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ASYMPTOTIC OPTIMALITY AND ARE OF CERTAIN RANK-ORDER TESTS UNDER CONTIGUITY

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1. Summary. In this paper we will derive asymptotically optimal rank-order tests for independence against suitable classes of nonparametric alternatives and give asymptotic relative efficiencies (ARE's) of such tests under general contiguous alternatives of positive quadrant dependence (cf. Lehmann [14]). From Lehmann [14] one can also see that such alternatives in some respects are more general than the alternatives considered in Bhuchongkul [1], Konijn [12], Hájek and Šidák [8] page 221), and others.

For the problem of symmetry and for the two-sample problem we can get completely analogous results with similar proofs. Details are omitted.

The paper is based on the theory of contiguity that was introduced by LeCam [13] and Hájek [6].

The results of this paper complement results obtained by Hodges and Lehmann [9], [10], Chernoff and Savage [3], Hájek [6], van Eeden [4], Bhuchongkul [1], Gokhale [5], and others.

2. Rank-order tests for independence. For each integer $n \geqq 1$ let $(Y_{n1}, Z_{n1}), \cdots, (Y_{nn}, Z_{nn})$ be n independent pairs of random variables, and suppose that all pairs (Y_{ni}, Z_{ni}), $i = 1, \cdots, n$, have the same continuous two-dimensional distribution function F_n, $n \geqq 1$. Then we will derive asymptotically optimal rank-order tests at level α, $0 < \alpha < 1$, (cf. Neyman [15]) for the hypothesis of independence

$$H: F_n(y, z) = F(y, z) = F(y, \infty)F(\infty, z) \qquad \text{for all} \quad y, z \in \mathbb{R}_1, n \geqq 1$$

against a suitable subclass of the alternative of positive quadrant dependence

$$K: F_n(y, z) \geqq F_n(y, \infty)F_n(\infty, z) \qquad \text{for all} \quad y, z \in \mathbb{R}_1,$$
$$\neq \text{for at least one pair} \quad (y, z), \qquad y, z \in \mathbb{R}_1, n \geqq 1$$

and give results on the ARE of such tests.

Now let b_j, $j = 1, 2$, be real-valued measurable functions on $]0, 1[$ such that

$$(1) \qquad \int_0^1 b_j(z)\, dz = 0, \qquad 0 < \int_0^1 b_j^2(z)\, dz = \sigma_j^2 < \infty, \qquad j = 1, 2,$$

and let g_{jn}, $j = 1, 2$, $n \geqq 1$ be real-valued measurable functions on $[0, 1]$ such that

$$\int_0^s g_{1n}(y)\, dy \int_0^t g_{2n}(z)\, dz \geqq 0 \quad \text{for all} \quad s, t \in [0, 1]$$
$$\neq 0 \quad \text{for at least one pair} \quad s, t \in [0, 1],$$

$$(2) \qquad \int_0^1 g_{jn}(z)\, dz = 0, \qquad \int_0^1 (g_{jn}(z) - b_j(z))^2\, dz \to 0, \qquad j = 1, 2,$$
$$\sup_{0 \leqq y, z \leqq 1} (g_{1n}(y) g_{2n}(z))^4 / n \to 0.$$

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To such b_j, $j = 1, 2$, we define a sequence $\{\varphi_n\}$ of rank-order tests φ_n, $n \geqq 1$, in the following way: Let $U_{n1}, \cdots, U_{nn}$ be independent random variables, each uniformly distributed over $]0, 1[$, and let $(U_{n[1]}, \cdots, U_{n[n]})$ denote the order statistic of $(U_{n1}, \cdots, U_{nn})$, $n \geqq 1$. Then we put

$$b_{jn}(i) = Eb_j(U_{n[i]}), \qquad i = 1, \cdots, n, n \geqq 1, j = 1, 2$$

and

$$\varphi_n = I_{\{T_n > c_n\}}, \qquad T_n = \sum_{i=1}^n b_{1n}(r_{ni}) b_{2n}(s_{ni}) / n^{\frac{1}{2}} \sigma_1 \sigma_2, \qquad n \geqq 1,$$

in which $c_n \to u_\alpha = \Phi^{-1}(1-\alpha)$, $0 < \alpha < 1$, $\Phi(t) = \int_{-\infty}^t \exp(z^2/2)/(2\pi)^{\frac{1}{2}} dz$, $t \in \mathbb{R}_1$, holds, and $r_n = (r_{n1}, \cdots, r_{nn})$, $s_n = (s_{n1}, \cdots, s_{nn})$ resp. are the ranks in $y_n = (y_{n1}, \cdots, y_{nn})$, $z_n = (z_{n1}, \cdots, z_{nn})$ resp.; $x_n = (x_{n1}, \cdots, x_{nn}) = (y_{n1}, z_{n1}, \cdots, y_{nn}, z_{nn}) \in \mathbb{R}_{2n}$.

Because of Lemma V.1.6.a of Hájek and Šidák [8], one may also work with

$$b_{jn}(i) = b_j(i/(n+1)), \qquad 1 \leqq i \leqq n, j = 1, 2.$$

If we let $b_1(u) = b_2(u) = u - \frac{1}{2}$ we get the Spearman rank correlation, for $b_1(u) = b_2(u) = \Phi^{-1}(u)$ one gets the Fisher–Yates or the van der Waerden version of the normal scores test, and finally, by letting $b_1(u) = b_2(u) = \operatorname{sign}(u - \frac{1}{2})$, we get the quadrant statistic for correlation, considered in Blomqvist [2].

Moreover, to such $b = (b_1, b_2)$ (or exactly to g_{jn}, $n \geqq 1, j = 1, 2$) we define a subclass K_b of alternatives such that $\{\varphi_n\}$ is an asymptotically optimal test for H against K_b at level α: To each $F \in H$ and each $\eta \in]0, \Delta[$, $\exists \Delta \in]0, 1[$, we can define an $F_{n\eta} \in K$, $n \geqq 1$, in the following way: Because of (2), by

$$f_{n\eta}(y, z) = 1 + (\eta / n^{\frac{1}{2}}) g_{1n}(F(y, \infty)) g_{2n}(F(\infty, z)), \qquad y, z \in \mathbb{R}_1,$$

an F-density with corresponding distribution function $F_{n\eta} \in K$ is defined for $n \geqq 1$. Thus we define K_b as the set of all such sequences $\{F_{n\eta}\}_{n \geqq 1}$, $F \in H$, $\eta \in]0, \Delta[$. One can show that for each $\{F_n\} \in K_b$ there exists an $F \in H$ such that the sequence $\{Q_n\}$ is contiguous to the sequence $\{P_n\}$ (cf. Hájek and Šidák [8] page 202), where Q_n and P_n resp. are the probability measures on $(\mathbb{R}_{2n}, \mathfrak{B}_{2n})$ corresponding to $G_n(y_1, z_1, \cdots, y_n, z_n) = F_n(y_1, z_1) \cdots F_n(y_n, z_n)$ and $H_n(y_1, z_1, \cdots, y_n, z_n) = F(y_1, z_1) \cdots F(y_n, z_n)$ respectively, $n \geqq 1$. If $\{Q_n\}$, defined by F_n, $n \geqq 1$, in the above way, is contiguous to $\{P_n\}$, defined by F in the above way, we shall say that $\{F_n\}$ is contiguous to $\{F\}$.

THEOREM 1. (a) $F \in H \Rightarrow \mathscr{L}[T_n \mid F] \to \mathfrak{N}(0, 1)$.

(b) $\{\varphi_n\}$ *is an asymptotically optimal test for* H *against* K_b *at level* α.

(c) $F \in H$ *and* $\{F_n\}$ *contiguous to* $\{F\}$ *imply*

$$\mathscr{L}[T_n - (n^{\frac{1}{2}}/\sigma_1 \sigma_2) \textstyle\int g_{1n}(F(y, \infty)) g_{2n}(F(\infty, z))\, dF_n(y, z) \mid F_n] \to \mathfrak{N}(0, 1).$$

PROOF. $F \in H$, (1), (2), and Lindeberg–Feller theorem entail

$$\mathscr{L}[S_n \mid F] \to \mathfrak{N}(0, 1) \tag{3}$$

with

$$S_n(x_n) = \sum_{i=1}^n g_{1n}(F(y_{ni}, \infty)) g_{2n}(F(\infty, z_{ni})) / n^{\frac{1}{2}} \sigma_1 \sigma_2, \qquad x_n \in \mathbb{R}_{2n}, n \geqq 1.$$

Moreover, contiguity of $\{Q_n\}$ to $\{P_n\}$ implies

$$\limsup_{n\to\infty} (\sup_{B\in\mathfrak{B}_{2n}} |P_n(B)-Q_n(B)|) < 1,$$

and therefore we get (cf. Kellerer [11] page 209) from contiguity of $\{F_n\}$ to $\{F\}$

(4) $$\limsup_{n\to\infty} (n^{\frac{1}{2}} \sup_{B\in\mathfrak{B}_2} |\int_B dF - \int_B dF_n|) < \infty.$$

Thus

$$\mathscr{L}[S_n-(n^{\frac{1}{2}}/\sigma_1\,\sigma_2)\int g_{1n}(F(y,\infty))g_{2n}(F(\infty,z))\,dF_n(y,z)\,|\,F_n] \to \mathfrak{N}(0,1),$$

according to Lindeberg–Feller theorem.

Contiguity of $\{Q_n\}$ to $\{P_n\}$, Q_n and P_n respectively defined by F_n and F respectively, as mentioned above, $n \geqq 1$, concludes the proof of part (a) and part (c), if we prove

(5) $$\int (T_n-S_n)^2\,dP_n \to 0.$$

On the other hand, we get for $\{F_n\}\in K_b$ from the definition of K_b an $F\in H$ such that the likelihood ratio of the corresponding Q_n and P_n has the form

$$L_n(x_n) = dQ_n/dP_n = \prod_{i=1}^n (1+(\eta/n^{\frac{1}{2}})g_{1n}(F(y_{ni},\infty))g_{2n}(F(\infty,z_{ni}))), \quad \exists\eta > 0.$$

An easy application of a Taylor-expansion then implies

$$\eta\sigma_1\,\sigma_2\,S_n-\log L_n-\eta^2\sigma_1{}^2\sigma_2{}^2/2 \to_{P_n} 0.$$

Therefore, by using (3) and Hájek and Šidák [8], Corollary VI.1.2, the proof of part (b) is concluded, too, if we prove (5). But (5) is a consequence of Hájek and Šidák [8], Theorem V.1.4a and of the independence of r_n and s_n under P_n.

Now we assume the existence of nonnegative (or nonpositive) real integrable functions g_{jn}^*, $j = 1, 2$, $n \geqq 1$, defined on [0, 1] such that

(6) $$g_{jn}(t) = g_{jn}(1)-\int_t^1 g_{jn}^*(z)\,dz \quad \text{for all} \quad t\in[0,1], \qquad j = 1,2, n \geqq 1.$$

Then we can prove the following lemma, which is very useful for deriving results on ARE's.

LEMMA 2. *If* $F\in H$ *and* $\{F_n\}$ *contiguous to* $\{F\}$, $F_n\in K$, $n \geqq 1$, *then*

(7) $$\delta_n \equiv (n^{\frac{1}{2}}/\sigma_1\,\sigma_2)\int (F_n(y,z)-F_n(y,\infty)F_n(\infty,z)) \cdot g_{1n}^*(F(y,\infty))g_{2n}^*(F(\infty,z))\,dF(y,z) \geqq 0$$

and

$$\delta_n-(n^{\frac{1}{2}}/\sigma_1\,\sigma_2)\int g_{1n}(F(y,\infty))g_{2n}(F(\infty,z))\,dF_n(y,z) \to 0.$$

PROOF. If we put $F_1(y) \equiv F(y,\infty)$, $F_2(z) \equiv F(\infty,z)$, $F_{n1}(y) \equiv F_n(y,\infty)$, $F_{n2}(z) \equiv F_n(\infty,z)$, then Fubini's theorem, (6), and $F_n(y,z) \geqq F_{n1}(y)F_{n2}(z)$, $g_{1n}^*(y)g_{2n}^*(z) \geqq 0$ imply

$$\begin{aligned}
&\int g_{1n}(F_1(y))g_{2n}(F_2(z))\,dF_n(y,z)-\int g_{1n}(F_1(y))\,dF_{n1}(y)\int g_{2n}(F_2(z))\,dF_{n2}(z)\\
&= \int (\int_{F_1(y)}^1 g_{1n}^*(s)\,ds)(\int_{F_2(z)}^1 g_{2n}^*(t)\,dt)(dF_n(y,z)-dF_{n1}(y)\,dF_{n2}(z))\\
&= \int\int\int I_{]-\infty,F_1{}^{-1}(s)[}(y)I_{]-\infty,F_2{}^{-1}(t)[}(z)(dF_n(y,z)-dF_{n1}(y)\,dF_{n2}(z))g_{1n}^*(s)g_{2n}^*(t)\,ds\,dt\\
&= \int (F_n(y,z)-F_{n1}(y)F_{n2}(z))g_{1n}^*(F_1(y))g_{2n}^*(F_2(z))\,dF(y,z) \geqq 0.
\end{aligned}$$

On the other hand, we get from (2) and (4)

$$n^{\frac{1}{2}}\left|\int g_{1n}(F_1(y))\,dF_{n1}(y)\int g_{2n}(F_2(z))\,dF_{n2}(z)\right|$$
$$= n^{\frac{1}{2}}\left|\int g_{1n}(F_1(y))(dF_{n1}(y)-dF_1(y))\right|\left|\int g_{2n}(F_2(z))(dF_{n2}(z)-dF_2(z))\right|$$
$$\leqq n^{\frac{1}{2}}\sup_{0\leqq s\leqq 1}|g_{1n}(s)|\sup_{A\in\mathfrak{B}_1}\left|\int_A dF_{n1}-\int_A dF_1\right|\sup_{0\leqq t\leqq 1}|g_{2n}(t)|\cdot\sup_{A\in\mathfrak{B}_1}\left|\int_A dF_{n2}-\int_A dF_2\right|$$
$$\leqq n^{\frac{1}{2}}\sup_{0\leqq s,t\leqq 1}|g_{1n}(s)g_{2n}(t)|\sup_{B\in\mathfrak{B}_2}\left|\int_B dF_n-\int_B dF\right|^2\to 0.$$

3. Examples.

3.1. For Spearman rank correlation ($b_1(u)=b_2(u)=u-\frac{1}{2}$) we can choose $g_{1n}=g_{2n}=b_1$ and $g_{1n}^*=g_{2n}^*=1$. Therefore we get from (7)

$$\delta_n^{(1)} = 12n^{\frac{1}{2}}\int(F_n(y,z)-F_n(y,\infty)F_n(\infty,z))\,dF(y,z). \tag{8}$$

3.2. For the normal scores test ($b_1=b_2=\Phi^{-1}$) we can choose $g_{1n}(u)=g_{2n}(u)=\Phi^{-1}(u)$ if $\varepsilon_n\leqq u\leqq 1-\varepsilon_n$; $=\Phi^{-1}(\varepsilon_n)+c(u-\varepsilon_n)$ if $0\leqq u<\varepsilon_n$; $=\Phi^{-1}(1-\varepsilon_n)+c(u-(1-\varepsilon_n))$ if $1-\varepsilon_n<u\leqq 1$, $\varepsilon_n=\Phi(-n^{\frac{1}{4}})$, $0\leqq c<\infty$, and $g_{1n}^*(u)=g_{2n}^*(u)=1/\Phi'(\Phi^{-1}(u))$ if $\varepsilon_n\leqq u\leqq 1-\varepsilon_n$; $=c$ if $0\leqq u<\varepsilon_n$ or $1-\varepsilon_n<u\leqq 1$. By putting $c\geqq 1/\Phi'(0)=(2\pi)^{\frac{1}{2}}$ we get $\delta_n^{(2)}\geqq\delta_n^{(1)}\pi/6$ from (7) and (8). This fact, together with Theorem 1, implies that the ARE of the Fisher–Yates (van der Waerden) rank correlation test to the Spearman test is not smaller than $(\pi/6)^2$ under contiguous alternatives from K. This result seems to contradict the results of Gokhale [5], who stated specially that there is no lower (positive) bound for the ARE of the Fisher–Yates test to the Spearman test. His result depends on the use of alternatives which do not correspond to positive (or negative) quadrant dependence. On the other hand, these tests are "one-sided" tests.

3.3. For the quadrant statistic ($b_1(u)=b_2(u)=\operatorname{sign}(u-\frac{1}{2})$) we can choose $g_{1n}(u)=g_{2n}(u)=c_n(u-\frac{1}{2})$ if $\frac{1}{2}-1/c_n<u<\frac{1}{2}+1/c_n$; $=-1$ if $0\leqq u\leqq\frac{1}{2}-1/c_n$; $=+1$ if $\frac{1}{2}+1/c_n\leqq u\leqq 1$, $c_n\geqq 2$, $c_n\to\infty$ and $g_{1n}^*(u)=g_{2n}^*(u)=c_n$ if $u\in A_n$; $=0$ if $u\notin A_n$, $A_n=]\frac{1}{2}-1/c_n,\frac{1}{2}+1/c_n[$, $n\geqq 1$. Therefore we get from (7)

$$\delta_n^{(3)}=n^{\frac{1}{2}}\int(F_n(y,z)-F_n(y,\infty)F_n(\infty,z))c_n^{\,2}I_{A_n}(F(y,\infty))I_{A_n}(F(\infty,z))\,dF(y,z).$$

From this formula we can easily derive the following result: If we assume that there is exactly one point (y_0,z_0) so that $F(y_0,\infty)=\frac{1}{2}$, $F(\infty,z_0)=\frac{1}{2}$ holds, then

$$(\delta_n^{(3)}\to 4D\Leftrightarrow n^{\frac{1}{2}}(F_n(y_0,z_0)-F_n(y_0,\infty)F_n(\infty,z_0))\to D).$$

Further consequences of the stated facts are the following ones:

(I.) For each positive real number z there exists a subset K_z of the set of all sequences from K so that the ARE of the Fisher–Yates (van der Waerden) test to the t-test is equal to z.

(II.) The same is true for the ARE of the Spearman test to the Quadrant test, the Spearman test to the t-test, the Fisher–Yates (van der Waerden) test to the Quadrant test, the Quadrant test to the t-test respectively.

(III.) For the ARE of the Fisher–Yates (van der Waerden) test to the Spearman test an analogous result is true, if we choose z from $[(\pi/6)^2, \infty]$.

(IV.) For the set $\{\{F_n\}: \{F_n\}$ contiguous to $\{F\}$, $F_n \in K$, $n \geqq 1$, $F \in H$, F being a distribution function of a normal distribution$\}$ of approximately normal alternatives the ARE of the Fisher–Yates (van der Waerden) test to the t-test is equal to 1, if the ARE exists.

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