \nu)$. □

Theorems 2.4, 2.5 and 2.6 are quite close to Theorems 2.1, 2.2 and 2.3 except that they involve essential limits rather than limits and ordinary Markov processes rather than strong Markov processes. Questions of the existence of essential limits can be reduced to questions of the existence of ordinary limits along countable sets, making it necessary to look only at processes with denumerable parameter sets. We begin with a lemma to the effect that increasing σ-fields, like increasing functions, have at most countably many discontinuities. It could be easily avoided but as it is so simple there seems no compelling reason to do so.

LEMMA 4.2. *Let $(M, \mathscr{G}, \mu)$ be a measure space, where μ is finite and $\mathscr{G}$ is the completion of a separable σ-field. Let $\{\mathscr{G}_t, -\infty < t < \infty\}$ be a set of complete sub-σ-fields of $\mathscr{G} \ni s < t \Rightarrow \mathscr{G}_S \subset \mathscr{G}_t$. Then for t not in some countable set, $\bigwedge_{s>t} \mathscr{G}_s = \mathscr{G}_t = \bigvee_{s<t} \mathscr{G}_s$.*

PROOF. $L^1(\mathscr{G}, \mu)$ is separable, so let $\{f_i\}$ be a countable dense subset. For each i, let $Y_t^i = E\{f_i \mid \mathscr{G}_t\}$—we take a separable version—and note that for each t, $\{Y_t^i\}$ is dense in $L^1(\mathscr{G}_t, \mu)$. This follows since for $g \in L^1(\mathscr{G}_t, \mu)$, $E\{|g - Y_t^i|\} \leqq E\{|g - f_i|\}$, and the fact that f_i is dense. But now, $\{Y_t^i, -\infty < t < \infty\}$ is a separable martingale and so has no oscillatory discontinuities w.p. 1. This implies that it must be continuous at all but countably many t. Thus, for t not in some countable set A, $P\{Y_t^i$ continuous at t, all $i\} = 1$. If $t \notin A$, $\lim_{s \uparrow t} Y_s^i = Y_t^i$ is measurable $\bigvee_{s<t} \mathscr{G}_s$ and $\lim_{s \downarrow t} Y_s^i = Y_t^i$ is measurable $\bigwedge_{s>t} \mathscr{G}_s$ for all i. By density of Y_t^i, this proves the lemma. □

Except for the continuity condition on the fields, which is trivial in view of Lemma 4.2, the following lemma was proved in [3].

LEMMA 4.3. *Let X_t be a measurable Markov process, and let g be a Borel measurable function on E. Then there is a countable parameter set Γ with the property that for* a.e. ω *and each interval* $[\alpha, \beta]$ *in* $[0, \infty)$:

$$\operatorname{ess\,sup}_{\alpha \leqq t \leqq \beta} g \circ X_t = \sup_{\alpha \leqq t \leqq \beta,\, t \in \Gamma} g \circ X_t$$

$$\operatorname{ess\,inf}_{\alpha \leqq t \leqq \beta} g \circ X_t = \inf_{\alpha \leqq t \leqq \beta,\, t \in \Gamma} g \circ X_t$$

and such that $t \in \Gamma \Rightarrow \mathscr{F}_t = \bigwedge_{s>t} \mathscr{F}_s$.

As is well known, if $\Gamma \subset (0, \infty)$ is a countable set such that $\bigcap_{s>t} \mathscr{F}_s = \mathscr{F}_t$ for each $t \in \Gamma$, $\{X_t, t \in \Gamma\}$ is strongly Markov. With Lemma 4.3, Theorems 2.4–2.6 follow from Theorem 4.2 as in the previous case. Theorem 2.4 follows directly upon taking $\mathscr{X}$ in Theorem 4.2 to be $(-\infty, \infty)$.

PROOF OF THEOREM 2.5. Let $f \in \mathscr{D}$. For each $x \in E$, $P.f(x)$ and—since the Chapman–Kolmogorov equations are assumed—$P_t P.f(x)$ are continuous on $(0, \infty)$ and $t \to P_t P.f(x)$ is continuous in $C[1/n, n]$ for each $n \geqq 1$. Now for each rational r, by Lemma 4.4 we can choose a parameter set Γ_r such that for each $\alpha < \beta$

$$\operatorname{ess\,sup}_{\alpha \leqq t \leqq \beta} P_r f \circ X_t = \sup_{\alpha \leqq t \leqq \beta,\, t \in \Gamma_r} P_r f \circ X_t$$

and

$$\operatorname{ess\,inf}_{\alpha \leqq t \leqq \beta} P_r f \circ X_t = \inf_{\alpha \leqq t \leqq \beta,\, t \in \Gamma_r} P_r f \circ X_t,$$

such that the fields $\mathscr{F}_t$ are continuous at each $t \in \Gamma_r$. Take $\Gamma = \bigcup_{r \text{ rational}} \Gamma_r$. For each $n > 1$ and $\xi \in C[1/n, n]$, $\|\xi\| = \sup\{\xi(r), r$ rational in $(1/n, n)\}$. Thus with probability one, for each n and $t_0 \geqq 0$

$$\lim_{t \in \Gamma, t \to t_0} P.f \circ X_t \quad \text{exists} \quad \Rightarrow \operatorname{ess\,lim}_{t \to t_0} P.f \circ X_t \quad \text{exists}.$$

Because of the translation property, the same holds for the essential limit of $P_{\cdot + s} f \circ X_t$. Thus Theorem 2.5 will be proved if we verify the corresponding limits exist along Γ. But $\{X_t, t \in \Gamma\}$ is strongly Markov and $f \in \mathscr{D}(C[1/n, n])$ so that by Theorem 4.2, $s \to P.f \circ X_s$ has right and left limits along Γ in $C[1/n, 1/n]$. This is true simultaneously for $n = 2, 3, \cdots$, which implies existence of essential right and left limits in $\mathscr{C}$ w.p. 1. □

PROOF OF THEOREM 2.6. Let $f \in \mathscr{B}_b$. Then $P.f(x) \in L^p((0, \infty), \nu)$. For each s, t, the set $A_{s,t}$ of x for which $P_{s+t}f(x) = P_sP_tf(x)$ has the property that $P\{X_u \in A_{st}\} = 1$ for all $u > 0$, and hence by Fubini's theorem, the set B_s for which

$$P_sP_tf(x) = P_{s+t}f(x) \quad \text{for} \quad \text{a.e. } (\nu) \qquad t > 0$$

has the property that $P\{X_u \in B_s\} = 1$, all $u > 0$. By modifying the transition probabilities off B_s if necessary, we can and will assume $B_s = E$, $\forall s > 0$, so that, considered as elements of $L^p((0, \infty), \nu)$, $P_sP.f(x) = P._{+s}f(x)$. But as ν is dominated by Lebesgue measure and $P.f(x)$ is bounded, the translation operator $s \to P._{+s}f(x)$ is continuous in $L^p((0, \infty), \nu)$, $1 \leqq p < \infty$ (but *not* for $p = \infty$!). Thus $f \in \mathscr{D}(L^p[(0, \infty), \nu])$.

As in the proof of Theorem 2.3, we consider the spaces $\mathscr{X}_n = L^p([1/(n+1), 1/n), \nu)$, $n = 0, 1, 2, \cdots$.

Fix n for the moment. Certainly $f \in \mathscr{D}(\chi_n)$. Let $\{\hat{\mu}_i\}$ be a countable determining set of functionals on $\mathscr{X}_n$. For each i, apply Lemma 4.4 to find a countable set $\Gamma_i \subset (0, \infty)$ with the property that

$$\operatorname{ess\,sup}_{\alpha \leqq t \leqq \beta} \hat{\mu}_i(P.f \circ X_t) = \sup_{\alpha \leqq t \leqq \beta,\, t \in \Gamma} \hat{\mu}_i(P.f \circ X_t)$$
$$\operatorname{ess\,inf}_{\alpha \leqq t \leqq \beta} \hat{\mu}_i(P.f \circ X_t) = \inf_{\alpha \leqq t \leqq \beta,\, t \in \Gamma} \hat{\mu}_i(P.f \circ X_t)$$

for all $\alpha < \beta$ and such that $\mathscr{F}_t$ is continuous at each $t \in \Gamma_i$. Let $\Gamma = \bigcup_i \Gamma_i$, and note that existence of $\mathscr{X}_n$-limits along Γ implies existence of essential $\mathscr{X}_n$-limits. But by Theorem 4.2, $s \to P.f \circ X_s$ has right and left $\mathscr{X}_n$-limits along Γ w.p. 1. This is true simultaneously for all n; hence, as $P_tf(x)$ is bounded by $\sup_{x \in E} |f(x)| < \infty$, we have right and left L^p-limits w.p. 1 along Γ, which implies the theorem. □

5. Markov chains. For a strong Markov process some of the results we have been discussing can be stated in topological terms; for example, if $f \in \mathscr{D}_R(E)$ then P_tf is fine continuous. This elegant statement of the Feller property is generally not valid without the strong Markov property so if X is merely Markov, the translation of path-continuity of $P_tf \circ X_s$ into continuity of $x \to P_tf(x)$ may be difficult. Markov chains, however, give an example where such translation is readily accomplished.

Let $\{X_t, t \geqq 0\}$ be a homogeneous Markov chain with state space $E = \{0, 1, 2, \cdots\}$ and transition matrix $(p_{ij}(t))$. We assume the matrix is standard, that is $\sum_j p_{ij}(t) = 1$ and $p_{ii}(t) \to 1$ as $t \to 0$ for all i. Such a process is not necessarily strongly Markov, and indeed its paths may be totally discontinuous. To bring notation into line with that of the preceding sections, let us define the operator P_t by

$$P_tf(i) = \textstyle\sum_j p_{ij}(t)f(j).$$

Then $p_{ij}(t) = P_tI_j(i)$, I_j being the indicator function of j.

Chung [2] has defined a "fine" topology on E: a base of neighborhoods at a point i is given by sets of the form $\{k : p_{ki}(\delta) > 1 - \delta'\}$ as δ, δ' go to zero. (It should be noted that this is *not* necessarily analogous to the fine topologies we have defined for strong Markov processes.) If we write

$$P_tI_j(k) = p_{ki}(\delta)P_{t-\delta}I_j(i) + \textstyle\sum_{l \neq i} p_{kl}(\delta)P_{t-\delta}I_j(l)$$

and let $k \to i$ and $\delta \to 0$, remembering that $t \to p_{ij}(t)$ is continuous on $t \geqq 0$, we see that $P_t I_j$ is continuous in this topology; in fact this topology is the coarsest topology making all such functions continuous. It follows from the obvious uniform convergence argument that if f is bounded on E, $P_t f$ is continuous in this topology. This conclusion is rather trivial—the topology was expressly constructed to make it true. More interesting is the fact, not immediately obvious from the above remarks, that Chung's fine topology is an intrinsic topology for the process in a certain sense.

There is a version of X_t taking values in $E \cup \{\infty\}$ such that $X_t = \lim \inf_{s \downarrow t} X_s$ for each s. Things are slightly complicated by the fact that X_t can take on the value ∞. However, it does so at each fixed t with zero probability, and the Lebesgue measure of $S_\infty(\omega) = \{t : X_t(\omega) = \infty\}$ is almost surely zero.

For $i \in E$ let $S_i(\omega) = \{t : X_t(\omega) = i\}$ and $S_E(\omega) = \bigcup_{i \in E} S_i(\omega)$. Then the restriction $X_t \mid_{t \in S_E}$ is almost surely right continuous.

This result is due to Chung (see [2] page 190) but it can be readily shown from Theorem 2.4. It is necessary to show that $s \to P_t I_j \circ X_s \mid_{s \in S_E}$ is right continuous, and this is quite easy given the fact [2], that for each i the set $S_i(\omega)$ has the property that $t \in S_i(\omega) \Rightarrow m([t, t+\delta] \cap S_i(\omega)) > 0$, m being Lebesgue measure. Thus, if $s \in S_E$, we have:

$$\text{if} \quad \operatorname{ess\,lim}_{u \downarrow s} P_t I_j \circ X_u = P_t I_j \circ X_s, \qquad \text{then} \quad \lim_{u \downarrow s,\, s \in S_E} P_t I_j \circ X_u = P_t I_j \circ X_s.$$

But $t \to P_t I_j(i)$ is continuous, so for $t > 0$ Theorem 2.4 implies $s \to P_t I_j \circ X_s$ has essential right limits at all $s \geqq 0$. In particular, if $s \in S_E(\omega)$, then $s \in S_i(\omega)$ for some i; as the Lebesgue measure of $S_i(\omega) \cap (s, s+\delta)$ is strictly positive for each $\delta > 0$, $P_t I_j(i) = P_t I_j \circ X_s$ must be an essential limit point of $P_t I_j \circ X_u$ as $u \downarrow s$, and hence the essential limit. This implies essential right continuity at each point of S_E, and, by our above remarks, right continuity of $P_t I_j \circ X_s \mid_{S_E}$.

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