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NOTES

A NOTE ON THE MULTIVARIATE t-RATIO DISTRIBUTION[1]

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1. Introduction. Press [2] has studied the distribution of the ratio of two random variables which follow the bivariate t-distribution. He has presented the density function of a linear function of the ratio. It is the purpose of this paper to extend these results to the multivariate case. That is to say, suppose that the random vector X, where $X' = (x_1, x_2, \cdots, x_p)$, has the multivariate Student t density function

$$f(x; \theta, \tau) = c[\nu + (X-\theta)'\tau(X-\theta)]^{-(\nu+p)/2}$$

where

$$c = \pi^{-p/2}|\tau|^{\frac{1}{2}}\nu^{\nu/2}\Gamma[\tfrac{1}{2}(\nu+p)]/\Gamma[\tfrac{1}{2}\nu].$$

We shall find the density function of the random vector Y, where $Y' = (y_1, y_2, \cdots, y_{p-1}) = (x_2/x_1, x_3/x_1, \cdots, x_p/x_1)$.

2. The density function when p is odd. Let $z = x_1$, $y_1 = x_2/x_1$, $y_2 = x_3/x_1, \cdots, y_{p-1} = x_p/x_1$. The Jacobian of this transformation is z^{p-1}. Also, $X = zW$, where $W' = (1, y_1, y_2, \cdots, y_{p-1})$. The joint density function of Y and z is

$$h(z, Y) = c[\nu + (zW-\theta)'\tau(zW-\theta)]^{-(\nu+p)/2}z^{p-1}$$

and the density function of Y is

$$g(Y) = cM^{-(\nu+p)/2}\int_{-\infty}^{\infty}[z^2 + (K/M)z + (L/M)]^{-(\nu+p)/2}z^{p-1}\,dz$$

where $M = W'\tau W$, $K = -2W'\tau\theta$, and $L = \nu + \theta'\tau\theta$.

Now $a = (L/M) - (K^2/4M^2)$ is always positive for $(L/M) - (K^2/4M^2) = \nu(W'\tau W)^{-1} + (W'\tau W)^{-2}[(W'\tau W)(\theta'\tau\theta) - (W'\tau\theta)^2]$. But τ is positive definite and $(W'\tau W)(\theta'\tau\theta) - (W'\tau\theta)^2 \geqq 0$ (see Rao [3] page 43).

Let $u = (z-b)/a$, where $b = -K/2M$. Then

$$\begin{aligned} g(Y) &= c(Ma)^{-(\nu+p)/2}a\int_{-\infty}^{\infty}[au^2+1]^{-(\nu+p)/2}[au+b]^{p-1}\,du \\ &= c(Ma)^{-(\nu+p)/2}a\int_{-\infty}^{\infty}[au^2+1]^{-(\nu+p)/2}\textstyle\sum_{i=0}^{p-1}\binom{p-1}{i}(au)^{p-1-i}b^i\,du. \end{aligned}$$

Since p is an odd integer

$$\int_{-\infty}^{\infty}[au^2+1]^{-(\nu+p)/2}u^{p-2j}\,du = 0$$

for $j = 1, 2, \cdots, (p-1)/2$.

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The substitution $v=(1+au^2)^{-1}$ allows one to verify that

$$\int_{-\infty}^{\infty}[au^2+1]^{-(v+p)/2}\,du = a^{-\frac{1}{2}}\Gamma[\tfrac{1}{2}(v+p-1)]\Gamma[\tfrac{1}{2}]/\Gamma[\tfrac{1}{2}(v+p)].$$

Finally, this integral and the integration-by-parts technique yield a value for $\int_{-\infty}^{\infty}[au^2+1]^{-(v+p)/2}u^{2j}\,du$ for $j=1,2,\cdots,(p-1)/2$.

Thus, we have that the density function of Y is

$$g(Y)=c(Ma)^{-(v+p)/2}a\sum_{j=0}^{(p-1)/2}\binom{p-1}{p-1-2j}a^{2j}b^{p-1-2j}\int_{-\infty}^{\infty}[au^2+1]^{-(v+p)/2}u^{2j}\,du.$$

3. The case where p is even. Using the same transformation as before, we find that the Jacobian is $|z|^{p-1}$. The density function of Y is given by

$$\begin{aligned}
g(Y) &= cM^{-(v+p)/2}\{\textstyle\int_0^{\infty}[z^2+(K/M)z+(L/M)]^{-(v+p)/2}z^{p-1}\,dz \\
&\qquad -\textstyle\int_{-\infty}^{0}[z^2+(K/M)z+(L/M)]^{-(v+p)/2}z^{p-1}\,dz\} \\
&= cM^{-(v+p)/2}a\{\textstyle\int_{-b/a}^{\infty}[a^2u^2+a]^{-(v+p)/2}[au+b]^{p-1}\,du \\
&\qquad -\textstyle\int_{-\infty}^{-b/a}[a^2u^2+a]^{-(v+p)/2}[au+b]^{p-1}\,du\} \\
&= cM^{-(v+p)/2}b^{p-1}a\{\textstyle\sum_{i=0}^{p-1}\binom{p-1}{p-i-1}b^{-i}a^{i}[\int_{-b/a}^{\infty}[a^2u^2+a]^{-(v+p)/2}u^{i}\,du \\
&\qquad -\textstyle\int_{-\infty}^{-b/a}[a^2u^2+a]^{-(v+p)/2}u^{i}\,du]\} \\
&= cM^{-(v+p)/2}b^{p-1}a\{\textstyle\sum_{i=0}^{p-1}\binom{p-1}{p-i-1}b^{-i}a^{i}[\int_{-b/a}^{\infty}[a^2u^2+a]^{-(v+p)/2}u^{i}\,du \\
&\qquad -\textstyle\int_{b/a}^{-b/a}[a^2u^2+a]^{-(v+p)/2}u^{i}\,du-\int_{-\infty}^{b/a}[a^2u^2+a]^{-(v+p)/2}u^{i}\,du]\}, \\
&\qquad\qquad \text{if} \quad b<0. \\
&= cM^{-(v+p)/2}b^{p-1}a\{\textstyle\sum_{i=0}^{p-1}\binom{p-1}{p-i-1}b^{-i}a^{i}[\int_{b/a}^{\infty}[a^2u^2+a]^{-(v+p)/2}u^{i}\,du \\
&\qquad +\textstyle\int_{-b/a}^{b/a}[a^2u^2+a]^{-(v+p)/2}u^{i}\,du-\int_{-\infty}^{-b/a}[a^2u^2+a]^{-(v+p)/2}u^{i}\,du]\}, \\
&\qquad\qquad \text{if} \quad b>0 \\
&= 2cM^{-(v+p)/2}b^{p-1}a\{\textstyle\sum_{i=1}^{p/2}\binom{p-1}{p-2i}a^{2i-1}b^{-(2i-1)}\int_{-b/a}^{\infty}[a^2u^2+a]^{-(v+p)/2}u^{2i-1}\,du \\
&\qquad +\textstyle\sum_{i=0}^{(p-2)/2}\binom{p-1}{p-2i-1}a^{2i}b^{-2i}\int_0^{-b/a}[a^2u^2+a]^{-(v+p)/2}u^{2i}\,du\}, \quad \text{if} \quad b<0. \\
&= 2cM^{-(v+p)/2}b^{p-1}a\{\textstyle\sum_{i=1}^{p/2}\binom{p-1}{p-2i}a^{2i-1}b^{-(2i-1)}\int_{b/a}^{\infty}[a^2u^2+a]^{-(v+p)/2}u^{2i-1}\,du \\
&\qquad +\textstyle\sum_{i=0}^{(p-2)/2}\binom{p-1}{p-2i-1}a^{2i}b^{-2i}\int_0^{b/a}[a^2u^2+a]^{-(v+p)/2}u^{2i}\,du\}, \quad \text{if} \quad b>0.
\end{aligned}$$

$\int_{-b/a}^{\infty}[a^2u^2+a]^{-(v+p)/2}u^{2i-1}\,du$ and $\int_{b/a}^{\infty}[a^2u^2+a]^{-(v+p)/2}u^{2i-1}\,du$ are given by

$$\begin{aligned}
\int_{b/a}^{\infty}[a^2u^2+a]^{-(v+p)/2}u\,du &= \int_{-b/a}^{\infty}[a^2u^2+a]^{-(v+p)/2}u\,du \\
&= a^{-2}(v+p-2)^{-1}[b^2+a]^{-(v+p-2)/2}
\end{aligned}$$

and

$$\int_{-b/a}^{\infty}[a^2u^2+a]^{-(\nu+p)/2}u^{2i-1}\,du$$
$$= a^{-2}(\nu+p-2)^{-1}\{(-b/a)^{2i-2}[b^2+a]^{-(\nu+p-2)/2}$$
$$+(2i-2)\int_{-b/a}^{\infty}[a^2u^2+a]^{-(\nu+p-2)/2}u^{2i-3}\,du\}.$$

$\int_0^{-b/a}[a^2u^2+a]^{-(\nu+p)/2}u^{2i}\,du$ and $\int_0^{b/a}[a^2u^2+a]^{-(\nu+p)/2}a^{2i}\,du$ are given by

$$\int_0^{-b/a}[a^2u^2+a]^{-(\nu+p)/2}\,du = \{a^{-(\nu+p+1)/2}\Gamma[\tfrac{1}{2}]\Gamma[\tfrac{1}{2}(\nu+p-1)]/2\Gamma[\tfrac{1}{2}(\nu+p)]\}$$
$$\cdot\{1-I[(1+b^2/a)^{-1};\tfrac{1}{2}(\nu+p-1),\tfrac{1}{2}]\}$$
$$= \int_0^{b/a}[a^2u^2+a]^{-(\nu+p)/2}\,du$$

and

$$\int_0^{-b/a}[a^2u^2+a]^{-(\nu+p)/2}u^{2i}\,du = a^{-2}(\nu+p-2)^{-1}\{(b/a)^{2i-1}[b^2+a]^{-(\nu+p-2)/2}$$
$$+(2i-1)\int_0^{-b/a}[a^2u^2+a]^{-(\nu+p)/2}u^{2i-2}\,du\}.$$

$I[(1+b^2/a)^{-1};\tfrac{1}{2}(\nu+p-1),\tfrac{1}{2}]$ is the incomplete beta function with argument $(1+b^2/a)^{-1}$ and parameters $\tfrac{1}{2}(\nu+p-1)$ and $\tfrac{1}{2}$.

4. Applications. Some applications where the above distribution arises have been mentioned by Press. A Bayesian application where it arises is the following one. Suppose a random sample, $X_1, \cdots, X_N$, of vector observations comes from a multivariate normal population, $N(\mu, \Sigma)$, where $\mu' = (\mu_1, \cdots, \mu_p)$. If one assumes the non-informative prior density for μ and Σ^{-1}, i.e.

$$g(\mu, \Sigma^{-1}) \propto |\Sigma|^{(p+1)/2},$$

then the posterior density of μ is the multivariate Student t density function (see Geisser [1]). The posterior density of the vector η, where $\eta' = (\mu_2/\mu_1, \mu_3/\mu_1, \cdots, \mu_p/\mu_1)$, is then given in Section 2 and Section 3. This is a multivariate generalization of the Fieller–Creasy problem.

Consider, for example, the case $p = 3$. The posterior density of η, where $\eta' = (\mu_2/\mu_1, \mu_3/\mu_1)$ is given in Section 2. This density may be integrated numerically, using a computer, in order to find a rectangular posterior region for η. That is to say, one is able to obtain a probability statement of the following type.

$$\Pr[a_1 < \mu_2/\mu_1 < a_2, b_1 < \mu_3/\mu_1 < b_2] = 1-\alpha$$

for given α, $0 < \alpha < 1$. It would probably be advisable to center the rectangular region at the point $(\bar{x}_2/\bar{x}_1, \bar{x}_3/\bar{x}_1)$ in order to ensure smallness of the region obtained. Here $\bar{X}' = N^{-1}\sum_{i=1}^{N} X_i' = (\bar{x}_1, \bar{x}_2, \bar{x}_3)$.

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