The Annals of Mathematical Statistics
1971, Vol. 42, No. 1, 368–372

SIMPLE PROOFS OF SOME THEOREMS ON POINT PROCESSES

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0. Summary. For stationary point processes in R^n a (proper) distribution of group size is shown to exist and a simple proof of the result:

$$\text{intensity} = \text{parameter} \times \text{mean group size}$$

is given. A generalization of this result to higher order moments is proved by similar methods.

1. Introduction. We shall be concerned throughout with an *n-dimensional, perhaps non-orderly point process* (see Khinchin [5] and Goldman [4] for the terminology). Let $N(S)$, a random variable, denote the number of points of the process in the set $S \subset R^n$. The sets S we consider will always be half-open intervals in R^n i.e. sets of the form $[a_1, b_1) \times \cdots \times [a_n, b_n)$. We shall assume that our process is *stationary* in the sense that for all nonnegative integers k, $P\{N(S+\boldsymbol{\tau}) = k\} = P\{N(S) = k\}$ for all $\boldsymbol{\tau} \in R^n$ and all bounded sets $S \subset R^n$ where $S+\boldsymbol{\tau} = \{\mathbf{x}+\boldsymbol{\tau} : \mathbf{x} \in S\}$. This just expresses a property of invariance under translations for the one-dimensional distributions of the process. We can then let $S_t(\mathbf{x}) = [x_1, x_1+t) \times \cdots \times [x_n, x_n+t)$, $(\mathbf{x} \in R^n)$, $S_t = S_t(\mathbf{0})$, $p_k(t) = P\{N(S_t) = k\}$ $(k = 1, 2, \cdots)$, and the *intensity* $m = \mathscr{E}N(S_1) = \sum_{k=1}^{\infty} kp_k(1)$. Finally, we remark that a reasonable finiteness condition will always be necessary but we introduce such conditions later as required.

For point processes in R^1 it is well known that, when the process is orderly

$$\lim_{t \downarrow 0} t^{-1}[1 - p_0(t)] = m \tag{1}$$

("Koroliuk's theorem": see Khinchin [5] page 41–42, also Zitek [10] and Leadbetter [6]). Extensions of this result for point processes in R^1 with ancillary variables are implicit in the work of Matthes [7]. This case includes that of a non-orderly point process in R^1. Here the result analogous to (1), namely

$$\lim_{t \downarrow 0} t^{-1}[1 - p_0(t)] \cdot a = \lambda a = m \tag{2}$$

where a is the mean group size (we show later that the distribution of group size exists and is proper) has been proved independently by Beutler and Leneman [1], Fieger [3], and Slivnyak [8], [9]. Slivnyak's methods were measure theoretic making heavy use of his fundamental formula ([8] Equation 8, [9] Equation 13). Fieger, who was also interested in analogues of (2) for non-stationary processes, employed theorems from the theory of the Burkhill integral. The arguments of Beutler and Leneman [1] depended on extension of the elementary convexity properties noticed

Received October 20, 1969.

by Khinchin [5]. The results of Beutler and Leneman [1] also imply, for point processes in R^1, that the limit

$$\lim_{t\downarrow 0} t^{-1} V(t) = \lambda(a^2+v) \tag{3}$$

where $V(t) = \text{Var } N(S_t)$ and v is the variance of the group size, with similar results for the higher order moments. It is the purpose of this note to provide simple proofs of these results using extensions of a technique suggested by Leadbetter [6]. The proofs are given for point processes in R^n; in this case we are not even aware of a previous proof of the result analogous to (1).

2. Intensity related to parameter and group size. We first adopt some suitable notation. If we write N_k for the number of groups of size k ($k = 1, 2, \cdots$) in the unit cube S_1 and N^* for the total number of groups in S_1 then $N^* = \sum_{k=1}^{\infty} N_k$ and $N = \sum_{k=1}^{\infty} kN_k$ is the total number of events in this cube.

A *finiteness condition* is imposed on the process by assuming that with probability one the random variable N^* is finite. Where necessary later the stronger assumption $\mathscr{E}N^* < \infty$ will be employed. In the last theorem the still stronger assumption $\mathscr{E}N^\alpha < \infty$ ($\alpha \geqq 1$) will be needed.

We shall now prove the first of our results which extends the results of Theorem 1 of Leadbetter [6].

THEOREM 1. *The limit* $\lim_{t\downarrow 0} t^{-n}[1-p_0(t)]$ *exists and is always equal to* $\mathscr{E}N^*$. *Also, for* $k = 1, 2, \cdots$ *the limit* $\lim_{t\downarrow 0} t^{-n} p_k(t)$ *exists* (*whenever* $\mathscr{E}N^* < \infty$) *and is equal to* $\mathscr{E}N_k$.

In R^1 the existence of the limits can be deduced from elementary convexity properties as was originally done by Khinchin [5] Section 7, Section 8 for a specialized class of stationary point processes. However, our methods (cf. Leadbetter [6]) enable the existence of these limits, and their values, to be determined concurrently even for stationary point processes in R^n. None of the approaches of previous authors have seemed capable of such easy extension to R^n.

PROOF. Consider a subdivision of the unit cube S_1 into m^n equal small cubes and define indicator functions by

$$\chi_m^{(k)}(\mathbf{i}) = 1 \quad \text{if} \quad N(S_{1/m}(\mathbf{i})) = k,$$
$$= 0 \quad \text{otherwise;}$$

for $\mathbf{i} \in I$ where $I = \{\mathbf{i} \in R^n : i_j \in \{0, 1/m, \cdots, (m-1)/m\}, j = 1, \cdots, n\}$. Then

$$\sum_{k=r}^{\infty} \chi_m^{(k)}(\mathbf{i}) = 1 \quad \text{if} \quad N(S_{1/m}(\mathbf{i})) \geqq r,$$
$$= 0 \quad \text{otherwise;}$$

and $Y_m^{(r)} = \sum_{i\in I} \sum_{k=r}^{\infty} \chi_m^{(k)}(\mathbf{i})$ = the number of cubes of the subdivision containing at least r events.

We now prove that as $m \to \infty$, $Y_m^{(r)} \to \sum_{k=r}^{\infty} N_k$ with probability one. Indeed, since with probability one there are only a finite number of points in the unit cube S_1

at which groups of events occur, each of these must be an isolated point, and hence, with probability one, there exists an m_0 such that $Y_m^{(r)} = \sum_{k=r}^{\infty} N_k$ for all $m \geqq m_0$.

In the case when $\mathscr{E}N^* < \infty$, since $Y_m^{(r)} \leqq N^*$ for all m, r, it follows by dominated convergence that as $m \to \infty$, $\mathscr{E}Y_m^{(r)} \to \mathscr{E}\sum_{k=r}^{\infty} N_k$. But $\mathscr{E}Y_m^{(r)} = \sum_{i \in I} P\{N(S_{1/m}(\mathbf{i})) \geqq r\} = m^n P\{N(S_{1/m}) \geqq r\}$ by stationarity. Thus we have $\lim_{m\to\infty} m^n P\{N(S_{1/m}) \geqq r\} = \mathscr{E}\sum_{k=r}^{\infty} N_k$.

Now from the monotonicity of $P\{N(S_t) \geqq r\}$ as a function of t we deduce the inequalities

$$(4) \qquad \left(\frac{t^{-1}}{[t^{-1}]+1}\right)^n \frac{P\{N(S_{([t^{-1}]+1)^{-1}}) \geqq r\}}{([t^{-1}]+1)^{-n}} \leqq \frac{P\{N(S_t) \geqq r\}}{t^n}$$
$$\leqq \frac{P\{N(S_{[t^{-1}]^{-1}}) \geqq r\}}{[t^{-1}]^{-n}} \left(\frac{t^{-1}}{[t^{-1}]}\right)^n$$

(where $[u]$ = the greatest integer less than u). It then follows that

$$\lim_{t\downarrow 0} t^{-n} P\{N(S_t) \geqq r\} = \mathscr{E}\sum_{k=r}^{\infty} N_k \qquad r = 1, 2, \cdots$$

and hence that

$$(5) \qquad \lim_{t\downarrow 0} t^{-n} p_k(t) = \mathscr{E}N_k \qquad k = 1, 2, \cdots$$

and

$$(6) \qquad \lim_{t\downarrow 0} t^{-n}[1 - p_0(t)] = \mathscr{E}N^*.$$

The latter result (6) can be easily proved when $\mathscr{E}N^* = \infty$ by an application of Fatou's lemma. The theorem is thus proved.

By Theorem 1 we are able for a stationary point process in R^n, to define the *parameter* λ by

$$(7) \qquad \lambda = \lim_{t\downarrow 0} t^{-n}[1 - p_0(t)]$$

and, for $k = 1, 2, \cdots$, the clearly nonnegative quantities π_k by

$$(8) \qquad \pi_k = \lim_{t\downarrow 0} p_k(t)[1 - p_0(t)]^{-1}.$$

The case $\lambda = 0$ is trivial, for, by subadditivity, we have $0 \leqq 1 - p_0(t) \leqq \lambda t^n$ for all t and hence $p_0(t) \equiv 1$ if $\lambda = 0$, which implies that with probability one no events occur, a case we may readily exclude. We therefore assume $\lambda > 0$. From $1 - p_0(t) = \sum_1^{\infty} p_k(t) \leqq \sum_1^{\infty} k p_k(t) = mt^n$ it is clear that $\lambda \leqq m$. Since the quantities $p_k(t)[1 - p_0(t)]^{-1} k = 1, 2, \cdots$ obviously form a proper probability distribution for every $t > 0$, their limits π_k form a possibly improper distribution $\{\pi_k\}$. On the basis of the relation (8) we shall refer to π_k as the *probability that a group is of size* k (cf. Fieger [2] in R^1); we shall shortly show that in fact the π_k form a proper distribution, whose mean is equal to m/λ ($\lambda < \infty$).

THEOREM 2. *The parameter λ defined in* (7) *is equal to the overall mean rate of occurrence and, when this latter quantity is finite, the mean rate of occurrence of groups of size k is equal to $\lambda\pi_k$, where π_k is defined in* (8).

PROOF. From Theorem 1 we have immediately, using the definitions (7) and (8) that $\lambda = \mathscr{E}N^* \leqq \infty$ and $\mathscr{E}N_k = \lambda\pi_k$ when $\mathscr{E}N^* < \infty$.

When $\mathscr{E}N^* < \infty$ this theorem also implies (since the terms in the sums are non-negative) that $\lambda = \mathscr{E}N^* = \sum_1^\infty \mathscr{E}N_k = \lambda \sum_1^\infty \pi_k$, whence $\sum_1^\infty \pi_k = 1$ (since $\lambda > 0$) and

(9) $$m = \mathscr{E}N = \sum_1^\infty k\mathscr{E}N_k = \lambda \sum_1^\infty k\pi_k = \lambda \times \quad \text{mean group size.}$$

Also, we may remark that the property of *orderliness*, defined to mean $\sum_2^\infty p_k(t) = o(t^n)$ as $t \downarrow 0$ (cf. Khinchin [5] for the case $n = 1$), is now clearly seen to be equivalent (when $\mathscr{E}N^* < \infty$) to the assertion that $\pi_k = 0$ $(k \geqq 2)$ and hence to the more intuitive notion that events occur singly with probability one (Dobrushin's lemma and its converse).

3. Higher order moments. We now consider the moments $\sum_{k=1}^\infty k^\alpha p_k(t) = \mu_\alpha(t)$ say, where α is fixed. Since $N(S_t)$ is nonnegative and non-decreasing in t we obviously have $\mu_\alpha(t)$ monotonic non-decreasing in t for all $\alpha \geqq 1$. In fact we can readily prove the stronger property

(10) $$\mu_\alpha(t_1 + t_2) \geqq \mu_\alpha(t_1) + \mu_\alpha(t_2).$$

The next theorem proves an easy generalization of the result (9).

THEOREM 3. *For* $\alpha \geqq 1$ *the limit* $\lim_{t\downarrow 0} t^{-n}\mu_\alpha(t)$ *exists whenever* $\mathscr{E}N^\alpha < \infty$ *and is then equal to* $\lambda \sum_{k=1}^\infty k^\alpha \pi_k$.

PROOF. Using the same notation as before we see that $\sum_{k=1}^\infty k\chi_m^{(k)}(\mathbf{i}) = N(S_{1/m}(\mathbf{i}))$. In this case we start from the result

$$\sum_{\mathbf{i}\in I} \left(\sum_{k=1}^\infty k\chi_m^{(k)}(\mathbf{i})\right)^\alpha \to \sum_{k=1}^\infty k^\alpha N_k$$

which can be proved as earlier the two sides again being equal, for sufficiently large m, with probability one. If we assume $\mathscr{E}N^\alpha < \infty$, then it follows from a generalization of (10) that we can apply the dominated convergence theorem and take expectations in this limit relationship. But, by stationarity, the left-hand side has expectation $m^n\mu_\alpha(1/m)$ while the right-hand side (since all terms are nonnegative) has expectation $\sum_{k=1}^\infty k^\alpha \mathscr{E}N_k = \lambda \sum_{k=1}^\infty k^\alpha \pi_k$ by Theorem 2. The monotonicity of $\mu_\alpha(t)$ then facilitates the use of inequalities analogous to (4) thus yielding

(11) $$\lim_{t\downarrow 0} t^{-n}\mu_\alpha(t) = \lambda \sum_{k=1}^\infty k^\alpha \pi_k$$

and completing the proof.

In particular, we consider $V(t) = \operatorname{Var} N(S_t) = \mu_2(t) - [\mu_1(t)]^2$ and find

$$\lim_{t\downarrow 0} t^{-n}V(t) = \lim_{t\downarrow 0} t^{-n}\mu_2(t) = \lambda \sum_{k=1}^\infty k^2\pi_k = \lambda \times \quad \text{mean square group size.}$$

REMARK. At the expense of a more complicated notation all the above results could be proved for sets $S(\mathbf{x}, \mathbf{t}) = [x_1, x_1 + t_1) \times \cdots \times [x_n, x_n + t_n)$ instead of just for the special case $t_i \equiv t$. e.g. The first result of Theorem 1 would then read: the limit

$$\lim_{t\downarrow 0} \left(\prod_{i=1}^n t_i\right)^{-1}[1 - p_0(\mathbf{t})], \qquad t = \max_{i \leqq i \leqq n} t_i$$

exists and is equal to $\mathscr{E}N^*$.

Finally, we mention that the results of our three theorems are already known (or at least implied by known results) for the special case of stationary point processes which are also completely random i.e. which are such that, for all finite collections of disjoint subsets of R^n the numbers of events in these subsets are independent random variables (for R^1 see Khinchin [5] Section 8, Section 11; for R^n the generalization is obvious).

Acknowledgments. The author is especially grateful to Dr. D. Vere-Jones and Dr. M. Westcott for helpful discussions and to the referee for his comments.

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