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CONSTRUCTION OF MARKOV PROCESSES FROM HITTING DISTRIBUTIONS II[1]

BY C. T. SHIH

University of Michigan and University of California, San Diego

Let K be a compact metric space, Δ a fixed point in K, $\mathcal{O}$ a base for the topology of K closed under the formation of finite unions and finite intersections, and $\mathscr{D} = \{(K-U)\cup\Delta \mid U\in\mathcal{O}\}$ (here Δ stands for $\{\Delta\}$). Let $\{H_D(x,\cdot)\mid x\in K, D\in\mathscr{D}\}$ be a family of probability measures satisfying the obvious necessary conditions of being the hitting distributions (as suggested in the notation) of a Hunt process on K with Δ as the death point and the following conditions: (a) if $x\notin D$ there exists D' such that $\sup_{y\in D'-\Delta}\int H_D(y,dz)H_{D'}(z,D'-\Delta)<1$; (b) if $D_n\downarrow\Delta$ and $D-\Delta$ is compact $\int H_{D_n}(x,dy)H_D(y,D-\Delta)\to 0$ uniformly on compact subsets of $K-\Delta$; (c) there is a subclass $\mathscr{D}'$ of $\mathscr{D}$ such that the sets $K-D$, $D\in\mathscr{D}'$, have compact closure in $K-\Delta$ and form a base for the topology of $K-\Delta$, and for $D\in\mathscr{D}'$ and real continuous f on K $\int H_D(x,dy)f(y)$ is continuous on $K-D$. Then a Hunt process is constructed from the prescribed hitting distributions $H_D(x,\cdot)$. This improves an earlier result of the author in that the smoothness condition (c) is much weaker than before; in fact the smoothness condition we actually assume is somewhat weaker than (c).

Introduction. This is an improvement of the result in [3], which we shall refer to as [I] in the sequel. In [I] the following result is obtained. Let K be a compact metric space, Δ be a fixed point in K, $\mathcal{O}$ be a base for the topology of K closed under the formation of finite unions and finite intersections, and $\mathscr{D} = \{(K-U)\cup\Delta \mid U\in\mathcal{O}\}$ (here Δ stands for $\{\Delta\}$). Assume given a family $\{H_D(x,\cdot)\mid x\in K, D\in\mathscr{D}\}$ of probability measures on K that satisfy the obvious necessary conditions for being the hitting distributions (as suggested in the notation) of a Hunt process on the state space K with Δ as the death point. Then a Hunt process is constructed from these prescribed hitting distributions under the following two conditions: (i) the transience condition that if $x\notin D$, $D\in\mathscr{D}$ there exists $D'\in\mathscr{D}$ containing x as an interior point such that $\int H_D(x,dy)H_{D'}(y,D'-\Delta)<1$; (ii) the smoothness condition that for every $D\in\mathscr{D}$ and real continuous f on K the function $\int H_D(x,dy)f(y)$ is continuous. In the present article the transience condition is strengthened to the extent of requiring $\sup_{y\in D'-\Delta}\int H_D(y,dz)H_{D'}(z,D'-\Delta)<1$, and another condition is added that if D, $D_n\in\mathscr{D}$, $D_n\downarrow\Delta$ and $D-\Delta$ is compact $\int H_{D_n}(x,dy)H_D(y,D-\Delta)$ converges to 0 uniformly on compact subsets of $K-\Delta$, which are, however, immediate consequences of (i) and (ii); but (ii) is greatly weakened, at least to the following extent: there is a subclass $\mathscr{D}'$ of $\mathscr{D}$ such that the sets $K-D$, $D\in\mathscr{D}'$, have compact closure in $K-\Delta$ and form a base for the topology of $K-\Delta$, and for every $D\in\mathscr{D}'$ and real continuous f on K the function

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$\int H_D(x, dy)f(y)$ is continuous on $K-D$. This relaxation has two aspects. One is the reducing of $\mathscr{D}$ to $\mathscr{D}'$, so that smoothness is now required only locally rather than globally as before. The other is the lessening of the extent of smoothness for each $D \in \mathscr{D}'$; this means roughly that the boundary points of D are not required to be regular for D. The significance of this can be seen from the fact that for some natural processes there exist no regular neighborhoods for some points. The simple process of uniform motion to the right on the real line is one such example; a less trivial example is the process in the plane whose x- and y-components are independent, with the former being the uniform motion to the right and the latter being the Brownian motion (or a diffusion process) but having 0 as a trap. Also, the present result implies the result of Hansen ([2] Theorem 2); remarks concerning this can be found in Section 1.

1. Main results. K, Δ, $\mathscr{O}$, $\mathscr{D}$ are as in the introduction. Let ρ be the metric on K and $\mathscr{B}$ be the σ-algebra of Borel sets. Denote by $\mathscr{M}$ the Banach space of bounded real Borel measurable functions on K, $\mathscr{C}$ its subspace of continuous functions, $\mathscr{M}_0$ its subspace of functions vanishing at Δ, and $\mathscr{C}_0$ the subspace of $\mathscr{M}_0$ of functions continuous on $K-\Delta$ (note that this is not the same $\mathscr{C}_0$ as in [I]).

Let $J = \{k2^{-n} \mid n \geqq 1, 0 \leqq k \leqq 2^n\}$. Let $\mathscr{E}$ stand for the class of functions $D = D(r)$ from [0, 1] into the family of closed subsets of K such that $D(r) \in \mathscr{D}$ for $r \in J$ and $D(r) = \bigcap_{s<r} D(s)$ for $0 < r \leqq 1$ (in particular $D(r)$ is decreasing). Note that since we use D both as a set and as a function of sets, it will be written as $D(r)$ when the latter is the case. Let $\mathscr{E}_0$ denote the class of those $D(r)$ in $\mathscr{E}$ satisfying $D(r) \subset \operatorname{int} D(s)$ (interior of $D(s)$) for $s < r$.

Assume given for every $x \in K$, $D \in \mathscr{D}$ a measure $H_D(x, \cdot)$ on K. We introduce the following hypotheses:

(a) $H_D(x, \cdot)$ is a probability measure concentrated on D.
(b) $H_D(x, B)$ is Borel measurable in x ($B \in \mathscr{B}$).
(c) $H_D(x, \{x\}) = 1$ if $x \in D$.
(d) $H_D(x, B) = \int H_{D'}(x, dy)H_D(y, B)$ if $D \subset D'$.
(e) If $D_n \downarrow D$, $H_{D_n}(x, \cdot)$ converges vaguely to $H_D(x, \cdot)$, i.e., $\int H_{D_n}(x, dy)f(y)$ converges to $\int H_D(x, dy)f(y)$ for all $f \in \mathscr{C}$.
(f) If $x \notin D$, there exists D' containing x as an interior point such that $\sup_{y \in D'-\Delta} \int H_D(y, dz)H_{D'}(z, D'-\Delta) < 1$.

Write $H_D f(x)$ for $\int H_D(x, dy)f(y)$. Under the above hypotheses it will be shown (Proposition 2.3) that for $D(r) \in \mathscr{E}, f \in \mathscr{C}$, $x \in K$ the function $r \to H_{D(r)}f(x)$, which is defined on a dense subset of [0, 1] (including J), has a left continuous extension to [0, 1]. The value of this extension at any r will be written as $H_{D(r)}f(x)$ and its integral $\int_0^1 H_{D(r)}f(x)\,dr$ often denoted by $H_{D[0,1]}f(x)$.

(g) For $x \neq \Delta$ and a neighborhood U of x there exist $D(r) \in \mathscr{E}$ with $x \in K-D(0) \subset K-D(1) \subset U$ and a neighborhood V of x such that if $f \in \mathscr{C}$, $H_{D[0,1]}f$ is continuous on V.

(h) If $D_n \downarrow \Delta$ and $D-\Delta$ is compact, $\int H_{D_n}(x, dy)H_D(y, D-\Delta) \to 0$ uniformly on compact subsets of $K-\Delta$.

THEOREM 1.1. *Under hypotheses* (a) *through* (h), *there exists a Hunt process on K, with Δ as the death point, such that starting at any x its hitting distribution of any $D \in \mathscr{D}$ is $H_D(x, \cdot)$. Its resolvent operators R_λ, $\lambda \geqq 0$, map $\mathscr{C}_0$ into $\mathscr{C}_0$.*

For the definition of a Hunt process (X_t) see [1], although as in [I] our definition of the hitting distributions is slightly different in that the hitting time T_A of a set A is defined to be the infimum of the nonnegative (rather than strictly positive) t with $X_t \in A$. Hypotheses (a) through (d) are the same as in [I]; hypothesis (e) as well as hypotheses (f) and (h) are immediate consequences of the hypotheses for [I; Theorem 1], i.e. hypotheses (a) through (d) and conditions (i) and (ii) in the introduction. Hypothesis (g) is of course satisfied when there exists a subclass $\mathscr{D}'$ of $\mathscr{D}$ such that the sets $K-D$, $D \in \mathscr{D}'$, have compact closure in $K-\Delta$ and form a base for the topology of $K-\Delta$, and for every $D \in \mathscr{D}', f \in \mathscr{C}$, the function $H_D f$ is continuous on $K-D$. This hypothesis seems to be a necessary condition for a process whose resolvent operators R_λ, $\lambda > 0$, satisfy some continuity condition, say, mapping real continuous functions (on $K-\Delta$) with compact support into continuous functions.[2] It is easy to show that in hypothesis (h) the pointwise convergence is a consequence of hypothesis (f); thus the content of hypothesis (h) is only the convergence being uniform on compact subsets of $K-\Delta$. It is tempting to do without this hypothesis and to relax hypothesis (f) to requiring only $\int H_D(x, dy)H_{D'}(y, D'-\Delta) < 1$ (i.e. assuming only the original transience condition in [I]), but we have not been able to do this. However, more comments concerning these two hypotheses will be given later.

A comparison of this result with that of Hansen ([2] Theorem 2) is worthwhile. We shall state his result briefly in our notation: a Hunt process on K with death point Δ can be constructed from a prescribed family of hitting distributions $H_D(x, \cdot)$, $x \in K$, D in a class $\mathscr{D}'$ as described above, provided they satisfy, besides hypotheses (a) through (d) above and a less serious condition (that holding points are isolated), the following conditions: (A) the lower semi-continuous (l.s.c.) excessive functions (i.e. nonnegative functions f with $H_D f \leqq f$ for all $D \in \mathscr{D}'$) separate points in $K-\Delta$; (B) for $D \in \mathscr{D}'$, $H_D f \in \mathscr{C}_0$ if $f \in \mathscr{C}_0$ and $H_D f$ is continuous on $K-D$ if $f \in \mathscr{M}_0$. It is not difficult to expand the family $\{H_D(x, \cdot) \mid x \in K, D \in \mathscr{D}'\}$ to a family $\{H_D(x, \cdot) \mid x \in K, D \in \mathscr{D}\}$ where $\mathscr{D}$ is a class satisfying the condition for the $\mathscr{D}$ given above, and show that hypotheses (a) through (h) are satisfied by this expanded family. The usefulness of the first part of condition (B) above ($H_D f \in \mathscr{C}_0$ if $f \in \mathscr{C}_0$) is only in this expansion; for this purpose one of the following conditions may perhaps be an interesting

[2] In the case where R_λ, $\lambda > 0$, map continuous functions vanishing at infinity into such functions and $\lambda R_\lambda f \to f$ uniformly as $\lambda \to \infty$ for such functions f, it is easy to show, using a theorem of A. V. Skorokhod on weak convergence of processes, that if $D(r)$ is a function from [0,1] into closed sets with $D(r) \subset \text{int } D(s)$ for $s < r$ and if $x_n \to x$, the hitting distribution $H_{D(r)}(x_n, \cdot)$ converges vaguely to $H_{D(r)}(x, \cdot)$ for all but countably many r; thus hypothesis (g) holds. The proof is essentially given in ([I] Section 7).

replacement: (I) the continuous excessive functions separate points in $K-\Delta$; (II) there exists a real excessive p such that for every $D \in \mathscr{D}'$ and every sequence x_n converging to a point $x \in D$ either $H_D(x_n, \cdot)$ converges vaguely to the point mass at x or $\liminf_n (p(x_n) - H_D p(x_n)) > 0$. Also, if we assume $\mathscr{D}''$ is a class of closed sets containing Δ as an interior point and closed under the formation of finite unions and finite intersections such that the sets $K-D$, $D \in \mathscr{D}''$, form a base for the topology of $K-\Delta$, and $\{H_D(x, \cdot) \mid x \in K, D \in \mathscr{D}''\}$ is a family of probability measures on K satisfying hypotheses (a) through (e) and (i) $H_D f$ is continuous on $K-D$ for $D \in \mathscr{D}''$, $f \in \mathscr{M}_0$ and (ii) the l.s.c. excessive functions separate points in $K-\Delta$, then via Theorem 1.1 we can find a Hunt process with the hitting distributions $H_D(x, \cdot)$.

As in [I] the major part of the work is to construct a process with prescribed hitting distributions as well as a prescribed time scale. Let g be a nonnegative Borel measurable function on K vanishing at Δ. We introduce the following hypotheses:

(j) g is bounded.

(k) For $x \neq \Delta$, a neighborhood U of x and $\varepsilon > 0$, there exists $\delta > 0$ such that if $H_D(x, K-U) > \varepsilon$ then $g_D(x) \equiv g(x) - H_D g(x) > \delta$.

(l) For $D(r) \in \mathscr{E}_0$ the function $g_{D[0,1]}(x) \equiv \int_0^1 {}_{D(r)}(x)\, dr = g(x) - H_{D[0,1]} g(x)$ is in $\mathscr{C}$.

(m) If $D_n \downarrow \Delta$, $g_{D_n}(x) \to g(x)$.

Theorem 1.2. *Under hypotheses* (a) *through* (m) *excluding* (f) *and* (h), *there exists a unique Hunt process on K, with Δ as the death point, such that starting at any x its hitting distribution of any $D \in \mathscr{D}$ is $H_D(x, \cdot)$ and its expected lifetime equals $g(x)$. Its resolvent operators R_λ, $\lambda \geqq 0$, map nonnegative* l.s.c. *functions into such functions. If $g \in \mathscr{C}_0$, the R_λ map $\mathscr{C}_0$ into $\mathscr{C}_0$.*

As we have seen, the statement of hypothesis (g) depends on hypothesis (f) in that $H_{D[0,1]} f$ makes sense for $D(r) \in \mathscr{E}$, $f \in \mathscr{C}$. Although hypothesis (f) is missing in the conditions of Theorem 1.2, the existence of a nonnegative g satisfying hypotheses (j) through (m) also guarantees that for $D(r) \in \mathscr{E}$, $f \in \mathscr{C}$ the function $r \to H_{D(r)} f(x)$ has a left continuous extension to [0, 1] (see Section 2). As remarked in [I], hypothesis (k) is a necessary condition. The function g we shall construct for Theorem 1.1 turns out to be in $\mathscr{C}_0$. If $g \in \mathscr{C}_0$, it is easy to show that hypothesis (h) must be satisfied, and if hypothesis (h) is satisfied, it can be shown that hypothesis (f) must also be satisfied (assuming of course all other hypotheses). We describe below a process satisfying all conditions of Theorem 1.2 but neither hypothesis (f) nor (h), and a variation of it satisfying all conditions of Theorem 1.2 and hypothesis (f) but not (h). Let K be the unit square $\{(a, b) \mid 0 \leqq a, b \leqq 1\}$ in the plane, with $\Delta = (1, 0)$. Define a process on K with death point Δ as follows: on the set $\{(a, b) \mid a < 1\}$ it is uniform motion to the right; any point $(1, b)$, $b > 0$, is a holding point with expected holding time 1, and from there a jump is made to (0, 0) with probability $q(b)$ and to $\Delta = (1, 0)$ with probability $1 - q(b)$. Assume $q(b)$ is continuous. Then all conditions of Theorem 1.2 are satisfied. If $\limsup_{b \to 0} q(b) = 1$, neither hypothesis (f) nor (h) is satisfied; if $0 < \limsup_{b \to 0} q(b) < 1$, hypothesis (f) is satisfied but (h) is not.

Hypothesis (l) can be replaced by the following weakened form: for $x \neq \Delta$ and a neighborhood U of x there exists $D(r)\in\mathscr{E}_0$ such that $x\in K-D(0)\subset K-D(1)\subset U$ and $g_{D[0,1]}$ is continuous. Also, in place of hypotheses (g) and (l) one can assume the following: for some $\mathscr{D}'\subset\mathscr{D}$ the sets $K-D$, $D\in\mathscr{D}'$, have compact closure in $K-\Delta$ and form a base for the topology of $K-\Delta$, and for every $D\in\mathscr{D}'$ the function g_D is continuous on $K-D$ and if $f\in\mathscr{C}$ the function H_Df is continuous on $K-D$. However we shall not justify these statements.

The outline of the work is as follows. In Section 2 we prove the continuity of the measures $H_{D[0,1]}(x,\cdot)$ for all $D(r)\in\mathscr{E}_0$ under conditions of either theorem. In Section 3 a suitable function g is defined for Theorem 1.1 in a manner similar to that in [I]. In Section 4 we define approximating processes in the same manner as in [I] and obtain the resolvent of the to-be-constructed process as the limit (in a weaker sense than that in [I]) of those of the approximating processes; here modifications of the proofs in [1] are called for. In defining the process, this time there is difficulty in obtaining the transition operators from the resolvent (and even if they can be obtained there seems to be difficulty in constructing a process from them and proving it has the right hitting distributions). However we find it possible to construct the process directly, using the projective limit process of the discrete skeletons of the approximating processes defined in ([I] Section 6). Based on ([I] Section 6) this direct construction is not difficult. (D. A. Dawson in his paper referred to in [I] also uses a direct construction, but his proofs are quite different.) In the case when no holding points are present it is actually rather short, and we carry out the proofs for this case; in the general case we shall only sketch the construction. These are done in Section 5. We remark that if we assume for both theorems H_Df is continuous on $K-D$ for $D\in\mathscr{D}$ and $f\in\mathscr{C}$ (replacing hypothesis (g)) and for Theorem 2 the function g is continuous on $K-\Delta$ (replacing hypothesis (l)), then Section 2 can be completely omitted and Section 4 largely omitted.

2. Continuity of $H_{D[0,1]}f$ for $D(r)\in\mathscr{E}_0, f\in\mathscr{C}$. All results in this section except Corollary 2.6 hold under either the conditions of Theorem 1.1 excluding hypothesis (h) or the conditions of Theorem 1.2. The proofs will be given for the first case. With obvious modifications they are valid when hypothesis (f) is replaced by the following condition: for $D\in\mathscr{D}$, $x\notin D$ there exists $D'\in\mathscr{D}$ containing x as an interior point such that

$$\sup_{y\in D'-\Delta}\int H_D(y,dy_1)\int_{D'-\Delta}H_{D'}(y_1,dy_2)\cdots\int H_D(y_{2n},dy_{2n+1})\int_{D'-\Delta}H_{D'}(y_{2n+1},dy_{2n+2})\downarrow 0.$$

But this condition holds under the conditions of Theorem 2. For hypothesis (k) implies $g_F>0$ on $K-F$ for $F\in\mathscr{D}$; consequently from hypothesis (l) one has $\inf_{y\in D'-\Delta}g_D(y)>0$ whenever $D'-\Delta$ is a compact subset of $K-D$, and the above convergence is clear.

In Lemma 2.1 and Lemma 2.2 let D_{ni}, $n\geqq 0$, $i\geqq 1$, be sets in $\mathscr{D}$ and let a stochastic process $(Z_n, n\geqq 0)$ on K have conditional probabilities

$$P(Z_{n+1}\in B\,|\,Z_n=y)=H_{D_{ni}}(y,B)\quad\text{on}\quad\Lambda_{ni}$$

where for each n, $\{\Lambda_{ni}, i \geqq 1\}$ is a partition of the sample space by sets in $\sigma(Z_1, \cdots, Z_n)$, the σ-algebra generated by $Z_1, \cdots, Z_n$. Let P_x denote the probability measure with $P_x[Z_0 = x] = 1$ and E_x denote the expectation with respect to P_x. Note that if $\Lambda_{ni} \in \sigma(Z_n)$ the process is Markov. For $F, G \subset K$ let $\tau_F = \inf\{n \geqq 0 \mid Z_n \in F\}$ (as always this is ∞ if $Z_n \in F$ for no n) and $\tau_{G,F} = \inf\{n \geqq \tau_G \mid Z_n \in F\}$. We shall use the convention $Z_\infty \equiv \Delta$.

LEMMA 2.1. *For $F \in \mathscr{D}$, $H_F(x, F-\Delta) \geqq P_x[Z(\tau_F) \in F-\Delta]$.*

PROOF. From ([I] 2.1)(Lemma 2.1 of [I]) $H_F(y, F-\Delta) \geqq \int H_{D_{ni}}(y, dz) H_F(z, F-\Delta) = H_{D_{ni}}(y, F-\Delta) + \int_{D_{ni}-F} H_{D_{ni}}(y, dz) H_F(z, F-\Delta)$. This implies

$$\begin{aligned} &E_x\{H_F(Z_n, F-\Delta); \tau_F \geqq n\} \\ &\quad = \textstyle\sum_i (P_x[\tau_{F-\Delta} = n, \Lambda_{ni}] + E_x\{H_F(Z_n, F-\Delta); \tau_F > n, \Lambda_{ni}\}) \\ &\quad = P_x[\tau_{F-\Delta} = n] + E_x\{H_F(Z_n, F-\Delta); \tau_F \geqq n+1\} \end{aligned}$$

and the lemma follows by induction.

LEMMA 2.2. *For $F, G \in \mathscr{D}$, $\int H_G(x, dy) H_F(y, F-\Delta) \geqq P_x[Z(\tau_{G,F}) \in F-\Delta]$.*

PROOF. Let us "refine" (Z_n) by adding random variables $Z_{n+\frac{1}{2}}$ and requiring them to satisfy

$$P(Z_{n+\frac{1}{2}} \in B \mid Z_n = y) = H_{D_{ni} \cup G}(y, B) \quad \text{on} \quad \Lambda_{ni}$$

$$P(Z_{n+1} \in B \mid Z_{n+\frac{1}{2}} = z) = H_{D_{ni}}(z, B) \quad \text{on} \quad \Lambda_{ni}\,.$$

This means that if $(Z_{m/2}, m \geqq 0)$ is defined by the above conditional probabilities, then with the time set restricted to the nonnegative integers it is the (Z_n) defined above. That this refinement can be done, i.e. that we have

$$P(Z_{n+1} \in B \mid Z_n = y) = \int P(Z_{n+\frac{1}{2}} \in dz \mid Z_n = y) P(Z_{n+1} \in B \mid Z_{n+\frac{1}{2}} = z),$$

is of course because of hypothesis (d). Let $\sigma_G = \inf\{m/2 \mid Z_{m/2} \in G\}$ and $\sigma_{G,F} = \inf\{m/2 \geqq \sigma_G \mid Z_{m/2} \in F\}$. Obviously $P_x[Z(\tau_{G,F}) \in F-\Delta] \leqq P_x[Z(\sigma_{G,F}) \in F-\Delta]$. Now it is easy to show that $P_x[Z(\sigma_G) \in \cdot] \leqq H_G(x, \cdot)$ (using hypothesis (d) and the reasoning in the previous proof). From Lemma 2.1 we then have $P_x[Z(\sigma_{G,F}) \in F-\Delta] \leqq \int H_G(x, dy) H_F(y, F-\Delta)$. The lemma is proved.

NOTATION. Let $B(x, \varepsilon) = \{y \mid \rho(x, y) < \varepsilon\}$, $E(x, \varepsilon) = K - B(x, \varepsilon)$.

PROPOSITION 2.3. *If $D_n \in \mathscr{D}$ for $n \geqq 0$ and $D_n \downarrow D$, $H_{D_n}(x, \cdot)$ converges vaguely.*

PROOF. Let (Z_n) be as defined above with $D_{ni} = D_n$ and Λ_{n1} equal to the entire sample space, i.e., $P(Z_{n+1} \in B \mid Z_n = y) = H_{D_n}(y, B)$. From hypothesis (d) $P_x[Z_{n+1} \in B] = H_{D_n}(x, B)$. We show Z_n converges a.s. P_x. Let $\varepsilon > 0$. From hypothesis (f) there exist $F_1, \cdots, F_l, G_1, \cdots, G_l$ in $\mathscr{D}$ and $c < 1$ such that $\operatorname{diam}(K - G_j) < \varepsilon$, $F_j - \Delta \subset K - G_j$, $E(\Delta, \varepsilon/2) \subset \bigcup_j F_j$ and

$$\sup_{y \in F_j - \Delta} \int H_{G_j}(y, dz) H_{F_j}(z, F_j - \Delta) < c$$

for all j. For a fixed j let $\tau = \tau_{G_j, F_j - \Delta}$ and define τ_n to be the iterates of $\tau : \tau_0 = 0$, $\tau_{n+1} = \tau_n + \tau(\theta_{\tau_n})$ where θ_m are the shift operators. Then since (Z_n) is Markov we obtain from Lemma 2.2 $P_x[\tau_n < \infty] < c^n$ (note this is independent of x, a fact to be used in the next proof). It follows that P_x[the sequence Z_n has oscillation $\geqq \varepsilon] = 0$.

Because of the above proposition, for any closed D containing Δ but not in $\mathscr{D}$ we can define $H_D(x, \cdot)$ to be the vague limit of $H_{D_n}(x, \cdot)$ where $D_n \in \mathscr{D}$, $D_n \downarrow D$ (it is independent of D_n). In particular if $D(r) \in \mathscr{E}$ there is a measure $H_{D(r)}(x, \cdot)$ for each r; because of hypothesis (e), $H_{D(r)}f(x)$ is left continuous in r for $f \in \mathscr{C}$.

PROPOSITION 2.4. *For $D(r) \in \mathscr{E}, f \in \mathscr{C}$, $\sum_k 2^{-m} H_{D(k2^{-m})} f(x)$ converges uniformly to $H_{D[0,1]} f(x) \equiv \int_0^1 H_{D(r)} f(x)\, dr$ (the sum is over $k = 1, \cdots, 2^m$).*

PROOF. It suffices to show the sum is uniformly Cauchy. For a fixed m define $(Z_k, 0 \leqq k \leqq 2^m)$ by requiring $P(Z_k \in B \mid Z_{k-1} = y) = H_{D(k2^{-m})}(y, B)$; again P_x denotes the probability measure with $P_x[Z_0 = x] = 1$. Let $\varepsilon > 0$. Define τ_j inductively by setting $\tau_0 = 0$, $\tau_{j+1} = \inf\{k > \tau_j \mid |f(Z_k) - f(Z(\tau_j))| < \varepsilon\}$ (assume $Z_\infty = \Delta$). Then from the previous proof we see that there exists j_0 such that $P_x[\tau_{j_0} < \infty] < \varepsilon$ for all x, independent of m. Now let $2^n > j_0/\varepsilon$ and $m > n$. On the set $\{\tau_{j_0} = \infty\}$ the total number of i such that there exists τ_j with $i2^{m-n} < \tau_j \leqq (i+1)2^{m-n}$ is smaller than j_0; but if there is no τ_j with $i2^{m-n} < \tau_j \leqq (i+1)2^{m-n}$ then $|f(Z_k) - f(Z_{i2^{m-n}})| < 2\varepsilon$ for $i2^{m-n} < k \leqq (i+1)2^{m-n}$. Hence, because $H_{D(k2^{-m})} f(x) = E_x\{f(Z_k)\}$, we have

$$\begin{aligned} &\left|\sum_{k=1}^{2^m} 2^{-m} H_{D(k2^{-m})} f(x) - \sum_{i=1}^{2^n} 2^{-n} H_{D(i2^{-n})} f(x)\right| \\ &\quad \leqq \sum_{i=1}^{2^n} 2^{-m} E_x\Big\{\sum_{(i-1)2^{m-n} < k \leqq i2^{m-n}} |f(Z_k) - f(Z_{i2^{m-n}})|; \tau_{j_0} = \infty\Big\} \\ &\qquad\qquad + 2\|f\| P_x[\tau_{j_0} < \infty] \\ &\quad < 2\varepsilon + 4\varepsilon \|f\|. \end{aligned}$$

PROPOSITION 2.5. *For $D(r) \in \mathscr{E}_0, f \in \mathscr{C}$ the function $H_{D[0,1]} f \in \mathscr{C}$.*

PROOF. We shall carry out the proof assuming hypothesis (g) is strengthened to the following extent: for $x \neq \Delta$ and a neighborhood U of x there exist $D \in \mathscr{D}$ with $x \in K - D \subset U$ and a neighborhood V of x such that $H_D f_1$ is continuous on V for all $f_1 \in \mathscr{C}$. In the general case one extends the present proof without much difficulty (but with much complication in notation) by using the uniform convergence in Proposition 2.4. It is clear that in proving the continuity of $H_{D[0,1]} f$ at an arbitrary x we may assume $x \notin D(0)$. In view of Proposition 2.4 it suffices to find, for given $\varepsilon > 0$, a neighborhood V of x and a sufficiently large m such that for $y \in V$

$$\left|\sum_k 2^{-m} H_{D(k2^{-m})} f(x) - \sum_k 2^{-m} H_{D(k2^{-m})} f(y)\right| < \varepsilon. \tag{2.1}$$

Let m be fixed. For $\varepsilon_1 > 0$ (to be determined in some way by ε) let $\delta > 0$ be such that $|f(y) - f(z)| < \varepsilon_1$ if $\rho(y, z) < \delta$. Choose F_j, G_j, $j = 1, \cdots, j(1), j(1)+1, \cdots, j(2^m)$ in $\mathscr{D}$ such that $\operatorname{diam}(K - G_j) < \delta$, $F_j - \Delta \subset K - G_j \subset K - D(k2^{-m})$ for $j \leqq j(k)$, $K - \operatorname{int} D(k-1)2^{-m}) \subset \bigcup_{j \leqq j(k)} \operatorname{int}(F_j - \Delta)$, $H_{G_j} f_1$ is continuous on $F_j - \Delta$ for all $f_1 \in \mathscr{C}$, and finally $\sup_{y \in F_j - \Delta} \int H_{G_j}(y, dz) H_{F_j}(z, F_j - \Delta) < 1$. Assume as we may

$x \in \inf(F_1 - \Delta)$. Define a process $(Z_n, n \geqq 0)$ as follows. First set $P(Z_1 \in B \mid Z_0 = y) - H_{G_1}(y, B)$. After $Z_0, \cdots, Z_n$ are defined (as usual the initial distribution is left unspecified) let Borel sets A_{nj}, $1 \leqq j \leqq j(2^m)+1$, be a partition of K satisfying $A_{nj} \subset F_j - \Delta$ for $j \leqq j(2^m)$, $K - \operatorname{int} D((k-1)2^{-m}) \subset \bigcup_{j \leqq j(k)} A_{nj}$ and $P_x[Z_n \in \partial A_{nj}] = 0$ for all $j(\partial A$ denotes the boundary of A). Then set

$$P(Z_{n+1} \in B \mid Z_n = y) = H_{G_j}(y, B), y \in A_{nj}, j \leqq j(2^m)$$
$$= y, y \in A_{n,j(2^m)+1} \, .$$

By construction it is clear that the finite dimensional distributions of (Z_n) under P_y converge vaguely to the corresponding ones of (Z_n) under P_x as $y \to x$. From the previous proof we must have

$$P_y[\tau_{D(1-2^{-m})} > n] \to 0 \quad \text{uniformly in} \quad y, \qquad n \to \infty. \tag{2.2}$$

We now refine (Z_n) by adding $Z_{n+\frac{1}{2}}$ and requiring

$$P(Z_{n+\frac{1}{2}} \in B \mid Z_n = y) = H_{G_j \cup D(k2^{-m})}(y, B),$$
$$P(Z_{n+1} \in B \mid Z_n = y, Z_{n+\frac{1}{2}} = z) = H_{G_j}(z, B)$$

for $y \in A_{nj} \cap D((k-1)2^{-m}) - D(k2^{-m})$, $j \leqq j(2^m)$, and $Z_{n+\frac{1}{2}} = Z_n$ if $Z_n \in A_{n,j(2^m)+1}$. Let $\sigma_A = \inf\{n/2 \mid Z_{n/2} \in A\}$ (recall $\tau_A = \inf\{n \mid Z_n \in A\}$). Then, for any y, $\sigma_{D(k2^{-m})} < \infty$ a.s. P_y and $P_y[Z(\sigma_{D(k2^{-m})}) \in \cdot] = H_{D(k2^{-m})}(y, \cdot)$ for $k < 2^m$. Trivially $\sigma_{D(k2^{-m})} \leqq \tau_{D(k2^{-m})} \leqq \sigma_{D((k+1)2^{-m})}$. Now using (2.2), a calculation similar to that in the previous two proofs on the oscillation of the paths of $(Z_{n/2})$, and the vague convergence mentioned above, one can find for any sufficiently large m a neighborhood V of x such that (2.1) holds, after having chosen suitable ε_1. We omit this detail.

As mentioned earlier for any closed set D containing Δ and $x \in K$ we have a measure $H_D(x, \cdot)$. Does the family $\{H_D(x, \cdot) \mid x \in K,\ D$ closed and containing $\Delta\}$ satisfy hypothesis (d)? The answer is yes. Its proof depends on the above proposition. This proof is relatively easy and will not be given. However there will be occasions (not serious) that we shall use this fact.

COROLLARY 2.6. *Under conditions of Theorem* 1.1 (*including hypothesis* (h)) $H_{D[0,1]}f \in \mathscr{C}_0$ *if* $D(r) \in \mathscr{E}_0, f \in \mathscr{C}_0$.

PROOF. Using hypothesis (h) one can easily show that for any compact $F \subset K - \Delta$ there exists $f_1 \in \mathscr{C}$ which differs from f only on a small neighborhood of Δ so that $\sup_{x \in F} |H_{D[0,1]}f(x) - H_{D[0,1]}f_1(x)|$ is small. The corollary follows immediately from the previous proposition.

3. The function g. We now assume the conditions of Theorem 1.1 and define a function g satisfying hypotheses (j) through (m). Let $\mathscr{E}_1$ denote the class of $D(r)$ in $\mathscr{E}$ such that $D(r) - \Delta$ is compact for all r and $D(r) - \Delta \subset \operatorname{int}(D(s) - \Delta)$ for $s < r$. Choose $D_k(r) \in \mathscr{E}_1$, $k \geqq 1$, such that for $x \neq \Delta$, $\varepsilon > 0$ there is $D_k(r)$ with $x \in D_k(1) - \Delta \subset D_k(0) - \Delta \subset B(x, \varepsilon)$. Let $a_k > 0$ with $\sum_k a_k < \infty$ and define

$$g(x) = \sum_{k=1}^{\infty} a_k \int_0^1 H_{D_k(r)}(x, D_k(r) - \Delta)\, dr.$$

Of course g is nonnegative and bounded, and vanishes at Δ. If $f_k \in \mathscr{C}$ and $f_k = 1$ on $D_k(0)-\Delta$, $f_k(\Delta) = 0$, then $g(x) = \sum_k a_k H_{D_k[0,1]} f_k(x)$. g is therefore continuous on $k-\Delta$ from the following proposition.

PROPOSITION 3.1. *For $D(r) \in \mathscr{E}_1$ and $f \in \mathscr{C}$, $H_{D[0,1]}f$ is continuous on $K-\Delta$.*

PROOF. Let $F(r) \in \mathscr{E}_0$ be such that $F(1) = \{\Delta\}$ and let

$$F_k(r) = D(r) \cup F(1-2^{-k}(1-r)).$$

Then $H_{F_k[0,1]}f \in \mathscr{C}$ by Proposition 2.5. But using hypothesis (h) one easily shows that it converges to $H_{D[0,1]}f$ uniformly on compact subsets of $K-\Delta$.

If $D(r) \in \mathscr{E}_0$, Corollary 2.6 implies $H_{D[0,1]}g$ is continuous on $K-\Delta$; consequently $g_{D[0,1]} = g - H_{D[0,1]}g \in \mathscr{C}_0$, establishing hypothesis (l). The proof that g satisfies hypothesis (k) is similar to that in [I], the modification being obvious. It is easy to establish hypothesis (m); in fact we can show more.

PROPOSITION 3.2. *If $D_n \downarrow \Delta$, $H_{D_n}g \to 0$ (so that $g_{D_n} \to g$) uniformly on compact subsets of $K-\Delta$.*

PROOF. $\int H_{D_n}(x, dy)g(y) = \sum_k a_k \int_0^1 dr \int H_{D_n}(x, dy) H_{D_k(r)}(y, D_k(r)-\Delta)$. Now each term $\int_0^1 dr \int H_{D_n}(x, dy) H_{D_k(r)}(y, D_k(r)-\Delta)$, being no larger than $\int H_{D_n}(x, dy) H_{D_k(0)}(y, D_k(0)-\Delta)$, converges to 0 uniformly on compact subsets of $K-\Delta$ by hypothesis (h).

Thus we have shown that g satisfies hypotheses (j) through (m); consequently Theorem 1.1 follows from Theorem 1.2. We now begin the proof of Theorem 1.2. As remarked at the beginning of Section 2, all results in that section except Corollary 2.6 are valid.

PROPOSITION 3.3. *If $D_n \downarrow D$, $g_{D_n} \uparrow g_D$.*

PROOF. Assume first $\Delta \notin \operatorname{int} D$. Let $F(r) \in \mathscr{E}_0$ be such that $F(0) \subset D$. Then

$$\begin{aligned} &g_{F[0,1]}(x) - \int H_D(x, dy) g_{F[0,1]}(y) \\ &\qquad = g(x) - H_{F[0,1]}g(x) - \int H_D(x, dy)(g(y) - H_{F[0,1]}g(y)) = g_D(x) \end{aligned}$$

because of the remark preceding Corollary 2.6 (that hypothesis (d) holds for all closed sets D, D' containing Δ with $D \subset D'$). Similarly

$$g_{D_n}(x) = g_{F[0,1]}(x) - \int H_{D_n}(x, dy) g_{F[0,1]}(y).$$

Now the proposition follows from the vague convergence of $H_{D_n}(x, \cdot)$ to $H_D(x, \cdot)$ and the continuity of $g_{F[0,1]}$. If $\Delta \notin \operatorname{int} D$, the convergence follows easily from the above established case and hypothesis (m).

Note that for $D(r) \in \mathscr{E}_0$ letting $F_k(r) = D(1-2^{-k}(1-r))$ we have $g_{F_k[0,1]} \uparrow g_{D(1)}$. Hence g_D is l.s.c. for every $D \in \mathscr{D}$; in particular g is l.s.c.

4. Approximating processes and convergence of their resolvents. The approximating processes $X^{(n)}$ are the same as in [I]. We choose a sequence $(\mathfrak{U}_n, \mathscr{V}_n)$ where $\mathscr{V}_n = \{V_{n1}, \cdots, V_{mn}\}$ is a partition of K by Borel sets and $\mathfrak{U}_n = \{U_{n1}, \cdots, U_{nm_n}\}$ is a subclass of $\mathcal{O}$ such that $\max_i \operatorname{diam} U_{ni} < 1/n$, $V_{ni} = U_{ni} - \bigcup_{j<i} U_{nj}$ (in particular $V_{ni} \subset U_{ni}$), and $(\mathfrak{U}_n, \mathscr{V}_n) \subset (\mathfrak{U}_k, \mathscr{V}_k)$ for $k \leqq n$, i.e., whenever $V_{ni} \cap U_{kj} \neq \varnothing$ we have $U_{ni} \subset U_{kj}$. Denote by $\mathscr{D}_n$ the class of sets of the form $(K-U) \cup \Delta$, U a (finite) union of sets in $\mathfrak{U}_n$, and let $\mathscr{D}_\infty = \bigcup_n \mathscr{D}_n$. $\mathscr{D}_n$ is closed under the formation of finite unions and finite intersections, $\mathscr{D}_n \uparrow \mathscr{D}_\infty$ and $\mathscr{D}_\infty \subset \mathscr{D}$. Let $D(n, x) = (K - U_{ni}) \cup \Delta$ for $x \in V_{ni}$; it is the largest set in $\mathscr{D}_n$ not containing x, and so if $x \notin D$ where $D \in \mathscr{D}_k$ for some $k \leqq n$, we have $D \subset D(n, x)$. Let

$$Q_n(x, B) = H_{D(n,x)}(x, B), \quad h_n(x) = g_{D(n,x)}(x), \quad e_n(x) = 1/h_n(x).$$

We construct first as in ([I] Section 4) the discrete skeletons $Z^{(n)} = (Z_\alpha, Q_x{}^{(n)})$ of the jump processes $X^{(n)}$. The $Z^{(n)}$ have as their common sample space the product space $\mathscr{W} = \prod_{\alpha<\pi} K_\alpha$, where $K_\alpha = K$ and π denotes the ordinal ω^ω (ω is the first infinite ordinal), and Z_α, $\alpha < \pi$, is the αth coordinate. $Z^{(n)}$ has one-step transition probability $Q_n(x, B)$ and satisfies the left continuity

$$\text{(4.1)} \qquad \text{if} \quad \alpha_m \uparrow \alpha, \quad Z_{\alpha_m} \to Z_\alpha \quad \text{a.s.} \quad \text{(i.e. a.s. } Q_x{}^{(n)} \text{ for all } x).$$

Since the conditions are now different the construction must be justified. Again this is done by successive extension of the measures $Q_x{}^{(n)}$ on the σ-algebras $\sigma(Z_\alpha, \alpha < \beta)$. For a fixed $\beta < \pi$ we assume the $Q_x{}^{(n)}$ are defined on $\sigma(Z_\alpha, \alpha < \beta)$ and (4.1) and

$$\text{(4.2)} \quad H_D(x, B) = Q_x{}^{(n)}[Z(\tau_D) \in B; \tau_D < \alpha] + \int Q_x{}^{(n)}[Z_\alpha \in dy; \tau_D \geqq \alpha] H_D(y, B), D \in \mathscr{D}_n$$

(where $\tau_A = \inf\{\gamma < \pi \mid Z_\gamma \in A\}$ if there is such γ, and $= \pi$ otherwise) hold for all $\alpha < \beta$, and show that

$$\text{(4.3)} \qquad \text{if} \quad \alpha_m \uparrow \beta, \qquad Z_{\alpha_m} \quad \text{converges a.s.}$$

and that with $Q_x{}^{(n)}$ extended to $\sigma(Z_\alpha, \alpha \leqq \beta)$ by the obvious requirement (4.2) holds when α is replaced by β. There is nothing to show when $\beta = 0$, and little to show when β has a predecessor α: using the fact $D \subset D(n, y)$ for $y \notin D$, $D \in \mathscr{D}_n$ one can write the second term on the right of (4.2) as

$$Q_x{}^{(n)}[Z(\tau_D) \in B; \tau_D = \alpha] + \int Q_x{}^{(n)}[Z_{\alpha+1} \in dy; \tau_D \geqq \alpha + 1] H_D(y, B)$$

by hypothesis (d). Thus we assume β is a limit ordinal and may indeed assume β is such that $\gamma + \beta = \beta$ for $\gamma < \beta$.

PROPOSITION 4.1. *(4.3) is valid.*

PROOF. Suppose the contrary. Then it is easy to obtain (see [I; 4.2]) distinct x_1, x_2 with $x_1 \neq \Delta$ such that for any $F_1, F_2 \in \mathscr{D}$ with $x_1 \in \operatorname{int} F_1$, $x_2 \in \operatorname{int} F_2$ we have

$$\text{(4.4)} \qquad \sup_{x \in F_1 - \Delta} Q_x{}^{(n)}[Z(\tau_{2m}) \in F_1 - \Delta; \tau_{2m} < \beta] = 1, \qquad m \geqq 0$$

where the τ_m are defined inductively by: $\tau_0 = 0$, $\tau_{2m} = \tau_{2m-1} + \tau_{F_1}(\theta_{\tau_{2m-1}})$, and $\tau_{2m+1} = \tau_{2m} + \tau_{F_2}(\theta_{\tau_{2m}})$($\theta_\alpha$ is the shift operator satisfying $Z_\gamma(\theta_\alpha) = Z_{\alpha+\gamma}$; $\tau_{m+1} = \pi$ if $\tau_m = \pi$). Let F_1, F_2 satisfy the above conditions and be such that $F_1 - \Delta$ is a compact subset of $K-F_2$. Define $(Y_m, m \geqq 0)$ by requiring $P(Y_{2m+1} \in B \mid Y_{2m}=y) = H_{F_2}(y, B)$, $P(Y_{2m} \in B \mid Y_{2m-1}=y) = H_{F_1}(y, B)$. Then since $\inf\{g_{F_2}(g) \mid y \in F_1-\Delta\} > 0$ we must have

$$\sup_{x \in F_1-\Delta} P_x[Y_{2m} \in F_1-\Delta] \to 0$$

where P_x denotes the probability measure with $P_x[Y_0=x]=1$. We show that for any m and x

$$P_x[Y_{2m} \in F_1-\Delta] \geqq Q_x^{(n)}[Z(\tau_{2m}) \in F_1-\Delta; \tau_{2m} < \beta], \tag{4.5}$$

which contradicts (4.4). In a manner similar to that in the proof of Lemma 2.2 we can refine $Z^{(n)}$ by introducing symbolic times $\alpha+j/2m$, $1 \leqq j < 2m$, $\alpha < \pi$, and defining $Z(\alpha+j/2m)$ by the conditional probabilities

$$Q_x^{(n)}\left(Z\left(\alpha+\frac{2j+1}{2m}\right) \in B \mid Z(\alpha)=y, Z\left(\alpha+\frac{j}{m}\right)=z\right) = H_{D(n,y)\cup F_2}(z, B)$$

$$Q_x^{(n)}\left(Z\left(\alpha+\frac{j+1}{m}\right) \in B \mid Z(\alpha)=y, Z\left(\alpha+\frac{2j+1}{2m}\right)=z\right) = H_{D(n,y)\cup F_1}(z, B).$$

Let σ_k, $k \geqq 0$, be defined as follows:

$$\sigma_0 = 0,$$

$$\sigma_{2k+1} = \inf\{\alpha+j/(2m) \geqq \tau_{2k} \mid Z(\alpha+j/(2m)) \in F_2\}$$

(if there is no such $\alpha+j/(2m)$ its value is π), and

$$\sigma_{2k+2} = \inf\{\alpha+j/(2m) \geqq \tau_{2k+1} \mid Z(\alpha+j/(2m)) \in F_1\}.$$

We may assume that the induction hypothesis holds for this refined process, so that

$$Q_x^{(n)}(Z(\sigma_{2k+1}) \in B, \sigma_{2k+1} < \beta \mid Z(\sigma_{2k}) = y, \sigma_{2k} < \beta) \leqq H_{F_2}(y, B)$$

$$Q_x^{(n)}(Z(\sigma_{2k+2}) \in B, \sigma_{2k+2} < \beta \mid Z(\sigma_{2k+1}) = y, \sigma_{2k+1} < \beta) \leqq H_{F_1}(y, B)$$

and consequently $Q_x^{(n)}[Z(\sigma_{2m}) \in F_1-\Delta; \sigma_{2m} < \beta] \leqq P_x[Y_{2m} \in F_1-\Delta]$. But obviously the left-hand side of the last inequality dominates $Q_x^{(n)}[Z(\tau_{2m}) \in F_1-\Delta; \tau_{2m} < \beta]$, proving (4.5) and therefore establishing the proposition.

Now extend the $Q_x^{(n)}$ to $\sigma(Z_\alpha, \alpha \leqq \beta)$ by requiring $Z_{\alpha_m} \to Z_\beta$ a.s. $Q_x^{(n)}$ if $\alpha_m \uparrow \beta$.

PROPOSITION 4.2. (4.2) *holds when* α *is replaced by* β.

PROOF. Let $D(r) \in \mathscr{E}_0$ with $D(1)=D$ and let $D_k(r) = D(1-2^{-k}(1-r))$. For a fixed k define $\tilde{Z}^{(n)} = (Z_\alpha, \tilde{Q}_x^{(n)})_{\alpha \leqq \beta}$ as follows: let $\tilde{D}(n, x) = D(n, x) \cup D_k(0)$ for $x \notin D_k(0)$, and $= D(n, x)$ otherwise; then require $\tilde{Z}^{(n)}$ to have one-step transition probability $\tilde{Q}_n(x, B) = H_{\tilde{D}(n,x)}(x, B)$. Of course we may assume $\tilde{Z}^{(n)}$ satisfies the properties established for $Z^{(n)}$. Now clearly $\tilde{Z}^{(n)}$ killed at the time $\tau_{D(0)}$ is dominated by $(Z_\alpha, Q_x^{(n)})_{\alpha \leqq \beta}$ killed at τ_D, and using hypothesis (e) it is not difficult to show that it actually increases to the latter (i.e. its finite dimensional distributions increase to those of the latter as $k \to \infty$). Also, using hypotheses (e) and (d) it is easy to show that, as $k \to \infty$, $\tilde{Q}_x^{(n)}[Z(\tau_{D_k(0)}) \in dy;\ \tau_{D_k(0)} \leqq \beta]$ converges vaguely to $Q_x^{(n)}[Z(\tau_D) \in dy;\ \tau_D \leqq \beta]$, and for $f \in \mathscr{C} \int \tilde{Q}_x^{(n)}[Z(\tau_{D_k(0)}) \in dy;\ \tau_{D_k(0)} \leqq \beta] H_{D_k[0,1]} f(y) \to \int Q_x^{(n)}[Z(\tau_D) \in dy;\ \tau_D \leqq \beta] f(y)$. Now for $f \in \mathscr{C}$ using the continuity of $H_{D_k[0,1]} f$ (Proposition 2.5) we have

$$H_{D_k[0,1]} f(x) = \int \tilde{Q}_x^{(n)}[Z(\tau_{D_k(0)} \wedge \beta) \in dy] H_{D_k[0,1]} f(y).$$

Letting $k \to \infty$ in this equality we obtain from the above statements the relation $H_D f(x) = \int Q_x^{(n)}[Z(\tau_D \wedge \beta) \in dy] H_D f(y)$, which is (4.2) when α is replaced by β.

The construction of $Z^{(n)}$ is thus justified.

PROPOSITION 4.3. *For* $D \in \mathscr{D}_n$, $g_D(x) \geqq \hat{E}_x^{(n)}\{\sum_{\alpha < \tau_D} h_n(Z_\alpha)\}$ ($\hat{E}^{(n)}$ *denotes the expectation with respect to* $Q_x^{(n)}$).

PROOF. We use the same technique as in the previous proof. Let $D(r)$, $D_k(r)$ and $\tilde{Z}^{(n)}$ be as in there. Then with $\tilde{h}_n(y) = g_{\tilde{D}(n,y)}(y)$ and using the fact that, for $x \notin D_k(0)$, $g_{D_k[0,1]}(x) = g_{\tilde{D}(n,x)}(x) + \int H_{\tilde{D}(n,x)}(x, dy) g_{D_k[0,1]}(y)$ we obtain from (4.1) (applied to $\tilde{Z}^{(n)}$) and the continuity of $g_{D_k[0,1]}$

$$g_{D_k[0,1]}(x) = \tilde{E}_x^{(n)}\{\textstyle\sum_{\alpha < \tau_{D_k(0)}} \tilde{h}_n(Z_\alpha)\} + \tilde{E}_x^{(n)}\{g_{D_k[0,1]}(\lim_{\alpha \uparrow \pi} Z_\alpha);\ \tau_{D_k(0)} = \pi\}$$

where $\tilde{E}_x^{(n)}$ denotes the expectation with respect to $\tilde{Q}_x^{(n)}$; note that from the proof of Proposition 4.1 Z_{α_m} converges a.s. if $\alpha_m \uparrow \pi$ and we denote by $\lim_{\alpha \uparrow \pi} Z_\alpha$ such a limit. Since as $k \to \infty$ $\tilde{Z}^{(n)}$ killed at $\tau_{D_k(n)}$ increases to $Z^{(n)}$ killed at τ_D and $\tilde{h}_n \uparrow h_n$ we have from Proposition 3.3

$$g_D(x) = \hat{E}_x^{(n)}\{\textstyle\sum_{\alpha < \tau_D} h_n(Z_\alpha)\} + \hat{E}_x^{(n)}\{g_D(\lim_{\alpha \uparrow \pi} Z_\alpha);\ \tau_D = \pi\}. \tag{4.6}$$

The proposition follows.

Since g_D is l.s.c. and strictly positive on $K - D$, we again have as in ([I] 4.5) that $Z_\alpha = \Delta$ a.s. for all $\alpha \geqq \omega^{m_n}$ (m_n is the cordinality of $\mathfrak{U}_n$). The construction of the jump process $X^{(n)} = (X_t, P_x^{(n)})$ from $Z^{(n)}$ is the same as in [I]. Let $T_A = \inf\{t \geqq 0 \mid X_t \in A\}$, $H_A^{(n)}(x, B) = P_x^{(n)}[X(T_A) \in B;\ T_A < \infty] = Q_x^{(n)}[Z(\tau_A) \in B;\ \tau_A < \pi]$ for Borel A. Then from (4.2)

$$H_D^{(n)}(x, \cdot) = H_D(x, \cdot), \qquad D \in \mathscr{D}_n. \tag{4.7}$$

Also since $Q_x^{(n)}[\tau_D = \pi] = 0$ for $D \in \mathscr{D}_n$ (4.6) gives

$$g_D(x) = \hat{E}_x^{(n)}\{\textstyle\sum_{\alpha < \tau_D} h_n(Z_\alpha)\} = E_x^{(n)} T_D, \qquad D \in \mathscr{D}_n, \tag{4.8}$$

($E_x^{(n)}$ denotes the expectation with respect to $P_x^{(n)}$), in particular

$$g(x) = \hat{E}_x^{(n)}\{\textstyle\sum_{\alpha < \pi} h_n(Z_\alpha)\} = E_x^{(n)} T_\Delta. \tag{4.9}$$

Let $\{R_\lambda^{(n)}, \lambda \geqq 0\}$ be the resolvent of $X^{(n)}$ on $\mathscr{M}_0$:

$$R_\lambda^{(n)} f(x) = E_x^{(n)} \int_0^\infty e^{-\lambda t} f(X_t)\, dt = \hat{E}_x^{(n)}\left\{\sum_{\alpha < \pi}\left(\prod_{\beta < \alpha} \frac{e_n(Z_\beta)}{\lambda + e_n(Z_\beta)}\right) \frac{1}{\lambda + e_n(Z_\alpha)} f(Z_\alpha)\right\}.$$

(See [I] Section 4 for the second equality; the infinite product in the last integrand is defined in the obvious manner, with the understanding that its value is 1 if $\alpha=0$ and 0 if $Z_\beta=\Delta$ for some $\beta<\alpha$.) Then $\|R_\lambda^{(n)}\|\leqq\min\{\|g\|, 1/\lambda\}$.

PROPOSITION 4.4. *For $f\in\mathscr{C}$, $R_0^{(n)}f$ converges uniformly. In the case $g\in\mathscr{C}_0$, $R_0^{(n)}f$ converges uniformly on compact subsets of $K-\Delta$ for $f\in\mathscr{C}_0$.*

PROOF. The proof is the same as that of [I, 5.1]; for the second statement one uses the fact $H_{D_m}g\to 0$ uniformly on compact subsets of $K-\Delta$ if $D_m\downarrow\Delta$.

PROPOSITION 4.5. *For $f\in\mathscr{C}$, let $R_0f=\lim_n R_0^{(n)}f$; then R_0f is* l.s.c. *if $f\geqq 0$. In the case $g\in\mathscr{C}_0$, $R_0f\in\mathscr{C}_0$ for $f\in\mathscr{C}_0$.*

PROOF. If we make the assumption that for $x\neq\Delta$ and neighborhood U of x there exists $D\in\mathscr{D}_\infty$ with $x\in K-D\subset U$ such that g_D is continuous on $K-D$ and H_Df is continuous on $K-D$ for all $f\in\mathscr{C}$, the proof is essentially the same as that of [I; 5.2], using the fact that $H_{D_m}g\to 0$ pointwise (and uniformly on compact subsets of $K-\Delta$ if $g\in\mathscr{C}_0$) for $D_m\downarrow\Delta$. The extension of this proof to the general case does not pose real difficulty and will not be given (see also the proof of Proposition 2.5).

In the case $g\in\mathscr{C}_0$ one obtains easily that $R_\lambda^{(n)}f$ converges, uniformly on compact subsets of $K-\Delta$, to a limit $R_\lambda f$ in $\mathscr{C}_0$ by induction based on the above results, the equation

$$R_\lambda^{(n)}f=\textstyle\sum_{l=0}^\infty(\mu-\lambda)^l(R_\mu^{(n)})^{l+1}f,\qquad |\lambda-\mu|<\|g\|^{-1}$$

and the fact that for $D_m\downarrow\Delta$, $R_\mu^{(n)}1_{D_m}\leqq H_{D_m}g\to 0$ uniformly on compact subsets of $K-\Delta$. In the general case a delicate consideration is needed for obtaining the limit resolvent $\{R_\lambda\}$ and its properties.

PROPOSITION 4.6. (i) *for $\lambda\geqq 0$, $f\in\mathscr{C}_0$, $R_\lambda^{(n)}f$ converges pointwise to a limit $R_\lambda f$;* (ii) *if $f\geqq 0$, $R_\lambda f$ is* l.s.c.; (iii) $R_\lambda f-R_\mu f=(\mu-\lambda)R_\lambda R_\mu f$; (iv) $\|R_\lambda f\|\leqq\|f\|\min\{\|g\|, 1/\lambda\}$.

PROOF. Choose $D(r)\in\mathscr{E}_0$ with $D(1)=\{\Delta\}$ and $D(r)\in\mathscr{D}_\infty$ for $r\in J$. By considering a subsequence of $X^{(n)}$ we may assume $D(k2^{-n})\in\mathscr{D}_n$ for all k. Let $T_{nk}=T_{D(k2^{-n})}$ and $S_\lambda^{n,k}f(x)=E_x^{(n)}\int_0^{T_{nk}}e^{-\lambda t}f(x_t)\,dt$. We shall prove that for $f\in\mathscr{C}_0$ with $f\geqq 0$

(4.10) $\quad\sum_{k=1}^{2^n}2^{-n}S_\lambda^{n,k}f(x)$ converges uniformly to a continuous function.

If $D(0)$ is sufficiently small, then for a fixed x and $\varepsilon>0$

$$\left|R_\lambda^{(n)}f(x)-\textstyle\sum_k 2^{-n}S_\lambda^{n,k}f(x)\right|\leqq\|f\|H_{D(0)}g(x)<\varepsilon$$

for all n; (i) and (ii) thus follow immediately. We prove (4.10) by induction. Consider first the case $\lambda=0$. Let $r\in J$ with $r<1$, and let $k_n2^{-n}=r$ from some n_0 on. Since f is continuous on $K-\operatorname{int}D(r)$ it follows from Proposition 4.4. and Proposition 4.5 that $S_0^{n,k_n}f$ converges uniformly to an l.s.c. function, to be denoted by S_0^rf. For a fixed x, $S_0^rf(x)$ is of course increasing with r; hence we extend it to all $r\in[0, 1]$ by taking left limits. Proposition 3.3 implies that $S_0^rf(x)$ is in fact left continuous. If r, $s\in J$ with $s<r$, then since $g_{D(r)}(y)\geqq\int_s^r(r-s)^{-1}g_{D(u)}(y)\,du\geqq g_{D(s)}(y)$ and the middle term is continuous in y by hypothesis (l), it follows from the proof

of Proposition 4.5 that $S_0{}^r f(x) \geqq \limsup_{y\to x} S_0{}^s f(y)$. Now (4.10) clearly holds when $\lambda = 0$. To establish the induction step assume $0 < \lambda - \mu \leqq \|g\|^{-1}$ and there exists for each $r \in [0, 1]$ a nonnegative $S_\mu{}^r f$ such that (a) $S_0{}^r f$ is l.s.c., (b) $S_0{}^r f(x)$ is increasing and left continuous in r, (c) for $s < r$, $S_\mu{}^r f(x) \geqq \limsup_{y\to x} S_\mu{}^s f(y)$, (d) for $s < r$, $\sum_{s2^n < k \leqq r2^n} 2^{-n} S_\mu{}^{n,k} f(x)$ converges to $\int_s^r S_\mu{}^u f(x)\,du$ uniformly in x, in particular if $k_n 2^{-n} = r \leqq 1$ for all large n, $S_\mu{}^{n,k_n} f(x) \to S_\mu{}^r f(x)$, and (e) for $s < r$, $\int_s^r S_\mu{}^u f(x)\,du$ is continuous. Then for $s < r$, s, $r \in J$, and with $k_n 2^{-n} = r$ for all large n

$$S_\mu{}^{n,k_n}\left(\sum_{s2^n<k\leqq r2^n} 2^{-n} S_\mu{}^{n,k} f\right)(x) \to S_\mu{}^r\left(\int_s^r S_\mu{}^u f\,du\right)(x)$$

uniformly in x. Let $(S_\mu{}^r)^2 f(x) = \lim_{s\uparrow r} S_\mu{}^r((r-s)^{-1}\int_s^r S_\mu{}^u f\,du)(x)$ and extend it to all $r \in [0, 1]$ by using left limits. Then it is easy to verify that (a) through (e) above are all valid when $(S_\mu{}^r)^2 f$ and $(S_\mu{}^{n,k})^2 f$ replace respectively $S_\mu{}^r f$ and $S_\mu{}^{n,k} f$. By induction we find $(S_\mu{}^r)^l f$ for $l > 1$ that satisfy (a) through (e) (with $(S_\mu{}^{n,k})^l f$ replacing $S_\mu{}^{n,k} f$). Now for $s < r$

$$\begin{aligned}\sum_{s2^n<k\leqq r2^n} 2^{-n} S_\lambda{}^{n,k} f(x) &= \sum_k 2^{-n} \sum_{l=0}^\infty (\mu-\lambda)^l (S_\mu{}^{n,k})^{l+1} f(x)\\ &= \sum_{l=0}^\infty (\mu-\lambda)^l \sum_k 2^{-n} (S_\mu{}^{n,k})^{l+1} f(x)\\ &\to \sum_{l=0}^\infty (\mu-\lambda)^l \int_s^r (S_\mu{}^u)^{l+1} f(x)\,du\end{aligned}$$

uniformly and the limit is continuous. Dividing the last expression by $r-s$ and letting $s \uparrow r$ we get a limit $S_\lambda{}^r f(x)$. (a) through (e) are satisfied when μ is replaced by λ. Hence by induction (a) through (e) are true for all μ, and we have established (4.10) and thus (i) and (ii). From the above reasoning it is also easy to get

$$R_\lambda{}^{(n)} R_\mu{}^{(n)} f = S_\lambda{}^{n,2^n} S_\mu{}^{n,2^n} f \to S_\lambda{}^1 S_\mu{}^1 f = R_\lambda R_\mu f$$

pointwise; (iii) then follows from the resolvent equation of $\{R_\lambda{}^{(n)}\}$. (iv) of course needs no proof.

COROLLARY 4.7. *Let $D \in \mathscr{D}_\infty$. Then for a nonnegative $f \in \mathscr{C}_0$ and $\lambda \geqq 0$ the function $E_x{}^{(n)} \int_0^{T_D} e^{-\lambda t} f(X_t)\,dt$ converges pointwise to an* l.s.c. *limit. In particular $E_x{}^{(n)} e^{-\lambda T_D}$ converges pointwise to an* u.s.c. *function.*

PROOF. This follows from Proposition 4.6 by considering the $X^{(n)}$ killed at T_D for all large n.

5. Construction of the process. As mentioned in Section 1 we shall construct the desired process directly. Let $Z^{(\infty)} = (\Omega, Z_\alpha{}^{(n)}, p^x)_{\alpha<\pi, n\geqq 1, x\in K}$ be the *projective limit process* of the $Z^{(n)}$ constructed in ([I] Section 6) (where the notation is $(\mathscr{W}_\infty, Z_\alpha{}^{(n)}, Q^x)_{\alpha<\pi, n\geqq 1, x\in K})$. Let us review some notation and facts. In $(\mathscr{W}, Z_\alpha)$, the common sample space of the $Z^{(n)}$, let $\sigma_n = \tau_{D(n,x)}$ if $Z_0 = x$ and let $\sigma_{n\alpha}$, $\alpha < \pi$, be the iterates of σ_n, i.e., $\sigma_{n0} = 0$, $\sigma_{n,\alpha+1} = \sigma_{n\alpha} + \sigma_n(\theta_{\sigma_{n\alpha}})$, $\sigma_{n\alpha} = \sup_{\beta<\alpha} \sigma_{n\beta}$ for a limit ordinal α. Then $\mathscr{W}_n = \{w \in \mathscr{W} \mid \sigma_{n\alpha}(w) = \alpha \text{ for all } \alpha < \pi\}$ has $Q_x{}^{(n)}$-probability 1 for all x. Let π_n be the projection from $\prod_m \mathscr{W}_m$ to $\mathscr{W}_n$ and let $Z_\alpha{}^{(n)} = Z_\alpha \circ \pi_n$. Then $\Omega = \{\omega \in \prod_n \mathscr{W}_n \mid Z_{\sigma_{n\alpha}}^{(m)}(\omega) = Z_\alpha{}^{(n)}(\omega) \text{ for all } \alpha < \pi \text{ and } m \geqq n \geqq 1\}$. The probability measures P^x are such that for every n $(\Omega, Z_\alpha{}^{(n)}, P^x)_{\alpha<\pi, x\in K}$ is equivalent to $Z^{(n)} = (\mathscr{W}, Z_\alpha, Q_x{}^{(n)})_{\alpha<\pi, x\in K}$. (This reflects the fact that for every pair $n < m$ there is a natural imbedding of $Z^{(n)}$ in $Z^{(m)}$ which is based on (4.7) and the inclusion $\mathscr{D}_n \subset \mathscr{D}_m$.) We shall use the Markov property ([I] (6.3)) of $Z^{(\infty)}$ without explicitly mentioning it.

For $m \geqq n$ let

$$\xi(m, n, \alpha) = \sum_{\alpha < \sigma_{n\alpha}^{(m)}} h_m(Z_\alpha^{(m)})$$

where $\sigma_{n\alpha}^{(m)} = \sigma_{n\alpha} \circ \pi_m$, (this is the same notation as in ([I] Section 6) if the set D there is $\{\Delta\}$). Then as shown in ([I] (6.1)) $\{\xi(m, n, \alpha), m \geqq n\}$ is a uniformly integrable martingale relative to any P^x. Let

$$T_{n\alpha} = \liminf_m \xi(m, n, \alpha).$$

In the case when no holding points[3] are present this will be the αth iterate of the stopping time in the to-be-constructed process which equals the hitting time of $D(n, x)$ when starting at x. It was the main concern in ([I] Section 6) to show $T_{n1} > 0$ a.s. P^x for $x \neq \Delta$. There a set $D \in \mathscr{D}_\infty$, say $D \in \mathscr{D}_1$, and a point $x \notin D$ were fixed. Let $\xi_m = \sum_{\alpha < \tau_D^{(m)}} h_m(Z_\alpha)$ and $\xi = \liminf_m \xi_m$. Then it was shown that $\xi > 0$ a.s. P^x. The proof relied on two facts. The first is the upper semi-continuity of $u(y) = P^y[\xi = 0]$, which follows in turn from the fact that $E_y^{(n)} e^{-\lambda T_D}$ converges pointwise to a u.s.c. function. But this again holds here (Corollary 4.7). The second is the following: let F be a compact subset of $K - \Delta$ consisting exclusively of instantaneous points and D_n be the largest set in $\mathscr{D}_n$ disjoint from F; then $\sup\{h_n(y) \mid y \notin D_n\} \downarrow 0$. This again can be easily proved as in ([I] (6.5), (6.6)), this time using the continuity of $g_{D[0,1]}$ for $D(r) \in \mathscr{D}_0$. Thus as in [I] we have, in the above set-up, $\xi > 0$ a.s. P^x. Applying this to T_{n1} we have $T_{n1} > 0$ a.s. P^x for all $x \neq \Delta$. Let

$$T_\infty = \sup\{T_{n\alpha} \mid \alpha < \pi, n \geqq 1\}.$$

Then it is easy to see that $P^x[T_{n\alpha} < T_{n,\alpha+1}] = P^x[T_{n\alpha} < T_\infty] = P^x[Z_\alpha^{(n)} \in K - \Delta]$ for all x, n, α. Note that $T_\infty = T_{n\alpha}$ a.s. for all large α, specifically for $\alpha \geqq \omega^{m_n}$. It is also clear that $T_\infty < \infty$ a.s.; in fact $E^x[T_\infty] = g(x)$ for all x (E^x stands for the expectation with respect to P^x).

We now assume that all points in $K - \Delta$ are instantaneous. Thus from what was said above $h_n \downarrow 0$ uniformly on compact subsets of $K - \Delta$. From this the following proposition is clear.

PROPOSITION 5.1. *For* $\omega \in \Omega$ *let* $\mathscr{T}(\omega) = \{T_{n\alpha}(\omega) \mid \alpha < \pi, n \geqq 1\}$. *Then* $\mathscr{T}(\omega)$ *is dense in* $[0, T_\infty(\omega)]$ a.s.

Assume $\mathscr{T}(\omega)$ is dense in $[0, T_\infty(\omega)]$ for every ω. Define $X(T_{n\alpha}(\omega), \omega) = Z_\alpha^{(n)}(\omega)$. From the fact $T_{m\beta} < T_{m,\beta+1}$ a.s. on $\{T_{m\beta} < T_\infty\}$ it is clear that this is independent of n and α except possibly on a subset of P^x-measure 0 for all x; we may define its value to be Δ on this exceptional set.

[3] A point $x \in K - \Delta$ is called a holding point if as $D \uparrow K - \{x\}$ the measure $H_D(x, \cdot)$ does not converge vaguely to the point mass at x. It is easy to show that this is equivalent to the condition that $\lim_{D \uparrow K - \{x\}} g_D(x)$ (which always exists) is positive. A point $x \in K - \Delta$ is called instantaneous if it is not a holding point.

PROPOSITION 5.2. *For almost every ω, $t \to X(t, \omega)$ has a right continuous extension on* $[0, T_\infty(\omega)]$ *with left limits.*

PROOF. The right continuity of $X_t(\omega)$ on $\mathscr{T}(\omega)$ for almost every ω is clear since if $T_{n\alpha}(\omega) \leqq T_{m\beta}(\omega) < T_{n,\alpha+1}(\omega)$, $X(T_{m\beta}(\omega), \omega)$ is of distance less than $1/n$ from $X(T_{n\alpha}(\omega), \omega)$. For any $\varepsilon > 0$ let $\tau = \inf\{\alpha \mid \rho(Z_\alpha, Z_0) > \varepsilon\}$ and let τ_k be the iterates of τ; then the reasoning in Section 2 shows that $Q_x^{(n)}[\tau_k < \pi] \to 0$ as $k \to \infty$ uniformly in n (and uniformly in x). Since $\{T_{n\alpha}(\omega) \mid \alpha < \pi\} \uparrow \mathscr{T}(\omega)$ a.s., this guarantees the existence of right and left limits on $[0, T_\infty(\omega)]$ for almost every ω.

Assume as we may that for every ω, $t \to X(t, \omega)$ has a right continuous extension to $[0, T_\infty(\omega)]$ which has left limits, and $X(T_\infty(\omega), \omega) = \Delta$. Let $X(t, \omega) = \Delta$ for $t \geqq T_\infty(\omega)$. Define $X_t(\omega) = X(t, \omega)$, $\mathscr{G}_t = \sigma(X_s, s \leqq t)$, and $\mathscr{G} = \sigma(X_t, t \geqq 0)$. It is easy to see that X_t is measurable with respect to the σ-algebra $\sigma(Z_\alpha^{(n)}, \alpha < \pi, n \geqq 1)$; therefore P^x is defined on $\mathscr{G}$ for all x. Let $P^\mu(\Lambda) = \int \mu(dx) P^x(\Lambda)$ for probability measures μ on K. Denote by $\mathscr{F}$ the completion of $\mathscr{G}$ with respect to the family of all such measures P^μ, and by $\mathscr{F}_t$ the completion of $\mathscr{G}_t$ with respect to the above family and $\mathscr{F}$ (see [I] page 26 for this terminology). We shall prove that $X = (\Omega, X_t, P^x)$ is the desired Hunt process. Let $P_t f(x) = E^x f(X_t)$ for $f \in \mathscr{M}$; it is Borel measurable in x and if $f \in \mathscr{C}_0$ it is right continuous in t.

PROPOSITION 5.3. *For $\lambda \geqq 0$ and $f \in \mathscr{M}_0$, $R_\lambda f(x) = E^x \int_0^\infty e^{-\lambda t} f(X_t)\, dt$.*

PROOF. We may assume f is continuous. Let $\varepsilon > 0$. Choose m so that $|f(y) - f(z)| < \varepsilon$ if $\rho(y, z) < 1/m$ and $H_D g(x) < \varepsilon$ for the smallest set $D \in \mathscr{D}_m$ with $\Delta \in \operatorname{int} D$. For $n \geqq m$

$$R_\lambda^{(n)} f(x) = \sum_{\gamma<\pi} E^x \left\{ \sum_{\sigma_{m\gamma}^{(n)} \leqq \alpha < \sigma_{m,\gamma+1}^{(n)}} \left(\prod_{\beta<\alpha} \frac{e_n(Z_\beta^{(n)})}{\lambda + e_n(Z_\beta^{(n)})} \right) \frac{1}{\lambda + e_n(Z_\alpha^{(n)})} f(Z_\alpha^{(n)}) \right\}.$$

Let $\tau_{m\gamma} = \min\{\sigma_{m\gamma}, \tau_D\}$. Then $R_\lambda^{(n)} f(x)$ differs from

$$\sum_{\gamma<\pi} E^x \left\{ f(Z_\gamma^{(m)}) \left\{ \sum_{\tau_{m\gamma}^{(n)} \leqq \alpha < \tau_{m,\gamma+1}^{(n)}} \left(\prod_{\beta<\alpha} \frac{e_n(Z_\beta^{(n)})}{\lambda + e_n(Z_\beta^{(n)})} \right) \frac{1}{\lambda + e_n(Z_\alpha^{(n)})} \right\} \right\} \tag{5.1}$$

by smaller than $\varepsilon(g(x) + \|f\|)$. By writing $(\lambda + a)^{-1} = \lambda^{-1}(1 - a/(\lambda + a))$ we have

$$\left(\prod_{\beta<\alpha} \frac{e_n(Z_\beta^{(n)})}{\lambda + e_n(Z_\beta^{(n)})} \right) \frac{1}{\lambda + e_n(Z_\alpha^{(n)})} = \frac{1}{\lambda} \left(\prod_{\beta<\alpha} \frac{e_n(Z_\beta^{(n)})}{\lambda + e_n(Z_\beta^{(n)})} - \prod_{\beta \leqq \alpha} \frac{e_n(Z^{(n)})}{\lambda + e_n(Z_\beta^{(n)})} \right)$$

for $\alpha < \tau_\Delta^{(n)}$. Since $\sum_\alpha h_n(Z_\alpha^{(n)}) < \infty$ a.s. the series in the integrand of each term of (5.1) telescopes a.s. Hence (5.1) equals

$$E^x \left\{ \sum_{\gamma<\pi} f(Z_\gamma^{(m)}) \frac{1}{\lambda} \left(\prod_{\beta<\tau_{m\gamma}^{(n)}} \frac{e_n(Z_\beta^{(n)})}{\lambda + e_n(Z_\beta^{(n)})} - \prod_{\beta<\tau_{m,\gamma+1}^{(n)}} \frac{e_n(Z_\beta^{(n)})}{\lambda + e_n(Z^{(n)})} \right) \right\}. \tag{5.2}$$

Now since $h_n \downarrow 0$ uniformly on $K - D$

$$\prod_{\beta<\tau_{m\gamma}^{(n)}} \frac{e_n(Z_\beta^{(n)})}{\lambda + e_n(Z_\beta^{(n)})} = \exp(-\textstyle\sum_{\beta<\tau_{m\gamma}^{(n)}} \log[\lambda h_n(Z_\beta^{(n)}) + 1]) \to \exp(-T'_{m\gamma})$$

where $T'_{m\gamma} = \min\{T_{m\gamma}, T_D\}(T_D = \inf\{t \geqq 0 \mid X_t \in D\})$; see the proof of ([I] 5.6) in [I] Section 6. For $f \geqq 0$ the integrand of (5.2) is uniformly integrable (being dominated by $\|f\| \sum_{\alpha<\pi} h_n(Z_\alpha^{(n)})$) and each term in the sum is nonnegative. Hence (5.2) converges to

$$E^x\{\textstyle\sum_{\gamma<\pi} f(X(T'_{m\gamma}))\lambda^{-1}(e^{-\lambda T'_{m\gamma}} - e^{-\lambda T'_{m,\gamma+1}})\}.$$

But the latter differs from $E^x \int_0^{T_D} e^{-\lambda t} f(X_t)\,dt$ by smaller than $\varepsilon g(x)$, and since $E^x\{T_\infty - T_D\} = \lim_n E^x\{\sum_{\tau_D^{(n)} \leqq \alpha < \pi} h_n(Z_\alpha^{(n)})\} = g(x) - g_D(x) = H_D g(x) < \varepsilon$, we have for $n \geqq m$

$$|R_\lambda^{(n)} f(x) - E^x \textstyle\int_0^\infty e^{-\lambda t} f(X_t)\,dt| < 2\varepsilon(g(x) + \|f\|).$$

The proposition now follows from Proposition 4.6.

We prove next that the process X is Markov.

PROPOSITION 5.4. *$P_{s+t} f(x) = P_s P_t f(x)$ for $x \in K$, $f \in \mathscr{M}$, s, $t \geqq 0$.*

PROOF. Since both sides are equal to 1 when $f = 1$ we may assume $f \in \mathscr{C}_0$ and $f \geqq 0$. Let $D(r) \in \mathscr{E}_0$ with $D(1) = \{\Delta\}$ and $D(r) \in \mathscr{D}_\infty$ for $r \in J$. Define $P_t^r f_1(x) = E^x\{f_1(X_t);\ t < T_{D(r)}\}$ for $f_1 \in \mathscr{C}_0$. Let $S_\lambda^r f(x) = \int_0^\infty e^{-\lambda t} P_t^r f(X_t)\,dt$. It is the same notation as in the proof of Proposition 4.6; there we have shown $\int_0^1 S_\lambda^r f(x)\,dr$ is continuous in x. Since $P_{s+t} f(x)$ and $P_s P_t f(x)$ are right continuous in t it suffices to show their Laplace transforms are equal. Obviously we need only show for $\lambda > 0$

$$\int_0^1 dr \int_0^\infty e^{-\lambda t} P_{s+t}^r f(x)\,dt = \int_0^1 dr \int_0^\infty e^{-\lambda t} P_s^r P_t^r f(x)\,dt; \tag{5.3}$$

for letting the function $D(r)$ go through an appropriate sequence we then obtain the desired equality of the Laplace transforms. Let $S_n = \inf\{T_{n\alpha} \mid T_{n\alpha} \geqq s\}$. Then $S_n \downarrow s$ on $\{s \leqq T_\infty\}$. Now the right-hand side of (5.3) equals

$$E^x\{\textstyle\int_0^1 S_\lambda^r f(X_s)\,dr;\ s < T\} = \lim_n E^x\{\int_0^1 S_\lambda^r f(X_{S_n})\,dr;\ S_n < T\}$$

where $T = T_{D(r)}$. On the other hand the left-hand side equals

$$\textstyle\int_0^1 E^x\{\int_s^{T \vee s} e^{-\lambda t} f(X_{s+t})\,dt\}\,dr = \lim_n \int_0^1 E^x\{\int_{S_n}^{T \vee S_n} e^{-\lambda t} f(X_{S_n+t})\,dt\}\,dr$$

and applying the Markov property of $Z^{(\infty)}$ we reduce the above to

$$\begin{aligned} &\lim_n \textstyle\int_0^1 E^x\{E^{X(S_n)} \int_0^T e^{-\lambda t} f(X_t)\,dt;\ S_n < T\}\,dr \\ &= \lim_n E^x\{\textstyle\int_0^1 S_\lambda^r f(X_{S_n})\,dr;\ S_n < T\}. \end{aligned}$$

Thus (5.3) is established and the proof is complete.

PROPOSITION 5.5. *$X = (\Omega, X_t, P^x)$ is a Hunt process.*

PROOF. The proof of the strong Markov property (with respect to the σ-algebras $\mathscr{F}_{t+}$) is modelled after that of the theorem on ([1] page 41), using the continuity of $\int_0^1 S_\lambda^r f(x)\,dr$ defined in the previous proof and a similar technique. For the proof of the quasi-left-continuity we refer the reader to the last part of the proof of the theorem on ([1] page 46); again the continuity of $\int_0^1 S_\lambda^r f(x)\,dr$ plays the central role.

PROPOSITION 5.6. *$E^xT_\Delta = g(x)$ and $P^x[X(T_D)\in B;\ T_D < \infty] = H_D(x, B)$ for all x, $D\in\mathscr{D}$.*

PROOF. The first part and the second part for $D\in\mathscr{D}_\infty$ are obvious from the construction. That the second part holds for arbitrary $D\in\mathscr{D}$ follows from hypothesis (e) and the quasi-left-continuity of X.

For the case when all points in $K-\Delta$ are instantaneous the proof of Theorem 1.2 is now completed except for the uniqueness assertion. But this is easy since any Hunt process on K with the same hitting distributions on the sets $D\in\mathscr{D}$ and the same expected lifetimes as X must have the same resolvent as X (to see this consider the approximating processes $X^{(n)}$) and is thus equivalent to X.

For the case when holding points are present, we give only a sketch of the construction without proof. In this case Proposition 5.1 does not hold. For each ω let $(a_1(\omega), b_1(\omega))$, $(a_2(\omega), b_2(\omega)), \cdots$ be the disjoint intervals whose union is $[0, T_\infty(\omega)]-\overline{\mathscr{T}(\omega)}$ and arranged with their lengths decreasing; it is understood that if $[0, T_\infty(\omega)]-\overline{\mathscr{T}(\omega)}$ is empty or is the union of finitely many disjoint intervals, all or all but finitely many $(a_k(\omega), b_k(\omega))$ are $(0, 0)$. Almost surely $b_k(\omega)\in\mathscr{T}(\omega)$ for all k, but this is not necessarily so for a_k. However, define $X(t, \omega)$ for $t\in\mathscr{T}(\omega)$ as before; then for almost every ω it still has a right continuous extension on $[0, T_\infty(\omega)]$ with left limits, and furthermore this extension is left continuous at those $a_k(\omega)$ not in $\mathscr{T}(\omega)$. Again let $X(t, \omega) = \Delta$ for $t \geqq T_\infty(\omega)$. Now define for each ω a measure $\nu(\omega, \cdot)$ on the infinite product $R_+{}^\infty = \{(t_1, t_2, \cdots) \mid t_k \geqq 0 \text{ for all } k\}$ to be the product measure $\nu_\omega{}^1\times\nu_\omega{}^2\times\cdots$ where $\nu_\omega{}^k$ is the exponential distribution with mean $b_k(\omega)-a_k(\omega)$ (point mass at 0 if $b_k(\omega) = a_k(\omega) = 0$). Let $\hat\Omega = \Omega\times R_+{}^\infty$ and $\hat P^x$ be the measure on Ω defined from P^x and the transition probability $\nu(\omega, \cdot)$. For each $\hat\omega = (\omega, (t_1, t_2, \cdots))$ in $\hat\Omega$ define $\varphi_{\hat\omega}$: $\overline{\mathscr{F}(\omega)} \to [0, \infty]$ as follows:

$$\varphi_{\hat\omega}(s) = s+\textstyle\sum_{i:b_i(\omega)\leqq s}(t_i - b_i(\omega)+a_i(\omega)).$$

Almost surely (i.e. a.s. $\hat P^x$ for all x) $\varphi_{\hat\omega}$ is an increasing homeomorphism of $\overline{\mathscr{T}(\omega)}$ onto a compact subset of $[0, \infty]$, hence has an extension $\varphi_{\hat\omega}$: $[0, \infty]\to[0, \infty]$ which is an increasing homeomorphism. Now define $\hat X_t(\hat\omega) = \hat X(t, \hat\omega)$ for such $\hat\omega$ to be $X(\varphi_{\hat\omega}{}^{-1}(t), \omega)$. The process $(\hat\Omega, \hat X_t, \hat P^x)$ is then the desired process.

REFERENCES

[1] BLUMENTHAL, R. M. and GETOOR, R. K. (1968). *Markov Processes and Potential Theory*, Academic Press, New York.

[2] HANSEN, W. (1968). Charakterisierung von Familien exzessiver Funktionen. *Invent. Math.* **5** 335–348.

[3] SHIH, C. T. (1971). Construction of Markov processes from hitting distributions. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **18** 47–72.