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F-SQUARE AND ORTHOGONAL F-SQUARES DESIGN: A GENERALIZATION OF LATIN SQUARE AND ORTHOGONAL LATIN SQUARES DESIGN[1]

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0. Summary. In this paper we are concerned with a generalization of the concept of Latin squares and orthogonality of Latin squares. The condition that every element appears once in each row and each column is replaced by the condition that it appears the same number of times in each row and each column. We call such squares F-squares. The usefulness of introducing and investigating the properties of F-squares could be justified in two directions. F-squares have meaningful applications in laying out experimental designs as exhibited previously by some authors. Their properties prove to be a useful tool in the studies of existence of orthogonal Latin squares and other combinatorial problems.

1. Introduction. The concept of F-squares is not entirely new. It has been considered directly or indirectly by Finney [3], [4], [5], Federer [2], Freeman [6], and Addelman [1]. However, these authors considered the F-squares as a by-product of their general interest in experimental designs and were not concerned with the theory of F-squares *per se*. The first author of the present paper was inspired by the examples of F-squares brought forward by the previous authors. Having in mind the usefulness of the F-squares for research in the theory of the designs and its application in practical problems, he defined the concept of F-squares and mutually orthogonal F-squares and obtained some results concerning them [7]. The purpose of this paper is to develop further the theory of F-squares and bring it to closer attention of mathematical statisticians. We plan to further the research in this area and present more results for publication shortly.

It may be worthwhile to point out at this stage that the theorems proved thus far aim at classifying some types of Latin squares making use of their relation to F-squares. The concept of orthogonality of Latin squares is generalized to F-squares and some of its implications to the problem of existence of orthogonal Latin squares are investigated. It is pointed out here that using the concept of F-squares one can distinguish between two types of Latin squares both mateless in respect to orthogonality but different in respect to their use in a broader sense of experimental designs.

2. F-Squares.

DEFINITION 2.1. Let $A = [a_{ij}]$ be an $n \times n$ matrix and let $\Sigma = (c_1, c_2, \cdots, c_m)$ be the ordered set of distinct elements of A. In addition, suppose that for each

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$k = 1, 2, \cdots, m$, c_k appears precisely λ_k times ($\lambda_k \geqq 1$) in each row and in each column of A. Then, A will be called a *frequency square* or, more concisely, an F-square on Σ of order n and frequency vector $(\lambda_1, \lambda_2, \cdots, \lambda_m)$.

We now introduce some notation. A matrix A will be said to be an $F(n; \lambda_1, \lambda_2, \cdots, \lambda_m)$ square if A is an F-square of order n and frequency vector $(\lambda_1, \lambda_2, \cdots, \lambda_m)$. This notation may be contracted by the use of powers to denote successive equal values of λ's. Thus $F(n; \lambda^m)$ represents $F(n; \lambda, \lambda, \cdots, \lambda)$ while $F(n; \lambda_1{}^2, \lambda_3, \lambda_4{}^2, \lambda_6, \cdots, \lambda_m)$ represents $F(n; \lambda_1, \lambda_1, \lambda_3, \lambda_4, \lambda_4, \lambda_6, \cdots, \lambda_m)$. In particular, in an $F(n; \lambda^m)$ square, m is determined uniquely by n and λ; hence we will represent such a square simply by $F(n; \lambda)$.

EXAMPLE 2.1. Let $\Sigma = \{1, 2, 3\}$ then

$$\begin{matrix} 1 & 2 & 3 & 3 & 2 & 1 \\ 2 & 3 & 1 & 1 & 3 & 2 \\ 3 & 1 & 2 & 2 & 1 & 3 \\ 3 & 1 & 2 & 2 & 1 & 3 \\ 2 & 3 & 1 & 1 & 3 & 2 \\ 1 & 2 & 3 & 3 & 2 & 1 \end{matrix}$$

is an $F(6; 2)$ square on Σ.

EXAMPLE 2.2. Let $\Sigma = \{1, 2, 3, 4\}$ then

$$\begin{matrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 1 & 2 \\ 4 & 3 & 2 & 1 \end{matrix}$$

is an $F(4; 1)$ square on Σ, while

$$\begin{matrix} 1 & 2 & 3 & 4 & 1 \\ 1 & 1 & 2 & 3 & 4 \\ 4 & 1 & 1 & 2 & 3 \\ 3 & 4 & 1 & 1 & 2 \\ 2 & 3 & 4 & 1 & 1 \end{matrix}$$

is an $F(5; 2, 1^3)$ on Σ.

Note that $F(n; 1)$ square is simply a Latin square of order n and thus exists for all n. Consequently $F(n; \lambda_1, \lambda_2, \cdots, \lambda_m)$ square on $\Sigma = (a_1, \cdots, a_m)$ exists if and only if $n = \sum_{i=1}^m \lambda_i$.

The necessity of this condition is obvious by the definition of an F-square. Sufficiency can be proved as follows: Construct an $F(n; 1)$ square on an order set $\Omega = (b_1, b_2, \cdots, b_n)$. Partition Ω into m disjoint subsets $S_1, S_2, \cdots, S_m$ such that S_i contains λ_i elements. Define a many-one map σ from Ω into Σ as follows:

$$\sigma(x) = a_i \quad \text{if and only if} \quad x \in S_i, \qquad i = 1, 2, \cdots, m.$$

If we now apply σ to the elements of $F(n; 1)$ square then we obtain an $F(n; \lambda_1, \lambda_2, \cdots, \lambda_m)$ square on Σ.

Before proceeding further, it should be noted that this idea is not entirely without practical importance. For example, if we let the elements of the m-set Σ to be treatments then an $F(n; \lambda_1, \lambda_2, \cdots, \lambda_m)$ square on Σ is an experimental design having the properties that:

(a) Treatment effects $\perp$ row effects (read treatment effects are orthogonal to row effects),

(b) Treatment effects $\perp$ column effects, and if $\lambda_i = \lambda$, $i = 1, 2, \cdots, m$, then we also have

(c) Treatment arrangement is balanced within rows and columns

(d) Row effects $\perp$ column effects.

An experimenter may consider such a design for two reasons. One of the reasons may be that he learned from past experience that the difference between some of the treatments considered in previous experimentation as distinct is negligible and could be ignored. This means that he would consequently change the structure of his design from a Latin square to a proper F-square. An F-square may also to begin with be preferred to a Latin square especially when the number of treatments is smaller than the order of the square and one would like to take advantage of the available experimental units in order to improve the precision of the estimates of at least some of the treatment effects.

The practical importance of a Graeco–Latin square design or in general a design consisting of t mutually orthogonal Latin squares for the elimination of 3 or $t+1$ sources of variations is well known. However, there are two severe limitations attached to any such design. Firstly there may not exist such a design for the given number of treatments. Secondly the experimenter is unjustifiably forced to use the same number of treatments in each Latin square. To overcome these difficulties the experimenter may have available orthogonal F-squares which could exist even when not that many mutually orthogonal Latin squares are known to exist.

3. On the orthogonality of F-squares.

We now introduce the concept of the orthogonality of F-squares and then of Latin squares as a special case of F-squares.

DEFINITION 3.1. Given an F-square $F_1(n; \lambda_1, \lambda_2, \cdots, \lambda_k)$ on a k-set $\Sigma = \{a_1, a_2, \cdots, a_k\}$ and an F-square $F_2(n; u_1, u_2, \cdots, u_t)$ on a t-set $\Omega = \{b_1, b_2, \cdots, b_t\}$. Then we say F_2 is an *orthogonal mate* for F_1 (and write $F_2 \perp F_1$) if upon superposition of F_2 on F_1, a_i appears $\lambda_i u_j$ times with b_j.

EXAMPLE 3.1.

$$F_1(6; 1) \qquad\qquad F_2(6; 1, 2, 1^3)$$

$$\begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 6 & 5 & 4 & 3 \\ 3 & 4 & 1 & 2 & 6 & 5 \\ 4 & 6 & 5 & 1 & 3 & 2 \\ 5 & 3 & 2 & 6 & 1 & 4 \\ 6 & 5 & 4 & 3 & 2 & 1 \end{matrix} \quad \perp \quad \begin{matrix} 1 & 2 & 3 & 4 & 2 & 5 \\ 4 & 5 & 2 & 3 & 1 & 2 \\ 2 & 3 & 2 & 5 & 4 & 1 \\ 2 & 1 & 4 & 2 & 5 & 3 \\ 5 & 4 & 1 & 2 & 3 & 2 \\ 3 & 2 & 5 & 1 & 2 & 4. \end{matrix}$$

DEFINITION 3.2. Let S_i be an n_i-set, $i = 1, 2, \cdots, t$. Let F_i be an F-square of order n on the set S_i with frequency vector $\lambda_i = (\lambda_{i1}, \lambda_{i2}, \cdots, \lambda_{il_i})$. Then we say $\{F_1, F_2, \cdots, F_t\}$ is a set of t mutually (pair-wise) orthogonal F-squares if $F_i \perp F_j$, $i \neq j$, $i, j = 1, 2, \cdots, t$. In particular, if $n_i = n$, $i = 1, 2, \cdots, t$, and every F_i is of type $F(n; 1)$, i.e. a Latin square of order n, then we denote such a set as a $0(n, t)$ set.

EXAMPLE 3.2. The following three F-squares are mutually orthogonal.

$$\begin{array}{ccccc} \multicolumn{5}{c}{F_1(5; 2^2, 1)} \\ 1 & 2 & 3 & 1 & 2 \\ 2 & 1 & 2 & 3 & 1 \\ 1 & 2 & 1 & 2 & 3 \\ 3 & 1 & 2 & 1 & 2 \\ 2 & 3 & 1 & 2 & 1 \end{array} \perp \begin{array}{ccccc} \multicolumn{5}{c}{F_2(5; 1^2, 3)} \\ 1 & 2 & 3 & 3 & 3 \\ 3 & 3 & 1 & 2 & 3 \\ 2 & 3 & 3 & 3 & 1 \\ 3 & 1 & 2 & 3 & 3 \\ 3 & 3 & 3 & 1 & 2 \end{array} \perp \begin{array}{ccccc} \multicolumn{5}{c}{F_3(5; 1^3, 2)} \\ 1 & 2 & 3 & 4 & 4 \\ 3 & 4 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 4 & 1 \\ 4 & 4 & 1 & 2 & 3. \end{array}$$

DEFINITION 3.3. Let Σ be an n-set. Let F be an F-square of order n on the set Σ with frequency vector $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_t)$. Then, we say F is of *degree* r with respect to the decomposition $n = u_1 + u_2 + \cdots + u_s$ if there exist $r-1$ F-squares $F_1, F_2, \cdots, F_{r-1}$ on an s-set Ω with frequency vector $\mathbf{u} = (u_1, u_2, \cdots, u_s)$ such that $\{F, F_1, \cdots, F_{r-1}\}$ is a set of r mutually orthogonal F-squares and r is the largest such integer. In particular, if $\lambda_i = 1$; $u_j = 1$, $i = 1, 2, \cdots, t = n, j = 1, 2, \cdots, s = n$, i.e. $F, F_1, F_2, \cdots, F_{r-1}$ are Latin squares we say that F is of type $E(n, r)$. F is said to be *orthogonally mateless* with respect to the decomposition $n = u_1 + u_2 + \cdots + u_s$ if its degree is one. F is said to be *orthogonally* rich if its degree is at least two with respect to every decomposition of n.

DEFINITION 3.4. A set of r F-squares $\{F_1, F_2, \cdots, F_r\}$ is said to be a mutually (pair-wise) orthogonally rich set if F_i is orthogonally rich and $F_i \perp F_j$, $i \neq j$, $i, j = 1, 2, \cdots, r$.

Now we give some examples to clarify the above definition.

EXAMPLE 3.3. The degree of

$$F = \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 6 & 1 & 4 & 5 \\ 3 & 6 & 2 & 5 & 1 & 4 \\ 4 & 1 & 5 & 2 & 6 & 3 \\ 5 & 4 & 1 & 6 & 3 & 2 \\ 6 & 5 & 4 & 3 & 2 & 1 \end{array}$$

is at least 2 with respect to the decomposition $6 = 1+1+1+1+2$. An orthogonal mate with respect to this decomposition is

$$F = \begin{array}{cccccc} 1 & 2 & 5 & 5 & 3 & 4 \\ 5 & 1 & 3 & 2 & 4 & 5 \\ 3 & 5 & 4 & 1 & 5 & 2 \\ 5 & 4 & 2 & 3 & 1 & 5 \\ 4 & 3 & 5 & 5 & 2 & 1 \\ 2 & 5 & 1 & 4 & 5 & 3. \end{array}$$

Indeed the degree of the above *F*-square is at least 2 with respect to any decomposition of 6 except $6 = 1+1+\cdots+1$.

EXAMPLE 3.4. The following *F*-square

$$F = \begin{matrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \\ 3 & 4 & 1 & 2 \\ 2 & 3 & 4 & 1 \end{matrix}$$

is orthogonally mateless with respect to the decomposition $4 = 1+1+1+1$. However, its degree is at least 3 with respect to the decomposition $4 = 2+2$. A pair of mutually orthogonal mates for *F* with respect to this decomposition is:

$$F_1 = \begin{matrix} 1 & 1 & 2 & 2 \\ 1 & 2 & 2 & 1 \\ 2 & 2 & 1 & 1 \\ 2 & 1 & 1 & 2 \end{matrix} \quad \text{and} \quad \begin{matrix} 1 & 2 & 1 & 2 \\ 1 & 2 & 1 & 2 \\ 2 & 1 & 2 & 1 \\ 2 & 1 & 2 & 1. \end{matrix}$$

The following property is often useful in the development of the theory of orthogonality of *F*-squares.

An F-square R on an n-set Σ *is orthogonally rich if and only if it has an orthogonal mate with respect to the decomposition* $n = 1+1+\cdots+1$. To prove the sufficiency let A be an $F(n; 1)$ square such that $A \perp R$. Then it is easy to see that all other orthogonal squares for R can be derived from A by the transformations introduced in Section 2.

DEFINITION 3.5. A sub-*F*-square of order t and frequency vector $(\lambda_1, \lambda_2, \cdots, \lambda_k)$ denoted by $\mathrm{SFS}(t; \lambda_1, \lambda_2, \cdots, \lambda_k)$ is an *F*-square of order t and frequency vector $(\lambda_1, \lambda_2, \cdots, \lambda_k)$ embedded in a larger *F*-square. If an *F*-square has only the trivial $\mathrm{SFS}(1; 1)$, then we say F contains no SFS.

EXAMPLE 3.5. The **bold face** cells in the following *F*-square form an $\mathrm{SFS}(2; 1)$.

$$\begin{matrix} \mathbf{1} & 1 & \mathbf{2} & 2 \\ 1 & 2 & 2 & 1 \\ 2 & 2 & 1 & 1 \\ \mathbf{2} & 1 & \mathbf{1} & 2. \end{matrix}$$

EXAMPLE 3.6. The following *F*-square has no SFS of any order.

$$\begin{matrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \\ 3 & 4 & 5 & 1 & 2 \\ 4 & 5 & 1 & 2 & 3 \\ 5 & 1 & 2 & 3 & 4. \end{matrix}$$

THEOREM 3.1. *If* L *is an* $F(n; 1)$ *on an n-set* Σ *then*: (a) L *cannot have an* $\mathrm{SFS}(t; 1)$ *if n is odd and* $t \geqq \frac{1}{2}(n+1)$, (b) L *cannot have an* $\mathrm{SFS}(t; 1)$ *if n is even and* $t \geqq \frac{1}{2}n+1$.

PROOF. (a) Suppose that L has an SFS$(t;\,1)$. We may assume without loss of generality that this SFS$(t;\,1)$ is formed by the first t rows and columns of L. Then the rectangle formed by the first t rows and the remaining $n-t$ columns could not include any element of the SFS$(t;\,1)$. On the other hand, $t \geqq \frac{1}{2}(n+1)$ implies $n-t \leqq \frac{1}{2}(n-1)$. Hence in each row some of the elements would have to appear more than once which contradicts the property of L. The proof of (b) is analogous.

It is obvious that if an F-square $F(n;\,\lambda_1, \lambda_2, \cdots, \lambda_t)$ is orthogonally mateless with respect to the decomposition $n = 1+1+\cdots+1$, then *it is not* necessarily mateless with respect to a coarser decomposition of n. However, if F is orthogonally mateless with respect to any decomposition then it is orthogonally mateless with respect to the decomposition $n = 1+1+\cdots+1$.

Mann [9] among other results proved that if a Latin square L of order $n = 4t+2$ has a sub-Latin square of order $2t+1$ then L is orthogonally mateless. In the language of the theory of F-squares Mann's result states that any $F(n;\,1)$, $n = 4t+2$, containing an SFS$(2t+1;\,1)$ is orthogonally mateless with respect to the decomposition $n = 1+1+\cdots+1$. The following theorem provides us with a much stronger result, viz. it says that such an F-square is orthogonally mateless with respect to a coarser decomposition of n than $n = 1+1+\cdots+1$. Therefore, Mann's [9] result turns out to be a special case of this theorem.

THEOREM 3.2. *Let L be an $F(n;\,1)$ square on an n-set Σ, $n = 4t+2$, t a positive integer. Then L is orthogonally mateless with respect to the decomposition $n = x+y$ if L contains an* SFS$(2t+1;\,1)$ *and x is an odd integer.*

PROOF. Let $m = 2t+1$. Denote the given SFS$(m;\,1)$ by L_1. With no loss of generality we can assume that L_1 occupies the square formed by the first m rows and columns of L. Partition L as follows:

$$L = \begin{array}{c|c} L_1 & L_2 \\ \hline L_3 & L_4 \end{array}.$$

Note that L_2, L_3, and L_4 are also SFS$(m;\,1)$. This is so because L is an $F(n;\,1)$ square. Note also that, the squares L_1 and L_4 (and similarly L_2 and L_3) contain the same elements of Σ. Now if L has an orthogonal mate with respect to the decomposition $n = x+y$, then this will mean that there is an $L' = F(n;\,x, y)$ on say $\Omega = \{A, B\}$ such that $L' \perp L$. This implies that upon the superposition of L' on L, A will appear x times on each row, column and element of Σ. Assume that r A's appear on L_1. This implies that r A's will also appear on L_4. Since the contents of L_1 and L_4 are the same, this means that, under the orthogonality assumption of L and L', $x(2t+1)$ A's should appear together on L_1 and L_4 or $2r = x(2t+1)$. Therefore, $r = x(2t+1)/2$. But if x is odd, $x(2t+1)/2$ has no integer solution, hence a contradiction.

As an immediate consequence to this theorem, we have:

COROLLARY 3.1. *Any Latin square of order* $n = 4t+2$ *containing a sub-Latin square of order* $n/2$ *is orthogonally mateless with respect to the decomposition* $n = 1+1+\cdots+1$.

A very natural way of writing $F(n; 1)$ squares are the cyclic ones. For this reason this family of F-squares has received a considerable amount of attention. For example, it is known that if L is an $F(n; 1)$ square based on a cyclic permutation group of order n, then (a) L is orthogonally rich if n is odd, (b) L is orthogonally mateless with respect to any decomposition of n as long as "1" is in the decomposition (for instance see Hedayat and Federer [8]). While cyclic $F(n; 1)$ squares are orthogonally mateless with respect to many decompositions of n if n is even, they are at least of degree 2 with respect to the decomposition $n = 2+2+\cdots+2$ as the following theorem shows.

THEOREM 3.3. *If* L *is an* $F(n; 1)$ *square and if* L *is based on a cyclic permutation group of order* n *(even) then* L *is at least of degree* 2 *with respect to the decomposition* $n = 2+2+\cdots+2$.

PROOF. By construction. There is no loss of generality if we let G be the cyclic permutation group generated by

$$\begin{pmatrix} 1 & 2 & \cdots & n \\ n & 1 & \cdots & n-1 \end{pmatrix}.$$

If we now consider the entries of the cells on the main diagonal and the diagonals parallel to the main one we see that the entries of each diagonal together with its complement are occupied by the same elements. Considering each diagonal followed by its complement as an entity we shall construct presently a square L' orthogonal to L with respect to the decomposition $n = 2+2+\cdots+2$. Filling in its n diagonals parallel to the main diagonal as follows: Take an ordered tuple of $n/2$ distinct elements say $\Omega = (a_1, a_2, \cdots, a_{n/2})$. Fill in the n spaces of the main diagonal of L' repeating each element of Ω twice in the prescribed order. Permute cyclically the elements of Ω and fill in the diagonal starting with the second position in the first row as before. Continue the process until all the n diagonals are completed.

The following example elucidates the content of the above procedure.

$$L = \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 1 & 2 & 3 & 4 & 5 \\ 5 & 6 & 1 & 2 & 3 & 4 \\ 4 & 5 & 6 & 1 & 2 & 3 \\ 3 & 4 & 5 & 6 & 1 & 2 \\ 2 & 3 & 4 & 5 & 6 & 1 \end{matrix}$$

is an $F(6; 1)$ square generated by

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 1 & 2 & 3 & 4 & 5 \end{pmatrix}.$$

An orthogonal mate for L with respect to the decomposition $6 = 2+2+2$ on the set $\Omega = \{A, B, C\}$ is

$$L' = \begin{array}{cccccc} A & B & C & A & B & C \\ C & A & B & C & A & B \\ C & A & B & C & A & B \\ B & C & A & B & C & A \\ B & C & A & B & C & A \\ A & B & C & A & B & C. \end{array}$$

COROLLARY 3.2. *Let n be an even integer. If L is an $F(n; 1)$ square and if L is based on a cyclic permutation group G of order n, then L is at least of degree 2 with respect to any decomposition of n as long as every component of the decomposition is even.*

THEOREM 3.4. *Let L be an $F(n; 1)$ square based on a cyclic permutation group G of order n on an n-set Σ. Then L contains an* SFS$(t; 1)$ *if and only if t divides n.*

PROOF. Clearly the first column of L forms a group isomorphic to G provided that we make each element of G correspond to the element of L into which it maps the unity of L. Obviously the subgroup of L corresponding to a subgroup of G of order t will form an SFS$(t; 1)$ within the rows formed by the subgroup of G. The same will apply to the cosets of the subgroup of L.

REMARKS. If $n = 4t+2$, t a positive integer, then any $F(n; 1)$ square based on a cyclic permutation group of order n has an SFS$(2t+1; 1)$. Such an F-square by Theorem 3.2 is orthogonally mateless with respect to the decomposition $n = x+(n-x)$ as long as x is odd, however, it is of degree 2 for those decompositions of n considered in Corollary 3.2.

The result of Theorem 3.3 gives a temptation to conclude that a similar result might hold for the family of $F(n; 1)$ squares, $n = 4t+3$, t a positive integer. Namely, if L is an $F(n; 1)$ square, $n = 4t+3$, then L is orthogonally mateless with respect to the decomposition $n = (2t+1)+(2t+2)$ if L has an SFS$(2t+1; 1)$. We found out that this is not the case as the following example shows:

$$L = \begin{array}{|ccc|cccc} \hline 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 3 & 1 & 6 & 7 & 4 & 5 \\ 3 & 1 & 2 & 5 & 4 & 7 & 6 \\ \hline 4 & 6 & 5 & 7 & 3 & 2 & 1 \\ 5 & 7 & 4 & 3 & 6 & 1 & 2 \\ 6 & 4 & 7 & 2 & 1 & 5 & 3 \\ 7 & 5 & 6 & 1 & 2 & 3 & 4 \end{array}$$

which is an $F(7; 1)$ square with an SFS$(3; 1)$. Note that L is not based on a cyclic group (see Theorem 3.4). Indeed this square has an orthogonal mate with respect to the finest decomposition of 7 viz., $7 = 1+1+\cdots+1$, and the following $F(7; 1)$ square is an example.

$$L' = \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 1 & 7 & 3 & 2 & 5 & 6 \\ 5 & 6 & 1 & 2 & 7 & 3 & 4 \\ 2 & 5 & 4 & 1 & 6 & 7 & 3 \\ 3 & 4 & 6 & 7 & 1 & 2 & 5 \\ 7 & 3 & 5 & 6 & 4 & 1 & 2 \\ 6 & 7 & 2 & 5 & 3 & 4 & 1. \end{array}$$

Note however, that L' is based on a cyclic permutation group generated by

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 1 & 7 & 3 & 2 & 5 & 6 \end{pmatrix}.$$

DEFINITION 3.6. A directrix (transversal) of an $F(n; \lambda_1, \lambda_2, \cdots, \lambda_t)$ square on a t-set $\Sigma = \{a_1, a_2, \cdots, a_t\}$ is a collection of n cells such that the entries of these cells contain λ_i times a_i, and every row and column of F is represented in this collection.

EXAMPLE 3.7. The bold face cells in the following $F(5; 2, 1, 2)$ square form a directrix.

$$\begin{array}{ccccc} \mathbf{1} & 5 & 1 & 4 & 5 \\ 5 & \mathbf{4} & 5 & 1 & 1 \\ 1 & 1 & 4 & 5 & \mathbf{5} \\ 5 & 5 & 1 & \mathbf{1} & 4 \\ 4 & 1 & \mathbf{5} & 5 & 1. \end{array}$$

Note that not every F-square has a directrix. For instance, the following $F(4; 1)$ square has no directrix.

$$\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \\ 3 & 4 & 1 & 2 \\ 2 & 3 & 4 & 1. \end{array}$$

It is apparent by now that the concept of a directrix serves as a useful tool in the theory of construction of orthogonal F-squares as it did in the theory of special F-squares, the Latin squares. Moreover the search for directrices could be motivated because of their applications in actual experiments. Suppose e.g. that an experimenter is laying out an experiment in a Latin square of order n, and he would consider taking additional observations in order to obtain some information regarding the variance and interactions. He could achieve his goal with n extra observations attached to a directrix of his design as it was pointed out by Youden and Hunter [11] and also emphasized by Scheffé [10].

Next we shall state two propositions, the first one indicates a sufficient condition for the existence of an $F(4t+3; 1)$ square with a directrix. The second proposition rules out the existence of orthogonal F-squares of certain combinatorial structure. This can be used for the enumeration of orthogonal F-squares.

PROPOSITION 3.1. *Let L be an $F(n; 1)$ square, $n = 4t+3$, with an* SFS$(2t+1; 1)$. *Then L has a directrix if the* SFS$(2t+1; 1)$ *does.*

PROPOSITION 3.2. *There does not exist a pair of orthogonal $F(n; 1)$ squares, $n = 4t+3$, having a pair of orthogonal* SFS$(2t+1; 1)$.

These propositions can be easily verified.

CONCLUDING REMARKS. We would like to emphasize that we do believe that concepts introduced in this paper and akin to them could serve as useful tools in the theory of design of experiments and related combinatorial problems. For example, to determine whether or not a given $F(n; 1)$ square, viz., a Latin square of order n has an orthogonal mate, one can first, as a necessary condition, check whether L has an orthogonal mate with respect to some conveniently chosen coarser decomposition than $n = 1+1+\cdots+1$. Note that in general it is much easier to search for an orthogonal mate for L of order n with respect to a coarser decomposition of n than with respect to the finest one, viz., $n = 1+1+\cdots+1$.

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