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ON UNIVERSAL MEASURABILITY AND PERFECT PROBABILITY

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A subset E of the real numbers R is an element of the set $\mathscr{U}$ of universally measurable subsets of R if, and only if, $\mu^*(E) = \mu_*(E)$ for each probability measure μ on the Borel subsets $\mathscr{B}$ of R. A subset E of R is an element of the sigma ideal $\mathscr{N}$ of universal null sets if, and only if, $\mu^*(E) = 0$ whenever μ is a nonatomic probability measure on $\mathscr{B}$.

The purpose of this note is to recount some properties of the sigma algebra $\mathscr{U}$ and its sigma ideal $\mathscr{N}$.

When dealing with a finite, nonnegative measure μ on a sigma algebra $\mathscr{S}$ of subsets of a set X it suffices, for our purposes, to normalize and, hence, suppose that μ is an element of the set $\mathscr{P}(\mathscr{S})$ of probability measures on $\mathscr{S}$. An element μ of $\mathscr{P}(\mathscr{S})$ is said to be perfect if for each $\mathscr{S}$-measurable function f, there exists $B \in \mathscr{B}$ such that $B \subset f(X)$ and $\mu(f^{-1}(B)) = 1$.

If A is a subset of R, then $\mathscr{B}_A$ will denote the sigma algebra of Borel subsets of A.

D. Blackwell [1] used

(1) If A is an analytic subset of R and f is a $\mathscr{B}_A$-measurable function, then $f(A)$ is an analytic set, and

(2) The sigma algebra of subsets of R generated by the analytic sets in a subset of $\mathscr{U}$, to show

(3) If A is an analytic subset of the interval $I = [0, 1]$ and $\mu \in \mathscr{P}(\mathscr{B}_A)$, then μ is perfect.

Then he asked whether there be subsets A of I, other than analytic sets, with the property that every $\mu \in \mathscr{P}(\mathscr{B}_A)$ is perfect.

V. V. Sazonov ([6] Lemma 3) answered Blackwell's question by showing that

(4) The necessary and sufficient condition in order that every $\mu \in \mathscr{P}(\mathscr{B}_A)$ be perfect is that $A \in \mathscr{U}$.

Meanwhile, G. Kallianpur introduced the notion of a D-space (i.e., $(A, \mathscr{B}_A)$ is a D-space if, and only if, $f(A) \in \mathscr{U}$ for every $\mathscr{B}_A$-measurable function f) in [3]. He showed that

(5) The necessary and sufficient condition in order that $(A, \mathscr{B}_A)$ be a D-space is that for every separable subsigma algebra $\mathscr{A}$ of $\mathscr{B}_A$, every $\mu \in \mathscr{P}(\mathscr{A})$ is perfect.

Unaware of Kallianpur's paper, Sazonov ([6] Theorem 9) gave a proof of the sufficiency of Kallianpur's result and a proof of the necessity of a stronger result:

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If $(A, \mathscr{B}_A)$ is a D-space and $\mathscr{A}$ is a (not necessarily separable) subsigma algebra of $\mathscr{B}_A$, then every $\mu \in \mathscr{P}(\mathscr{A})$ is perfect.

It follows from (1) that if A is an analytic subset of I, then $(A, \mathscr{B}_A)$ is a D-space. In response to the question: Are there non-analytic subsets E of I for which $(E, \mathscr{B}_E)$ is a D-space?, [2] was written. It was shown in [2] that if L is a Lusin subset of I (i.e., L is an uncountable subset of I with the property that if F is a first category subset of I, then $L \cap F$ is a countable set) and f is a $\mathscr{B}_L$-measurable function, then $f(L) \in \mathscr{N}$ and, hence, $(L, \mathscr{B}_L)$ is a D-space. Recall that a Lusin set L is not an analytic set. (Any uncountable analytic set contains an uncountable, nowhere dense, perfect set.)

If one attempts to classify the elements of $\mathscr{N}$ by "thinness," one finds several distinct types. An uncountable subset G of I is said to be concentrated if there is a countable subset T of I such that if V is an open set containing T, then $G - V$ is a countable set. Concentrated subsets of I are relatively thin elements of $\mathscr{N}$ and Lusin sets are concentrated about any countable dense subset of I.

The existence of (uncountable) Lusin sets follows from the continuum hypothesis, and it is not known whether the existence of uncountable concentrated sets can be proved without the continuum hypothesis ([4] page 527, footnote). In summary,

(6) The continuum hypothesis implies there exist (uncountable) Lusin subsets of I.

(7) If L is a Lusin subset of I, then $(L, \mathscr{B}_L)$ is a D-space.

It is natural to ask whether $(A, \mathscr{B}_A)$ is a D-space for every $A \in \mathscr{U}$. We shall give an example to show that the continuum hypothesis implies that the answer to the preceding question is no and then conclude with a few remarks about the character of probability measures on Borel subsets of universal null sets.

F. Rothberger ([5] Theorem 3) has shown that if there exists a concentrated subset H of I such that H and I have the same cardinality (the continuum hypothesis implies that we can take $H = L$), then there is a concentrated subset G of I and a continuous function g on G such that $g(G)$ is the set J of irrationals in I. Let K be a subset of I satisfying $m^*(K) = 1$ and $m_*(K) = 0$, where m denotes Lebesgue measure. Let $C = g^{-1}(J \cap K)$. Then C is a concentrated subset of I, the restriction f of g to C is continuous on C, and $f(C) = J \cap K$ satisfies $m^*(f(C)) = 1$ and $m_*(f(C)) = 0$. Thus $(C, \mathscr{B}_C)$ is not a D-space:

(8) If there exists a concentrated subset of I which has the cardinality of I, then there exists a concentrated subset C of I such that $(C, \mathscr{B}_C)$ is not a D-space.

Let $\mathscr{N}_E = \{F \subset E;\ \mu^*(F) = 0$ if μ is a non-atomic element of $\mathscr{P}(\mathscr{B}_E)\}$, $E \subset I$.

(9) $$\mathscr{N}_E = \{E \cap F;\ F \in \mathscr{N}\}, \qquad E \subset I.$$

Proof of (9). Suppose $F \subset E$; if $F \in \mathscr{N}$, $\lambda \in \mathscr{P}(\mathscr{B}_E)$, and $\mu \in \mathscr{P}(\mathscr{B})$ is defined by $\mu(B) = \lambda(B \cap E)$, then $\mu^*(F) = 0$ implies that $\lambda^*(F \cap E) = 0$. Suppose there exists a non-atomic element α of $\mathscr{P}(\mathscr{B})$ such that $\alpha^*(F) > 0$. Let K be a Borel set containing F and satisfying $\alpha(K) = \alpha^*(F)$. If $\beta \in \mathscr{P}(\mathscr{B}_E)$ is defined by $\beta(B \cap E) = \alpha(B \cap K)/\alpha(K)$, then $\beta^*(F) = 1$.

From (9) we obtain

$$E \in \mathscr{N} \Leftrightarrow E \in \mathscr{N}_E \Leftrightarrow \text{all } \mu \in \mathscr{P}(\mathscr{B}_E) \text{ are atomic.} \tag{10}$$

In [2] we showed that if L is a Lusin subset of I, $\mathscr{T}$ is a separable subsigma algebra of $\mathscr{B}_L$, and α is a probability measure on $\mathscr{T}$, then α is atomic and, hence, has an extension to an atomic probability measure on $\mathscr{B}_L$. Such extensions are sometimes not available in the case of the concentrated set C: Since $(C, \mathscr{B}_C)$ is not a D-space, (5) implies that there exists a separable subsigma algebra $\mathscr{A}$ of $\mathscr{B}_C$ and $\mu \in \mathscr{P}(\mathscr{A})$ such that μ is not perfect. If μ were to have an extension, γ, to $\mathscr{P}(\mathscr{B}_C)$, then γ would be atomic by (10) and, hence, μ would be atomic. But, if μ were atomic, then μ would be perfect, which it is not. Thus we conclude that μ has no extension to $\mathscr{P}(\mathscr{B}_C)$.

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REFERENCES

[1] Blackwell, David. (1956). On a class of probability spaces. *Proc. Third Berkeley Symp. Math. Statistic Prob.* **2**. Univ. of California.

[2] Darst, R. B. (1969). Properties of Lusin sets with applications to probability. *Proc. Amer. Math. Soc.* **20** 348–350.

[3] Kallianpur, Gopinath. (1959). A note on perfect probability. *Ann. Math. Statist.* **30** 169–172.

[4] Kuratowski, K. (1966). *Topology* **1**. Academic Press, New York.

[5] Rothberger, Fritz (1941). Sur les familles indenombrables de suites de nombres naturels at les problemes concernant la properte C. *Proc. Cambridge Philos. Soc.* **37** 109–126.

[6] Sazonov, V. V. (1962). On perfect measures. *Izv. Akad. Nauk S.S.S.R. Ser. Mat.* **27** 391–414. *Amer. Math. Soc. Translation* Series 2 **48** (1965), 229–254.