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LIMIT PROCESSES FOR CO-SPECTRAL AND QUADRATURE SPECTRAL DISTRIBUTION FUNCTIONS[1]

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1. Introduction and summary. Limit processes for the sequences of stochastic processes defined by co-spectral and quadrature spectral distribution functions are found using the theory of weak convergence. The limit processes are shown to be Gaussian with independent increments and with covariance functions defined in terms of hypothesized spectral densities.

Section 3 contains a discussion of the moments of the processes. The first and second asymptotic moments, which characterize the limit processes, are computed giving results analogous to those of Grenander and Rosenblatt [6] and Ibragimov [8] for autospectra. We also evaluate the higher asymptotic moments and put bounds on the higher moments. The latter are required in demonstrating tightness of the measures generated in $C[0, \pi]$ by the co-spectral and quadrature spectral distribution functions. In Section 4, limit processes, under certain conditions listed in Section 2, are obtained and described in Theorem 4.5. Finally, Section 5 contains a discussion of asymptotic goodness-of-fit testing for spectral distribution functions.

2. Definitions and assumptions. We consider real, normal, discrete time parameter, zero mean and jointly weakly stationary time series $\{\underline{X}(t)\}_{t=-\infty}^{\infty} = \{X_k(t), k = 1, 2, \cdots, m, -\infty < t < \infty\}$ and we let $\mathbf{R}(v) = E[\underline{X}(t)\underline{X}'(t+v)]$ where $\mathbf{R}(v) = \{R_{jk}(v), j, k = 1, 2, \cdots, m\}$. We assume that the cross-spectral density matrix, $\mathbf{f}(\lambda)$, exists with $\mathbf{f}(\lambda) = (2\pi)^{-1}\sum_{v=-\infty}^{\infty} \exp(-iv\lambda)\mathbf{R}(v)$. Letting $f_{jk}(\lambda)$ be the j, kth element of $\mathbf{f}(\lambda)$ we define, for $j, k = 1, 2, \cdots, m$, the co-spectral density functions by $c_{jk}(\lambda) = \operatorname{Re}[f_{jk}(\lambda)]$, the quadrature spectral density functions by $q_{jk}(\lambda) = -\operatorname{Im}[f_{jk}(\lambda)]$, the corresponding distribution functions by $C_{jk}(\lambda) = \int_0^\lambda c_{jk}(l)\,dl$, $Q_{jk}(\lambda) = \int_0^\lambda q_{jk}(l)\,dl$ and define the cross-spectral distribution functions by $F_{jk}(\lambda) = \int_0^\lambda f_{jk}(l)\,dl$. If ${}_ER_{jk}(v)$ and ${}_OR_{jk}(v)$ are the odd and even parts of the cross-covariance function then

$$ {}_ER_{jk}(v) = \int_{-\pi}^{\pi} \cos(v\lambda)c_{jk}(\lambda)\,d\lambda \quad \text{and} \quad {}_OR_{jk}(v) = \int_{-\pi}^{\pi} \sin(v\lambda)q_{jk}(\lambda)\,d\lambda, \qquad j \neq k = 1, 2, \cdots, m. $$

If $\{\underline{X}(t), t = 1, 2, \cdots, N\}$ represents N observations from a realization of the series $\{\underline{X}(t)\}$ then the sample cross-covariance matrix of lag v is denoted by $\mathbf{R}_N(v)$ where the j, kth element is defined by $R_{jkN}(v) = N^{-1}\sum_{t=1}^{N-v} X_j(t)X_k(t+v)$. The sample cross-spectral density matrix is denoted by $\mathbf{f}_N(\lambda)$ and its j, kth element is defined by $f_{jkN}(\lambda) = (2\pi N)^{-1}\sum_{s=1}^{N} \exp(is\lambda)X_j(s)\sum_{t=1}^{N} \exp(-it\lambda)X_k(t)$. The sample

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co-spectral and quadrature spectral densities are defined by: $c_{jkN}(\lambda) = \mathrm{Re}\,[f_{jkN}(\lambda)]$ and $q_{jkN}(\lambda) = -\mathrm{Im}\,[f_{jkN}(\lambda)]$. Sample distribution functions are defined by $F_{jkN}(\lambda) = \int_0^\lambda f_{jkN}(l)\,dl$, $C_{jkN}(\lambda) = \mathrm{Re}\,[F_{jkN}(\lambda)]$ and $Q_{jkN}(\lambda) = -\mathrm{Im}\,[F_{jkN}(\lambda)]$.

The following notation is used extensively below:

$$\Phi_{jkN}(\lambda) = F_{jkN}(\lambda) - F_{jk}(\lambda), \qquad \Phi^c_{jkN}(\lambda) = C_{jkN}(\lambda) - C_{jk}(\lambda),$$
$$\Phi^q_{jkN}(\lambda) = Q_{jkN}(\lambda) - Q_{jk}(\lambda), \qquad \theta_{jkN}(\lambda) = F_{jkN}(\lambda) - E[F_{jkN}(\lambda)],$$
$$\theta^c_{jkN}(\lambda) = C_{jkN}(\lambda) - E[C_{jkN}(\lambda)] \quad \text{and} \quad \theta^q_{jkN}(\lambda) = Q_{jkN}(\lambda) - E[Q_{jkN}(\lambda)].$$

In addition we let

$$H_{jk}(\mu) = 2\pi \int_0^\mu |f_{jk}(l)|^2\,dl,$$
$$H^{cf}_{jk}(\mu) = 2\pi \int_0^\mu f_{jk}(l) c_{jk}(l)\,dl,$$
$$H^{qf}_{jk}(\mu) = 2\pi \int_0^\mu f_{jj}(l) q_{jk}(l)\,dl,$$
$$H^c_{jk}(\mu) = \pi \int_0^\mu [f_{jj}(l) f_{kk}(l) + c^2_{jk}(l) - q^2_{jk}(l)]\,dl,$$
$$H^q_{jk}(\mu) = \pi \int_0^\mu [f_{jj}(l) f_{kk}(l) + q^2_{jk}(l) - c^2_{jk}(l)]\,dl \qquad \text{and}$$
$$H^{cq}_{jk}(\mu) = \pi \int_0^\mu c_{jk}(l) q_{jk}(l)\,dl.$$

This last notation is used in specifying the covariance functions for the limiting processes.

The conditions required on the time series to prove the theorems in the sequel are now summarized. They will be referred to at the beginning of each theorem. The first set of conditions is a summary of those given above.

CONDITION 0. Time series are real, jointly normal, discrete time parameter, zero mean, jointly weakly stationary and possess spectral density functions.

CONDITION 1. Condition 0 is assumed and further it is assumed that $f_{jj}(\lambda) \in L^2[-\pi, \pi]$.

CONDITION 2. Condition 1 is assumed and it is also assumed that for some $\delta > 0$, $f_{jj}(\lambda) \in L^{2+\delta}[-\pi, \pi]$.

CONDITION 3. Condition 2 is assumed and in addition it is assumed that a pair of time series, $\{X(t)\}$ and $\{Y(t)\}$ have the following representation:

$$X(t) = \sum_{v=-\infty}^{\infty} a(v)\xi(t-v), \tag{2.1}$$
$$Y(t) = \sum_{\mu=-\infty}^{\infty}{}' \, b(\mu)\zeta(t-\mu),$$

where $\sum_{v=-\infty}^{\infty} |a(v)|^2 < \infty$ and $\sum_{\mu=-\infty}^{\infty} |b(\mu)|^2 < \infty$ and the elementary series satisfy Condition A.

CONDITION A. $\{\xi(t)\}$ and $\{\zeta(t)\}$ are jointly normal time series satisfying: (i) for every t, $\xi(t)$ and $\zeta(t)$ are normal with mean 0 and variance 1; (ii) there exists a positive integer M such that for $|m| > M$, $\mathrm{Cov}\,[\xi(t), \xi(t+m)] = \mathrm{Cov}\,[\mathrm{Cov}\,\zeta(t), \zeta(t+m)] = \mathrm{Cov}\,[\xi(t), \zeta(t+m)] = 0$.

NOTE. To understand the role of Condition A, it should be understood that the following results fall into three main categories: (1) theorems (proved in Section 4) asserting that the sequence of stochastic processes $\{N^{\frac{1}{2}}\Phi_{12}(\lambda),\ 0 \leqq \lambda \leqq \pi\}$ converges weakly to a Gaussian process, (2) theorems (proved in Section 3) determining the covariance kernel of the limit process, and (3) theorems (proved in Section 5) which evaluate the asymptotic distributions of various functionals of such stochastic processes as $\{N^{\frac{1}{2}}\Phi_{12}^{c}(\lambda), 0 \leqq \lambda \leqq \pi\}$. The theorems of category (1) are proved for any pair of processes $\{X(t)\}$ and $\{Y(t)\}$ which can be represented in the form (2.1) in terms of series $\{\xi(t)\}$ and $\{\zeta(t)\}$ satisfying Condition A. The theorems of category (2) are proved for normal processes. One could state analogous theorems for processes $\{X(t)\}$ and $\{Y(t)\}$ which are not normal, but then the covariance kernel of the limit normal process would not be of the simple form for which we give results of category (3).

CONDITION 4. Condition 2 is assumed and it is further assumed that the spectral distribution functions have no intervals of constancy.

CONDITION 5. Condition 3 and Condition 4 are assumed and, in addition, the coefficients in the linear scheme defined in Condition 3 are assumed to have the properties: $a(v) = O(v^B)$, $b(v) = O(v^B)$ where $B < -\frac{3}{2}$.

Grenander and Rosenblatt [6] obtained the limit processes for autospectra essentially under Condition 5 but with the normality assumption dropped. Ibragimov [8] obtained limit processes for autospectra essentially under Condition 4, i.e., assuming normal time series. Malevich [10] relaxed the condition on the spectral density to that of square integrability. Brillinger [3], under a different set of assumptions involving the near independence of widely separated values of strictly stationary time series, obtained limit processes for the matrix of cross-spectral distribution functions. Goodness-of-fit testing for autospectra is discussed by Grenander and Rosenblatt [6].

3. Moments for spectral distribution functions. In computing the moments of the processes under consideration certain kernels arise. We now define and state several properties of these kernels, omitting proofs which are available in [9]. The kernels and theorems are extensions of those in [8].

Let A be an interval in $[0, \pi]$ and define the following functions of two variables, $l_1, l_2 \in [-\pi, \pi]$:

$$\Phi_{1N}^{(A)}(l_1, l_2) = \frac{1}{2\pi N^{\frac{1}{2}}} \int_A \frac{\sin N(l_1-\alpha)/2}{\sin(l_1-\alpha)/2} \frac{\sin N(l_2-\alpha)/2}{\sin(l_2-\alpha)/2}\, d\alpha,$$

$$\Phi_{2N}^{(A)}(l_1, l_2) = \frac{1}{2\pi N^{\frac{1}{2}}} \int_A \frac{\sin N(l_1+\alpha)/2}{\sin(l_1+\alpha)/2} \frac{\sin N(l_2+\alpha)/2}{\sin(l_2+\alpha)/2}\, d\alpha,$$

$$\Phi_{3N}^{(A)}(l_1, l_2) = \frac{1}{2\pi N^{\frac{1}{2}}} \int_A \frac{\sin N(l_1-\alpha)/2}{\sin(l_1-\alpha)/2} \frac{\sin N(l_2+\alpha)/2}{\sin(l_2+\alpha)/2}\, d\alpha,$$

$$\Phi_{4N}^{(A)}(l_1, l_2) = \frac{1}{2\pi N^{\frac{1}{2}}} \int_A \frac{\sin N(l_1+\alpha)/2}{\sin(l_1+\alpha)/2} \frac{\sin N(l_2-\alpha)/2}{\sin(l_2-\alpha)/2}\, d\alpha.$$

These functions can be used to define a number of kernels. For intervals A, $B \subset [0, \pi]$, define

$$\psi_{ijN}^{[A,B]}(l_1, l_2) = \Phi_{iN}^{(A)}(l_1, l_2)\Phi_{jN}^{(B)}(l_1, l_2) \qquad i,j = 1,\cdots,4.$$

The first property of interest to us establishes certain bounds for the kernels.

THEOREM A. *If* $(a, b)\,(c, d) \subset (-\pi, \pi)$ *there exists* C *such that, for* $l_1 \in (-\pi, \pi)$ *and* $N \geqq 1$,

$$\int_{-\pi}^{\pi} |\psi_{ijN}^{[(a,b),(c,d)]}(l_1, l_2)|\, dl_2 < C.$$

The next result yields the most useful property of the kernels.

THEOREM B. *Assume* $h(l) \in L^2[-\pi, \pi]$ *and* $0 \leqq \lambda < \mu \leqq \pi$. *Then, for almost all* $l_1 \in [-\pi, \pi]$, *as* $N \to \infty$

$$\text{(i)} \quad \int_{-\pi}^{\pi} h(l_2)\psi_{iiN}^{[(0,\lambda),(0,\lambda)]}(l_1, l_2)\, dl_2 \to \begin{cases} 2\pi h(l_1) & \text{for } \begin{cases} l_1 \in (0,\lambda), i = 1 \\ l_1 \in (-\lambda, 0), i = 2 \end{cases} \\ 2\pi h(-l_1) & \text{for } \begin{cases} l_1 \in (0,\lambda), i = 3 \\ l_1 \in (-\lambda, 0), i = 4 \end{cases} \\ 0 & \text{for } \begin{cases} l_1 \notin [0,\lambda], i = 1,3 \\ l_1 \notin [-\lambda, 0], i = 2,4; \end{cases} \end{cases}$$

$$\text{(ii)} \qquad \int_{-\pi}^{\pi} h(l_2)\psi_{ijN}^{[(0,\lambda),(0,\lambda)]}(l_1, l_2)\, dl_2 \to 0 \qquad i,j = 1,2,3,4, i \neq j;$$

$$\text{(iii)} \qquad \int_{-\pi}^{\pi} h(l_2)\psi_{ijN}^{[(0,\lambda),(\lambda,\mu)]}(l_1, l_2)\, dl_2 \to 0 \qquad i,j = 1,2,3,4.$$

Theorem A and Theorem B can be extended to kernels involving more variables. Let $\psi_N(l_1, l_2, l_3, l_4) = \Phi_{i_1N}^{(A)}(l_{j_1}, l_{j_2})\Phi_{i_2N}^{(A)}(l_{j_3}, l_{j_4})\Phi_{i_3N}^{(A)}(l_{j_5}, l_{j_6})\Phi_{i_4N}^{(A)}(l_{j_7}, l_{j_8})$ where $i_k, j_k \in \{1, 2, 3, 4\}$ and $j_1 \neq j_2, j_3 \neq j_4, j_5 \neq j_6, j_7 \neq j_8$ and two of the subscripts are 1, two are 2, etc. A kernel such as

$$\psi_N(l_1, l_2, l_3, l_4) = \Phi_{1N}^{(A)}(l_1, l_2)\Phi_{1N}^{(A)}(l_1, l_2)\Phi_{1N}^{(A)}(l_3, l_4)\Phi_{1N}^{(A)}(l_3, l_4) \tag{3.1}$$

has properties similar to $\psi_{11N}^{(A,A)}(l_1, l_2)$. A kernel such as

$$\psi_N^{\,1}(l_1, l_2, l_3, l_4) = \Phi_{3N}^{(A)}(l_1, l_2)\Phi_{2N}^{(A)}(l_1, l_3)\Phi_{2N}^{(A)}(l_2, l_4)\Phi_{4N}^{(A)}(l_3, l_4) \tag{3.2}$$

has properties similar to $\psi_{12N}^{(A,A)}(l_1, l_2)$. Such kernels arise in the computation of the fourth moments. Further extensions of these kernels are useful in computing the $2n$th moments.

We now consider a pair of time series, $\{X_1(t)\}$ and $\{X_2(t)\}$, and discuss the first moments in the following theorem.

THEOREM 3.1. (a) *Under Condition* 0, $\lim_{N\to\infty} E[C_{12N}(\lambda)] = C_{12}(\lambda)$ *and* $\lim_{N\to\infty} E[Q_{12N}(\lambda)] = Q_{12}(\lambda)$. (b) *Under Condition* 1 *there exists a constant* K *such that* $\sup_{0\leqq\lambda\leqq\pi} N^{\frac{1}{2}} |E[C_{12N}(\lambda)] - C_{12}(\lambda)| \leqq K$ *and* $\sup_{0\leqq\lambda\leqq\pi} N^{\frac{1}{2}} |E[Q_{12N}(\lambda)] - Q_{12}(\lambda)| \leqq K$.

The proof of this theorem is obtained by applying Theorem 1.1 [8].

The next theorem uses Theorem A and Theorem B to find the asymptotic second moments for the processes.

THEOREM 3.2. *For* λ, $\mu \in [0, \pi]$ *and under Condition* 1,

(i) $\lim_{N\to\infty} NE[C_{12N}(\lambda)-E(C_{12N}(\lambda))][C_{12N}(\mu)-E(C_{12N}(\mu))] = H_{12}^{c}(\min(\lambda,\mu))$,

(ii) $\lim_{N\to\infty} NE[Q_{12N}(\lambda)-E(Q_{12N}(\lambda))][Q_{12N}(\mu)-E(Q_{12N}(\mu))] = H_{12}^{q}(\min(\lambda,\mu))$,

(iii) $\lim_{N\to\infty} NE[C_{12N}(\lambda)-E(C_{12N}(\lambda))][Q_{12N}(\mu)-E(Q_{12N}(\mu))] = H_{12}^{cq}(\min(\lambda,\mu))$.

PROOF. Considering part (i) we first note that

$$(3.3)\qquad \operatorname{Cov}[C_{12N}(\lambda), C_{12N}(\mu)] = \int_0^\lambda\int_0^\mu \{E[c_{12N}(l_1)c_{12N}(l_2)]-E[c_{12N}(l_1)]E[c_{12N}(l_2)]\}\,dl_2\,dl_1.$$

Using Isserlis' formula for products of normal random variables it can be shown that

$$E[c_{12N}(l_1)c_{12N}(l_2)] = (4\pi N)^{-2}\sum_{j_1j_2j_3j_4}\exp\{i(j_1\,l_1-j_2\,l_1+j_3\,l_2-j_4\,l_2)\}\{S_1+S_2+S_3\}$$

where:

$$\begin{aligned}
S_1 &= R_{11}(j_1-j_3)R_{22}(j_2-j_4)\\
&\quad +R_{11}(j_1-j_4)R_{22}(j_2-j_3)+R_{11}(j_2-j_3)R_{22}(j_1-j_4)\\
&\quad +R_{11}(j_2-j_4)R_{22}(j_1-j_3)\\
&= S_{11}+S_{12}+S_{13}+S_{14},\\
S_2 &= [R_{12}(j_1-j_3)+R_{12}(j_3-j_1)][R_{12}(j_4-j_2)+R_{12}(j_2-j_4)]\\
&\quad +[R_{12}(j_1-j_4)+R_{12}(j_4-j_1)][R_{12}(j_3-j_2)+R_{12}(j_2-j_3)]\\
&\quad -[R_{12}(j_1-j_3)R_{12}(j_2-j_4)+R_{12}(j_3-j_1)R_{12}(j_4-j_2)\\
&\quad +R_{12}(j_1-j_4)R_{12}(j_2-j_3)+R_{12}(j_4-j_1)R_{12}(j_3-j_2)]\\
&= S_{21}+S_{22}-[S_{23}+S_{24}+S_{25}+S_{26}],\\
S_3 &= [R_{12}(j_1-j_2)+R_{12}(j_2-j_1)][R_{12}(j_3-j_4)+R_{12}(j_4-j_3)].
\end{aligned}$$

Considering the first term of S_1 and using the spectral representation for the covariance function we obtain:

$$\begin{aligned}
(3.4)\qquad g_{11N}(l_1,l_2) &= (4\pi N)^{-2}\sum_{j_1j_2j_3j_4}\exp[i(j_1\,l_1-j_2\,l_1+j_3\,l_2-j_4\,l_2)]S_{11}\\
&= (4\pi N)^{-2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{\sin N(l_1+\alpha)/2}{\sin(l_1+\alpha)/2}\,\frac{\sin N(l_1-\beta)/2}{\sin(l_1-\beta)/2}\\
&\quad\cdot\frac{\sin N(l_2-\alpha)/2}{\sin(l_2-\alpha)/2}\,\frac{\sin N(l_2+\beta)/2}{\sin(l_2+\beta)/2}f_{11}(\alpha)f_{22}(\beta)\,d\alpha\,d\beta.
\end{aligned}$$

If we assume that $\lambda < \mu$, use some simple manipulations and integrate with respect to the other two variables, then

$$(3.5)\quad \int_0^\lambda \int_0^\mu g_{11N}(l_1, l_2)\, dl_1\, dl_2 = (4N)^{-1} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f_{11}(l_1) f_{22}(l_2) \{\psi_{43N}^{[(0,\lambda),(0,\lambda)]}(l_1, l_2) + \psi_{43N}^{[(0,\lambda),(\lambda,\mu)]}(l_1, l_2)\, dl_2\, dl_1\}.$$

We let $g_N(l_1) = f_{11}(l_1) \int_{-\pi}^{\pi} f_{22}(l_2) \psi_{43N}^{[(0,\lambda)(0,\lambda)]}(l_1, l_2)\, dl_2$ and consider $\int_{-\pi}^{\pi} g_N(l_1)\, dl_1$. Theorem B and the integrability of $f_{ii}(\cdot)$ enable us to assert that $\lim_{N\to\infty} g_N(l_1) = 0$ a.e. Using Theorem A and the fact that $f_{ii}(\cdot) \in L^2[-\pi, \pi]$ one can verify the condition of Theorem 6 [4] page 122 which enables us to assert that $\lim_{N\to\infty} \int_{-\pi}^{\pi} g_N(l_1)\, dl_1 = 0$. This, plus a similar argument applied to the expression containing $\psi_{43N}^{[(0,\lambda),(\lambda,\mu)]}(l_1, l_2)$ shows that

$$\lim_{N\to\infty} N \int_0^\lambda \int_0^\mu g_{11N}(l_1, l_2)\, dl_1\, dl_2 = 0.$$

A different result is obtained by the same methods by applying Theorem B to an expression like (3.5) obtained using S_{12}. In this case

$$\lim_{N\to\infty} \tfrac{1}{4} \int_{\pi}^{\pi} \int_{-\pi}^{\pi} f_{11}(l_1) f_{22}(l_2) \psi_{44N}^{[(0,\lambda),(0,\lambda)]}(l_1, l_2)\, dl_2\, dl_1 = \pi/2 \int_0^\lambda f_{11}(l_1) f_{22}(-l_1)\, dl_1\,.$$

Treating the remaining terms in S_1, S_2 and S_3 similarly and collecting non-zero limits we see that

$$(3.6)\quad \lim_{N\to\infty} N\, \mathrm{Cov}\,[C_{12N}(\lambda), C_{12N}(\mu)] = \pi \int_0^\lambda f_{11}(l_1) f_{22}(-l_1)\, dl_1 + 2\pi \int_{-\lambda}^{0} c_{12}^2(l_1)\, dl_1 - \pi/2 \int_{-\lambda}^{0} |f_{12}(l_1)|^2 dl_1 - \pi/2 \int_0^\lambda |f_{12}(l_1)|^2 dl_1 = H_{12}^c(\lambda).$$

Conclusion (ii) and Conclusion (iii) are proved in the same way.

In goodness-of-fit testing one is more interested in deviations of sample distribution functions from hypothesized distribution functions than from the expected values of sample distribution functions. These deviations are treated in the following corollary, the proof of which is an easy consequence of Theorem 3.1 and Theorem 3.2.

COROLLARY 3.3. *For $\lambda < \mu \in [0, \pi]$ and under Condition* 1

(i) $$\lim_{N\to\infty} NE[\Phi_{12N}^c(\lambda)\Phi_{12N}^c(\mu)] = H_{12}^c[(\lambda)],$$

(ii) $$\lim_{N\to\infty} NE[\Phi_{12N}^q(\lambda)\Phi_{12N}^q(\mu)] = H_{12}^q(\lambda),$$

(iii) $$\lim_{N\to\infty} NE[\Phi_{12N}^c(\lambda)\Phi_{12N}^q(\mu)] = H_{12}^{cq}(\lambda).$$

Using kernels of the form (3.1) and (3.2) one can show, for $\mu < \lambda \in [0, \pi]$ and under Condition 1, that

$$\lim_{N\to\infty} NE[\theta_{11N}(\lambda) - \theta_{11N}(\mu)]^4 = 3(H_{11}(\lambda) - H_{11}(\mu))^2,$$

$$\lim_{N\to\infty} NE[\theta_{12N}^c(\lambda) - \theta_{12N}^c(\mu)]^4 = 3(H_{12}^c(\lambda) - H_{12}^c(\mu))^2,$$

$$\lim_{N\to\infty} NE[\theta_{12N}^q(\lambda) - \theta_{12N}^q(\mu)]^4 = 3(H_{12}^q(\lambda) - H_{12}^q(\mu))^2.$$

In a similar way it can be shown that,

$$\lim_{N\to\infty} E[N^{\frac{1}{2}}(\theta_{11N}(\lambda)-\theta_{11N}(\mu)]^{2n} = M\{2\pi \int_\mu^\lambda f_{11}^2(l)\,dl\}^n$$

where $M = 1\cdot 3\cdot \cdots \cdot(2n-1)$ and that the corresponding results hold for the processes $\{\theta_{12N}^c(\lambda)\}$ and $\{\theta_{12N}^q(\lambda)\}$.

These results also hold for the processes $\{\Phi_{12N}^c(\lambda)\}$ and $\{\Phi_{12N}^q(\lambda)\}$. In addition using the kernels introduced at the beginning of this section, one can obtain the asymptotic covariance matrix found in Table 1.

TABLE 1

Covariances of spectral distribution functions ($\mu < \lambda$)

	$N^{\frac{1}{2}}\Phi_{11N}(\lambda)$	$N^{\frac{1}{2}}\Phi_{22N}(\lambda)$	$N^{\frac{1}{2}}\Phi_{12N}^c(\lambda)$	$N^{\frac{1}{2}}\Phi_{12N}^q(\lambda)$
$N^{\frac{1}{2}}\Phi_{11N}(\mu)$	$H_{11}(\mu)$	$H_{12}(\mu)$	$H_{12}^{cf}(\mu)$	$H_{12}^{qf}(\mu)$
$N^{\frac{1}{2}}\Phi_{22N}(\mu)$		$H_{22}(\mu)$	$H_{21}^{cf}(\mu)$	$H_{21}^{qf}(\mu)$
$N^{\frac{1}{2}}\Phi_{12N}^c(\mu)$			$H_{12}^c(\mu)$	$H_{12}^{cq}(\mu)$
$N^{\frac{1}{2}}\Phi_{12N}^q(\mu)$				$H_{12}^q(\mu)$

To apply limit theorems to processes such as $\{\theta_{12N}^c(\lambda), \lambda\in[0,\pi]\}_{N=1}^\infty$ one must show that the sequences of probability measures generated in $C[0,\pi]$ by these processes are tight. For the space $C[0,\pi]$ this amounts to showing that the sequences of measures are relatively compact or, equivalently, demonstrating the conditions of the Arzelà—Ascoli characterization for relative compactness. To demonstrate these conditions it is sufficient to obtain appropriate bounds on the moments of the processes and these are obtained in the following theorem. For definitions of tightness and relative compactness and for the statement of Prokhorov's theorem relating tightness and relative compactness one can refer to Billingsley [2].

THEOREM 3.4. *For $\alpha < \beta\in[0,\pi]$ and under Condition 2 there exist a_1, a_2 and a_3, $0 < a_i < \infty$, such that*

(i) $$E\left|N^{\frac{1}{2}}(\theta_{12N}^c(\beta)-\theta_{12N}^c(\alpha))\right|^{a_1} \leqq K_1(\beta-\alpha)^{1+b_1},$$

(ii) $$E\left|N^{\frac{1}{2}}(\theta_{12N}^q(\beta)-\theta_{12N}^q(\alpha))\right|^{a_2} \leqq K_2(\beta-\alpha)^{1+b_2},$$

(iii) $$E\left|N^{\frac{1}{2}}(\theta_{11N}(\beta)-\theta_{11N}(\alpha))\right|^{a_3} \leqq K_3(\beta-\alpha)^{1+b_3},$$

where the constants K_i and b_i are positive and do not depend upon α, β and N.

PROOF. We first prove (i). Just as (3.6) was obtained, one can obtain the following expression for the second moment. We let $\psi_{ijN}^{[(\alpha,\beta),(\alpha,\beta)]}(l_1,l_2) \equiv \psi_{ijN}$.

(3.7) $$\begin{aligned} &E[N^{\frac{1}{2}}(\theta_{12N}^c(\beta)-\phi_{12N}^c(\alpha))]^2 \\ &= \tfrac{1}{4}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f_{11}(l_1)f_{22}(l_2)\{\psi_{33N}+\psi_{44N}\}\,dl_1\,dl_2 \\ &\quad+\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} c_{12}(l_1)c_{12}(l_2)\psi_{22N}\,dl_1\,dl_2 \\ &\quad-\tfrac{1}{4}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f_{12}(l_1)\overline{f_{12}(l_2)}\{\psi_{11N}+\psi_{22N}\}\,dl_1\,dl_2 \\ &\quad+\text{terms that go to zero as}\quad N\to\infty. \end{aligned}$$

We consider the term involving ψ_{33N} and deal with it in detail as in [8]. Using the Cauchy–Schwarz inequality and the following result (essentially given by Ibragimov [8])

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}|\psi_{ijN}|\,dl_1\,dl_2 \leqq 4\pi(\beta-\alpha)$$

we can show that, for $\delta > 0$,

$$\begin{aligned} I_1 &= \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f_{11}(l_1)f_{22}(l_2)\psi_{33N}\,dl_1\,dl_2 \\ &\leqq (4\pi(\beta-\alpha))^{\delta/2+\delta}\Big(\int_{-\pi}^{\pi}|f_{11}(l)|^{2+\delta}\,dl\Big)^{1/2+\delta} \\ &\quad\cdot\Big(\int_{-\pi}^{\pi}\{\int_{-\pi}|f_{22}(l_2)|^{2+\delta}|\psi_{33N}|\,dl_2\int_{-\pi}^{\pi}|\psi_{33}|\,dl_2\}dl_1\Big)^{2/2+\delta}. \end{aligned}$$

Interchanging the order of integration and using the fact that there exists $A > 0$ such that $\int_{-\pi}^{\pi}|\psi_{ijN}|\,dl_2 \leqq A$ (Theorem A) we have

$$I_1 \leqq A(4\pi(\beta-\alpha))^{\delta/2+\delta}\Big(\int_{-\pi}^{\pi}|f_{11}(l)|^{2+\delta}\,dl\Big)^{1/2+\delta}\Big(\int_{-\pi}^{\pi}|f_{22}(l)|^{2+\delta}\Big)^{2/2+\delta}.$$

Since we assume that $f_{ii}(l)\in L^{2+\delta}[-\pi,\pi]$ we have $I_1 \leqq B_1(\beta-\alpha)^{\delta/2+\delta}$ where B_1 depends upon $\delta, f_{11}(l), f_{22}(l)$ but not on N. In the same way we can find bounds for the remaining terms in (3.7) (including the terms that go to zero). As a consequence,

$$E[N^{\frac{1}{2}}(\theta^c_{12N}(\beta)-\theta^c_{12N}(\alpha))]^2 \leqq B(\beta-\alpha)^{\delta/2+\delta}$$

where B is independent of N. Similar methods applied to terms involved in computing the $2n$th moment prove

$$E[N^{\frac{1}{2}}(\theta^c_{12N}(\beta)-\theta_{12N}(\alpha))]^{2n} \leqq B_n(\beta-\alpha)^{n\delta/2+\delta} \tag{3.8}$$

where B_n is independent of N. Since $\delta > 0$ it is possible to choose n so that $n\delta/(2+\delta) > 1+b_1$ for $b_1 > 0$. Letting $a_1 = 2n$ in (3.8), part (i) is proved.

Obvious modifications of the proof given above establish (ii) and (iii). (iii) yields an alternative method for proving Theorem 4.1 [8].

4. Limit processes for spectral functions. The derivation of the limit processes for cross-spectral distribution functions for time series defined by (2.1) requires a preliminary discussion of the elementary series $\{\xi(t)\}$ and $\{\zeta(t)\}$ satisfying Condition A. Covariance sequences and spectral functions for these processes will be subscripted by ξ and ζ; i.e., the sample cross-covariance of lag v will be denoted by $R_{\xi\zeta N}(v)$. In this section, we first find the asymptotic finite dimensional distributions for sequences of processes such as $\{N^{\frac{1}{2}}\theta^c_{\xi\zeta N}(\lambda)\}_{N=1}^{\infty}$. Then we use Theorem 3.4 to put probabilistic bounds on the moduli of continuity (see [2], page 54) of the processes. These are then used, as in [8], to obtain limit processes for sequences such as $\{N^{\frac{1}{2}}\Phi^c_{\xi\zeta N}(\lambda)\}_{N=1}^{\infty}$. Using this result and a technique of Grenander and Rosenblatt [6] adapted for our use, we obtain asymptotic finite-dimensional probability distributions for such processes as $\{N^{\frac{1}{2}}\Phi^c_{12N}(\lambda)\}_{N=1}^{\infty}$. Theorem 3.4 is again used to show tightness of sequences of the corresponding probability measures and hence to prove the basic result, Theorem 4.5.

To begin, we obtain approximations for certain bilinear forms defined by $\{\xi(t)\}$ and $\{\zeta(t)\}$. Let $S_{Nk}^c(\lambda) = (2\pi)^{-1}\sum'_{|v|<k}\{R_{\xi\zeta N}(v) - E[R_{\xi\zeta N}(v)]\}\{\exp(-i\lambda v)-1\}/(-iv)$, $S_{Nk}^c(\lambda) = \mathrm{Re}[S_{Nk}(\lambda)]$ and $S_{Nk}^q(\lambda) = -\mathrm{Im}[S_{Nk}(\lambda)]$. In the above notation and in the sequel, a prime on the summation sign indicates that when the index is zero the coefficient is understood to be λ. The following result demonstrates that $S_{Nk}^c(\lambda)$ and $S_{Nk}^q(\lambda)$ are adequate approximations for the bilinear forms, $\theta_{\xi\zeta N}^c(\lambda)$ and $\theta_{\xi\zeta N}^q(\lambda)$ or, more precisely:

THEOREM 4.1. *Under Condition* A, *for* $M < k \ll N$, $N^{\frac{1}{2}}[\theta_{\xi\zeta N}^c(\lambda) - S_{Nk}^c(\lambda)]$ *and* $N^{\frac{1}{2}}[\theta_{\xi\zeta N}^q(\lambda) - S_{Nk}^q(\lambda)]$ *converge in probability to zero uniformly in* λ *as* $N, k \to \infty$.

PROOF. $N^{\frac{1}{2}}\theta_{\xi\zeta N}(\lambda) = N^{\frac{1}{2}}S_{Nk}(\lambda) + N^{\frac{1}{2}}\gamma_{Nk}(\lambda)$ where, since $M<k$, $\gamma_{Nk}(\lambda) = (2\pi)^{-1}\sum_{k\leq|v|<N}R_{\xi\zeta N}(\lambda)\{\exp(-iv\lambda)-1\}/(-iv)$. The result is proved by showing that, for large k and N, $\gamma_{Nk}(\lambda)$ is small with high probability. This, in turn, is proved by the methods of Theorem 1 [6], page 188, somewhat complicated by the correlations permitted by Condition A and by the appearance of more than one series. We omit the details of the proof.

These approximations enable us to establish the asymptotic finite dimensional distributions of the processes $\{N^{\frac{1}{2}}\theta_{\xi\zeta N}^c(\lambda)\}$ and $\{N^{\frac{1}{2}}\theta_{\xi\zeta N}^q(\lambda)\}$ as given in the next theorem.

THEOREM 4.2. *For* $0 \leqq \lambda_1 < \lambda_2 < \cdots < \lambda_\rho \leqq \pi$ *and under Condition* A (i) *any* ρ*-vector* $N^{\frac{1}{2}}(\theta_{\xi\zeta N}^c(\lambda_1), \theta_{\xi\zeta N}^c(\lambda_2), \cdots, \theta_{\xi\zeta N}^c(\lambda_\rho))$ *has an asymptotic probability distribution that is normal with zero mean and correlation matrix* $\|H_{\xi\zeta}^c(\min(\lambda_i, \lambda_j))\|$, (ii) *any* ρ*-vector* $N^{\frac{1}{2}}(\theta_{\xi\zeta N}^q(\lambda_1), \cdots, \theta_{\xi\zeta N}^q(\lambda_\rho))$ *has an asymptotic probability distribution that is normal with zero mean and correlation matrix* $\|H_{\xi\zeta}^q(\min(\lambda_i, \lambda_j))\|$.

PROOF. To prove (i) it is sufficient to prove the asymptotic normality with zero mean and variance $(\sum_{i=1}^{\rho} t)^2 H_{\xi\zeta}^c(\lambda_1) + \sum_{j=2}^{\rho}(\sum_{i=j}^{\rho} t_i)^2(H_{\xi\zeta}^c(\lambda_j) - H_{\xi\zeta}^c(\lambda_{j-1}))$ of all random variables of the form $N^{\frac{1}{2}}\sum_{j=1}^{\rho} t_i\theta_{\xi\zeta N}^c(\lambda_j)$ where the t_j are arbitrary real numbers. This device reduces the problem from ρ-dimensions to a problem of one dimension. The unidimensional case is dealt with below and the same methods used apply to the more general case. Some random variables related to the bilinear forms, $S_{Nk}(\lambda)$ and $\theta_{\xi\zeta N}(\lambda)$, are defined as follows. For $k < N$ and $v = 1, 2, \cdots, N$ let

$$\begin{aligned}2\pi Z_\lambda(v) = {} & \lambda[\xi(v)\zeta(v) - E[\xi(v)\zeta(v)]] \\ & + \textstyle\sum_{j=1}^{k}[\xi(v)\zeta(v+j) - E[\xi(v)\zeta(v+j)]]\{\exp(-i\lambda j)-1\}/(-ij) \\ & + \textstyle\sum_{j=1}^{k}[\xi(v+j)\zeta(j) - E[\xi(v+j)\zeta(v)]]\{\exp(-i\lambda j)-1\}/(-ij).\end{aligned}$$

We note that, by Condition A, $\{Z_\lambda(v)\}$ are $k+M+1$ dependent identically distributed random variables. Also let $U_N(\lambda) = N^{-\frac{1}{2}}\sum_{v=1}^{N} Z_\lambda(v)$, $U_N{}^c(\lambda) = \mathrm{Re}[U_N]$ and $U_N{}^q(\lambda) = -\mathrm{Im}[U_N]$. Using a limit theorem for dependent random variables due to Hoeffding and Robbins [7] one can assert that $U_N{}^c(\lambda)$ and $U_N{}^q(\lambda)$ are asymptotically normally distributed. It is evident that for all $k > 0$, $U_N(\lambda) \to_p N^{\frac{1}{2}}S_{Nk}(\lambda)$ as $N \to \infty$ and similarly for the real and imaginary parts. Furthermore, Theorem 4.1 asserts that $N^{\frac{1}{2}}(\theta_{\xi\zeta N}(\lambda)-S_{Nk}(\lambda)) \to_p 0$ as $N, k \to \infty$ which implies that $N^{\frac{1}{2}}\theta^c_{\xi\zeta N}(\lambda)$ and $N^{\frac{1}{2}}\theta^q_{\xi\zeta N}(\lambda)$ are asymptotically normally distributed. The asymptotic variances are given by Theorem 3.2. This constitutes the proof for the unidimensional case, and some obvious modifications of the above argument together with the remarks made above concerning the Cramér–Wold device, establish the result for the ρ-dimensional case.

Now we discuss the moduli of continuity of the sample paths of our processes. We let $\omega_{\theta_{\xi\zeta N}}(\delta) = \sup_{|\lambda_2-\lambda_1|\leqq\delta} N^{\frac{1}{2}}\left|\theta_{\xi\zeta N}(\lambda_2)-\theta_{\xi\zeta N}(\lambda_1)\right|$ be the modulus of continuity of a realization of the process $\{N^{\frac{1}{2}}\theta_{\xi\zeta N}(\lambda)\}$ and define $\omega_{\theta^c_{\xi\zeta N}}(\delta)$ and $\omega_{\theta^q_{\xi\zeta N}}(\delta)$ similarly. Since the series $\{\xi(t)\}$ and $\{\zeta(t)\}$ certainly satisfy Condition 2, Theorem 3.4 applied to $\{N^{\frac{1}{2}}\theta^c_{\xi\zeta N}(\lambda)\}$ and $\{N^{\frac{1}{2}}\theta^q_{\xi\zeta N}(\lambda)\}$ gives conditions sufficient to imply that $\omega_{\theta^c_{\xi\zeta N}}(\delta)\downarrow 0$ and $\omega_{\theta^q_{\xi\zeta N}}(\delta)\downarrow 0$ in probability as $\delta \downarrow 0$ independent of N. More precisely, the theorem implies that, for every $\varepsilon > 0$ there exists a function $\omega_{\theta^c_{\xi\zeta}}(\delta, \varepsilon)\downarrow 0$ as $\delta\downarrow 0$ not depending on N such that

$$(4.1) \qquad P[\theta^c_{\xi\zeta N}(\cdot)\,|\,\omega_{\theta^c_{\xi\zeta N}}(\delta) \leqq \omega_{\theta^c_{\xi\zeta}}(\delta,\varepsilon), \delta > 0] > 1-\varepsilon/2,$$

and similarly for the process defined by the quadrature spectral distribution function. Now, $\omega_{\Phi_{\xi\zeta N}}(\delta) \leqq \omega_{\theta_{\xi\zeta N}}(\delta)+\omega_{[\Phi_{\xi\zeta N}-\theta_{\xi\zeta N}]}(\delta)$ and since $\mathrm{Cov}\,[\xi(j_1)\zeta(j_2)] = 0$ if $|j_1-j_2| > M$

$$\omega_{[\Phi_{\xi\zeta N}-\theta_{\xi\zeta N}]}(\delta) \leqq \sup_{|\lambda_2-\lambda_1|\leqq\delta}\left|1/2\pi \textstyle\sum_{|v|<M}(\exp(-i\lambda_2 v)-\exp(-i\lambda_1 v))R_{\xi\zeta}(v)\right|$$
$$= \omega_{\xi\zeta}(\delta).$$

It can be seen that $\omega_{\xi\zeta}(\delta)\downarrow 0$ as $\delta\downarrow 0$, at least for $0 < \delta < \pi/M$, and is bounded by $1/\pi\sum_{|v|<M}|R_{\xi\zeta}(v)|$ otherwise. This, together with (4.1), implies that, for every $\varepsilon > 0$, there exists a function $\omega_{\Phi^c_{\xi\zeta}}(\delta,\varepsilon)\downarrow 0$ as $\delta\downarrow 0$ not depending on N such that

$$(4.2) \qquad P[\Phi^c_{\xi\zeta N}(\cdot)\,|\,\omega_{\Phi^c_{\xi\zeta N}}(\delta) \leqq \omega_{\Phi^c_{\xi\zeta}}(\delta,\varepsilon), \delta > 0] > 1-\varepsilon/2$$

and similarly for the quadrature spectral distribution function.

If $\{P^c_{\xi\zeta N}\}_{N=1}^{\infty}$ and $\{P^q_{\xi\zeta N}\}_{N=1}^{\infty}$ are the sequences of probability measures generated in $C[0, \pi]$ by the real and imaginary parts of the processes $\{\Phi_{\xi\zeta N}(\lambda)\}_{N=1}^{\infty}$ then, since $\Phi_{\xi\zeta N}(0) = 0$ for all N, Lemma 2.1 [11] implies that the sequences are tight. This, together with (4.2) and Theorem 4.2, implies the following result.

THEOREM 4.3. *Under Condition* A, *as* $N\to\infty$, *the measures* $\{P^c_{\xi\zeta N}\}$ *and* $\{P^q_{\xi\zeta N}\}$ *converge weakly to the measures* $P^c_{\xi\zeta}$ *and* $P^q_{\xi\zeta}$ *generated in* $C[0, \pi]$ *by the Gaussian processes* $\{\Phi^c_{\xi\zeta}(\lambda)\}$ *and* $\{\Phi^q_{\xi\zeta}(\lambda)\}$ *where*

$$\Phi^c_{\xi\zeta}(0) = \Phi^q_{\xi\zeta}(0) = E[\Phi^c_{\xi\zeta}(\lambda)] = E[\Phi^q_{\xi\zeta}(\lambda)] = 0 \qquad \textit{and}$$
$$E[\Phi^c_{\xi\zeta}(\lambda)\Phi^c_{\xi\zeta}(\mu)] = H^c_{\xi\zeta}[\min(\lambda,\mu)],$$
$$E[\Phi^q_{\xi\zeta}(\lambda)\Phi^q_{\xi\zeta}(\mu)] = H^q_{\xi\zeta}[\min(\lambda,\mu)] \qquad \textit{and}$$
$$E[\Phi^c_{\xi\zeta}(\lambda)\Phi^q_{\xi\zeta}(\mu)] = H^{cq}_{\xi\zeta}[\min(\lambda,\mu)].$$

We now establish relationships between series such as those of (2.1) and those of Condition A that permit the analogue of Theorem 4.3 to be proved for series of the form (2.1) provided the conditions placed upon the coefficients $\{a(v)\}$ and $\{b(v)\}$ are those of Condition 5.

First note that

$$f_{12}(l) = 2\pi f''_{12}(l) f_{\xi\zeta}(l) \tag{4.3}$$

where $f''_{12}(l) = (2\pi)^{-1}\sum_{r=-\infty}^{\infty} a(r)\exp(irl)\sum_{s=-\infty}^{\infty} b(s)\exp(-isl)$ and $f_{\xi\zeta}(l) = (2\pi)^{-1}\sum_{v=-\infty}^{\infty} R_{\xi\zeta}(v)\exp(-ilv)$. We now give a result that links the elementary series to the linear series.

THEOREM 4.4. *Under Condition* 5

$$N^{\frac{1}{2}}\sup_{0\leq\lambda\leq\pi}\left|\int_0^\lambda [f_{12N}(l)-(f_{\xi\zeta}(l))^{-1}f_{12}(l)f_{\xi\zeta N}(l)]\,dl\right| \to_p 0.$$

PROOF. We first note that

$$\int_0^\lambda f_{12N}(l)\,dl$$
$$= (2\pi N)^{-1}\sum_{r,s=-\infty}^{\infty} a(r)b(s)\sum_{n,m=1}^{\prime N}\xi(n-r)\zeta(m-s)\frac{\exp(-i\lambda(n-m))-1}{-i(n-m)}$$

and by (4.3)

$$\int_0^\lambda (f_{\xi\zeta}(l))^{-1}f_{12}(l)f_{\xi\zeta N}(l)\,dl$$
$$= 2\pi\int_0^\lambda f''_{12}(l)f_{\xi\zeta N}(l)\,dl$$
$$= (2\pi N)^{-1}\sum_{r,s=-\infty}^{\infty} a(r)b(s)\int_0^\lambda\sum_{n,m=1}^{\prime N}\xi(n)\zeta(m)\exp(il(r-s-n+m))\,dl$$

and hence

$$2\pi N^{\frac{1}{2}}\int_0^\lambda [f_{12N}(l)-(f_{\xi\zeta}(l))^{-1}f_{12}(l)f_{\xi\zeta N}(l)]\,dl = N^{-\frac{1}{2}}\sum_{r,s=-\infty}^{\infty} a(r)b(s)d(r,s)$$

where

$$d(r,s) = \sum_{n,m=1}^{\prime N}\xi(n-r)\zeta(m-s)\frac{\exp(-i\lambda(n-m))-1}{-i(n-m)}$$
$$-\sum_{n,m=1}^{N-r}\sum_{m=1-s}^{\prime N-s}\xi(n+r)\zeta(m+s)\frac{\exp(-i\lambda(n-m))-1}{-i(n-m)}.$$

Applying the method of Theorem 2, page 192 [6] to this statistic it can be shown that it tends to zero in probability. Obviously the real and imaginary parts tend to zero in probability also.

Integrating by parts we see that

$$N^{\frac{1}{2}}[2\pi\int_0^\lambda f''_{12}(l)f_{\xi\zeta N}(l)\,dl - F_{12}(l)]$$
$$= N^{\frac{1}{2}}2\pi\int_0^\lambda f''_{12}(l)[f_{\xi\zeta N}(l)-f_{\xi\zeta}(l)]\,dl$$
$$= N^{\frac{1}{2}}2\pi f''_{12}(\lambda)\Phi_{\xi\zeta N}(\lambda) - N^{\frac{1}{2}}2\pi\int_0^\lambda f'''_{12}(l)\Phi_{\xi\zeta N}(l)\,dl.$$

Theorem 4.3 implies that the real and imaginary parts of $N^{\frac{1}{2}}\int_0 f_{12}^{\prime u}(l)\Phi_{\xi\zeta N}(l)\,dl$ are asymptotically normal variables. Combining this fact with the conclusion of Theorem 4.4, it follows that $N^{\frac{1}{2}}\Phi_{12N}(\lambda)$ has real and imaginary parts that are asymptotically normally distributed. The asymptotic variances of these variables were obtained in Corollary 3.3. Using the Cramér–Wold device and the argument used above we can show the following result.

THEOREM 4.5. *For* $0 \leqq \lambda_1 < \lambda_2 < \cdots < \lambda_p \leqq \pi$, *the* p-*vectors* $N^{\frac{1}{2}}(\Phi_{12N}^c(\lambda_1), \cdots, \Phi_{12N}^c(\lambda_p))$ *and* $N^{\frac{1}{2}}(\Phi_{12N}^q(\lambda_1), \cdots, \Phi_{12N}^q(\lambda_p))$ *have asymptotic probability distributions that are normal with zero mean and covariance matrices respectively*: $\|H_{12}^c(\min(\lambda_i, \lambda_j))\|$ *and* $\|H_{12}^q(\min(\lambda_i, \lambda_j))\|$.

We next examine the moduli of continuity of the processes $\{\Phi_{12N}^c(\lambda)\}$ and $\{\Phi_{12N}^q(\lambda)\}$. First of all, Theorem 3.4 applied to the processes $\{N^{\frac{1}{2}}\theta_{12N}^c(\lambda)\}$ and $\{N^{\frac{1}{2}}\theta_{\xi\zeta N}^q(\lambda)\}$ implies that $\omega_{\theta_{\xi\zeta N}^c}(\delta)\downarrow 0$ and $\omega_{\theta_{\xi\zeta N}^q}(\delta)\downarrow 0$ in probability as $\delta\downarrow 0$ provided the series satisfy Condition 2. Also, under the same conditions, it can be shown, as in the proof of Theorem 5.1 [8] that

$$\omega_{\Phi_{12N}^c}(\delta) \leqq \omega_{\theta_{12N}^c}(\delta)+\omega_{12}^c(\delta) \quad \text{and} \quad \omega_{\Phi_{12N}^q}(\delta) \leqq \omega_{\theta_{12N}^q}(\delta)+\omega_{12}^q(\delta)$$

where $\omega_{12}^c(\delta)\downarrow 0$ and $\omega_{12}^q(\delta)\downarrow 0$ as $\delta\downarrow 0$. These results, plus the fact that the sample paths are zero when the parameter is zero imply, by Lemma 2.1 [11], the compactness condition for the induced measures. Combining this with the results of Theorem 4.5 proves the following main result.

THEOREM 4.6. *Assume Condition 5. Then, as* $N\to\infty$ *the measures* $\{P_{12N}^c\}$ *and* $\{P_{12N}^q\}$ *generated in* $C[0, \pi]$ *by the processes* $\{N^{\frac{1}{2}}\Phi_{12N}^c(\lambda)\}$ *and* $\{N^{\frac{1}{2}}\Phi_{12N}^q(\lambda)\}$ *converge weakly to the measures* P_{12}^c *and* P_{12}^q *generated by the Gaussian processes* $\Phi_{12}^c(\lambda)$ *and* $\Phi_{12}^q(\lambda)$ *where* $\Phi_{12}^c(0) = \Phi_{12}^q(0) = E[\Phi_{12}^c(\lambda)] = E[\Phi_{12}^q(\lambda)] = 0$ *and* $E[\Phi_{12}^c(\lambda)\Phi_{12}^c(\mu)] = H_{12}^c[\min(\lambda,\mu)]$, $E[\Phi_{12}^q(\lambda)\Phi_{12}^q(\mu)] = H_{12}^q(\min(\lambda,\mu))$.

5. Goodness-of-Fit testing. The goodness-of-fit tests discussed below involve certain functionals on the paths of processes related to Wiener processes. If $\{\Phi(\lambda), \lambda\in[0, b]\}$ is a Gaussian process with $\Phi(0) = E[\Phi(\lambda)] = 0$ and $E[\Phi(\lambda)\Phi(\mu)] = H(\min(\lambda, \mu))$, then we establish the following notation for some common functionals on such a process. Let

$$D^+(\Phi, b) = \sup_{0\leqq\lambda\leqq b}\Phi(\lambda),$$

$$D(\Phi, b) = \sup_{0\leqq\lambda\leqq b}|\Phi(\lambda)|,$$

$$R(\Phi, b) = \sup_{0\leqq\lambda\leqq b}\Phi(\lambda) - \inf_{0\leqq\lambda\leqq b}\Phi(\lambda),$$

$$W_1^2(\Phi, b) = \int_0^b \Phi^2(\lambda)\,dH(\lambda) \qquad \text{and}$$

$$W_2^2(\Phi, b) = \int_0^b [\Phi(\lambda) - 1/H(b)\int_0^b \Phi(\mu)\,dH(\mu)]^2\,dH(\lambda).$$

If we assume $H(\cdot)$ is a continuous, strictly increasing function such that $H(0) = 0$ and $H(\pi) < \infty$, and if we let $\{B(t), t\in[0, 1]\}$ be a Wiener process with parameter $\sigma^2 = 1$, then probability distributions for the functionals on $\Phi(\cdot)$ are related to those on $B(\cdot)$ according to the following: for $\alpha > 0$,

$$\text{(i)} \quad P[D^+(\Phi, \pi) \leqq \alpha] = P[D^+(B, 1) \leqq \alpha/H(\pi)^{\frac{1}{2}}],$$

$$\text{(ii)} \quad P[D(\Phi, \pi) \leqq \alpha] = P[D(B, 1) \leqq \alpha/H(\pi)^{\frac{1}{2}}],$$

$$\text{(iii)} \quad P[R(\Phi, \pi) \leqq \alpha] = P[R(B, 1) \leqq \alpha/H(\pi)^{\frac{1}{2}}],$$

$$\text{(iv)} \quad P[W_1{}^2(\Phi, \pi) \leqq \alpha] = P[W_1{}^2(B, 1) \leqq \alpha/H(\pi)^2],$$

$$\text{(v)} \quad P[W_2{}^2(\Phi, \pi) \leqq \alpha] = P[W_2{}^2(B, 1) \leqq \alpha/H(\pi)^2].$$

For the Wiener process, the distributions for the Smirnov and Kolmogorov statistics are given by:

$$\text{(5.1)} \qquad P[D^+(B,1) \leqq \beta] = \Lambda(\beta) \equiv \Phi(\beta) - \Phi(-\beta),$$

$$\text{(5.2)} \qquad P[D(B,1) \leqq \beta) = \Delta(\beta) \equiv \textstyle\sum_{n=-\infty}^{\infty} (-1)^n[\Phi(\beta(2n+1)) - \Phi(\beta(2n-1))],$$

where $\Phi(\cdot)$, in this context, is the normal probability distribution function. $\Delta(\cdot)$ is tabulated in [6]. The distribution for the range of a Wiener process, obtained by Feller [5], is given by

$$\text{(5.3)} \qquad P[R(B,1) \leqq \beta] = \Xi(\beta) \equiv 1 - \textstyle\sum_{n=1}^{\infty}(-1)^{n+1}8n[1-\Phi(n\beta)].$$

We include a partial tabulation for $\Xi(\cdot)$ in Table 2. The distribution for the Cramér–Von Mises statistic is found by inverting the transform, $\phi_1(z) = (\cosh(2z)^{\frac{1}{2}})^{-\frac{1}{2}}$, of the probability density for the statistic $W_1{}^2(B, 1)$ to obtain

$$\text{(5.4)} \qquad P[W_1{}^2(B,1) \leqq \beta] = \Omega(\beta) \equiv 2^{\frac{1}{2}} \textstyle\sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n}[1 - \chi_1{}^2((4n+1)^2/4\beta)],$$

where $\chi_1{}^2(\cdot)$ is the probability distribution function of a $\chi_1{}^2$ random variable. A partial tabulation of $\Omega(\cdot)$ also appears in Table 2. The Watson-type statistic [12], $W_2{}^2(B, 1)$, can be shown to have a probability density with Laplace Transform

$$\phi_2(z) = \textstyle\prod_{k=1}^{\infty} [1+2z(k\pi)^{-2}]^{-\frac{1}{2}} = [(-2z)^{\frac{1}{2}}/\sin(-2z)^{\frac{1}{2}}]^{\frac{1}{2}}.$$

$\phi_2(z)$ also arises in connection with the Cramér–Von Mises statistic for probability distribution functions. Anderson and Darling [1], after inverting the transform,

TABLE 2

Tabulation of $\Xi(\cdot)$ *and* $\Omega(\cdot)$

α	$\Xi(\alpha)$	α	$\Omega(\alpha)$
0.7898	0.005	0.0294	0.005
0.8326	0.01	0.0345	0.01
0.9030	0.025	0.0444	0.025
0.9720	0.05	0.0565	0.05
1.0628	0.10	0.0765	0.10
2.2412	0.90	1.1958	0.90
2.4977	0.95	1.6557	0.95
2.7344	0.975	2.1347	0.975
3.0233	0.99	2.7875	0.99
3.2272	0.995	3.2918	0.995

obtained and tabulated a probability distribution function which they denoted by $a_1(z)$. Consequently

(5.5) $$P[W_2{}^2(B,1) \leqq \beta] = a_1(\beta).$$

Since each of the functionals under discussion is continuous in the uniform topology on $C[0,\pi]$, the weak convergence results of Theorem 4.5 imply the convergence in distribution of functionals of sample spectral distribution functions to those functionals on the limit processes. This, together with (i) to (v) above, and (5.1) to (5.5) imply the following result

THEOREM 5.1. *Assume Condition* 5 *and* $\alpha > 0$. *Then*

(a) $$\lim_{N\to\infty} P[D^+(N^{\frac{1}{2}}\Phi^c_{jkN}, \pi) \leqq \alpha] = \Lambda(\alpha/H^c_{jk}(\pi)^{\frac{1}{2}}),$$

$$\lim_{N\to\infty} P[D(N^{\frac{1}{2}}\Phi^c_{jkN}, \pi) \leqq \alpha] = \Delta(\alpha/H^c_{jk}(\pi)^{\frac{1}{2}}),$$

$$\lim_{N\to\infty} P[R(N^{\frac{1}{2}}\Phi^c_{jkN}, \pi) \leqq \alpha] = \Xi(\alpha/H^c_{jk}(\pi)^{\frac{1}{2}}),$$

$$\lim_{N\to\infty} P[W_1{}^2(N^{\frac{1}{2}}\Phi^c_{jkN}, \pi) \leqq \alpha] = \Omega(\alpha/H^c_{jk}(\pi)^2),$$

$$\lim_{N\to\infty} P[W_2{}^2(N^{\frac{1}{2}}\Phi^c_{jkN}, \pi) \leqq \alpha] = a_1(\alpha/H^c_{jk}(\pi)^2), \qquad j \neq k.$$

(b) *Similar results hold for processes such as* $\{N^{\frac{1}{2}}\Phi^q_{jkN}(\lambda), \lambda \in [0,\pi]\}$ *and* $\{N^{\frac{1}{2}}\Phi_{jjN}(\lambda), \lambda \in [0,\pi]\}$ *with the covariance parameters modified to* $H^q_{jk}(\pi)$ *and* $H_{jj}(\pi)$.

Note that if the cross-spectrum is hypothesized to be zero, then, by Corollary 3.3 (iii), the processes $\{\Phi^c_{jk}(\lambda)\}$ and $\{\Phi^q_{jk}(\lambda)\}$ are independent and hence joint tests based on the co-spectral and quadrature spectral distribution functions are available from Theorem 5.1.

Besides the above tests for one or two series one might wish to consider multiple series of order $m > 2$. If $\mathbf{c}' = (c_1, c_2, \cdots, c_m)$ is a vector of constants and $\mathbf{F}'(\lambda) = (F_{11}(\lambda), \cdots, F_{mm}(\lambda))$ and $\mathbf{F}_N'(\lambda) = (F_{11N}(\lambda), \cdots, F_{mmN}(\lambda))$ then, assuming N observations have been taken on each series, one can consider the statistic $L_N(\lambda) = N^{\frac{1}{2}}\mathbf{c}'(\mathbf{F}_N(\lambda) - \mathbf{F}(\lambda))$. The sequence of processes $\{L_N(\lambda), \lambda \in [0, \pi]\}_{N=1}^{\infty}$ converges to a Gaussian process $\{\Phi_L(\lambda), \lambda \in [0, \pi]\}$ with $\Phi_L(0) = E[\Phi_L(\lambda)] = 0$ and $\operatorname{Cov}[\Phi_L(\lambda), \Phi_L(\mu)] = 2\pi \int_0^{\min(\lambda,\mu)} (\sum_{j,k=1}^m c_j c_k |f_{jk}(l)|^2)\, dl = H_L(\min(\lambda, \mu))$. If one assumes that the autospectral densities do not vanish on any sets of positive measure, then $H_L(\lambda)$ is strictly monotone increasing. Hence one can obtain tests based on the following:

(5.6) $$\lim_{N\to\infty} P[R(L_N, \pi) \leqq \alpha] = \Xi(\alpha/H_L(\pi)^{\frac{1}{2}}), \qquad \alpha > 0$$

$$\lim_{N\to\infty} P[W_1{}^2(L_N, \pi) \leqq \alpha] = \Omega_1(\alpha/H_L(\pi)^2), \qquad \alpha > 0.$$

Similar results can be obtained for the co-spectral and quadrature spectral distribution functions using Table 1.

If the spectral distribution functions are not completely specified then the parameters defined by the variances are not available and so must be estimated from the data. Under Condition 1 we have that

$$H^c_{ij}(\pi) = 8^{-1}[R_{ii}(0)R_{jj}(0) + R_{ij}(0)] + 4^{-1}\sum_1^{\infty} [R_{ii}(v)R_{jj}(v) + R_{ij}(v)R_{ij}(-v)],$$

$$H_{ij}^{q}(\pi) = 8^{-1}[R_{ii}(0)R_{jj}(0) - R_{ij}^{2}(0)] + 4^{-1}\sum_{1}^{\infty} [R_{ii}(v)R_{jj}(v) - R_{ij}(v)R_{ij}(-v)]$$

and

$$H_{ij}(\pi) = 4^{-1}\sum_{-\infty}^{\infty} R_{ij}^{2}(v).$$

Assuming T_N, $N \to \infty$ in such a way that $T_N/N \to 0$ and letting

$$H_{ijT_N}^{c}(\pi) = 8^{-1}[R_{iiN}(0)R_{jjN}(0) + R_{ijN}(0)] + 4^{-1}\sum_{1}^{T_N} [R_{iiN}(v)R_{jjN}(v) + R_{ijN}(v)R_{ijN}(-v)],$$

$$H_{ijT_N}^{q}(\pi) = 8^{-1}[R_{iiN}(0)R_{jjN}(0) - R_{ijN}^{2}(0)] + 4^{-1}\sum_{1}^{T_N} [R_{iiN}(v)R_{jjN}(v) - R_{ijN}(v)R_{ijN}(-v)] \quad \text{and}$$

$$H_{ijT_N}(\pi) = 4^{-1}\sum_{|v|<T_N} R_{ijN}^{2}(v)$$

then, under Condition 1, $H_{ijT_N}^{c}(\pi)$, $H_{ijT_N}^{q}(\pi)$ and $H_{ijT_N}(\pi)$ may be shown to be consistent estimators for $H_{ij}^{c}(\pi)$, $H_{ij}^{q}(\pi)$ and $H_{ij}(\pi)$. This result and Theorem 5.1 yield the following result.

THEOREM 5.2. *Assume Condition 5. Then, for $\alpha > 0$,*

$$\lim_{T_N, N\to\infty; T_N/N\to 0} P[D(N^{\frac{1}{2}}\Phi_{ijN}^{c}, \pi) \leqq \alpha(H_{T_N}^{c}(\pi)^{\frac{1}{2}})]$$
$$= \lim_{T_N, N\to\infty; T_N/N\to 0} P[D(N^{\frac{1}{2}}\Phi_{ijN}^{q}, \pi) \leqq \alpha(H_{T_N}^{q}(\pi)^{\frac{1}{2}})] = \Delta(\alpha).$$

The next theorem is a two-sample Kolmogorov–Smirnov result for cross-spectral distribution functions. Let $\{X_{1i}\}_{i=1}^{N_1}$ and $\{X_{2j}\}_{j=1}^{N_2}$ be two independent series of observations from one time series and let $\{Y_{1i}\}_{i=1}^{N_1}$ and $\{Y_{2j}\}_{j=1}^{N_2}$ be two independent series of observations from another. Let $C_{XYN}^{(1)}(\lambda)$ be the sample co-spectral distribution function of the first sets of observations and let $C_{XYN}^{(2)}(\lambda)$ be the co-spectral distribution function of the second sets of observations. Let $Q_{XYN}^{(1)}(\lambda)$ and $Q_{XYN}^{(2)}(\lambda)$ be defined similarly and let $H_{T_N}^{c(1)}(\pi)$, $H_{T_N}^{c(2)}(\pi)$, $H_{T_N}^{q(1)}(\pi)$ and $H_{T_N}^{q(2)}(\pi)$ be consistent estimators of their population analogues.

THEOREM 5.3. *Assume Condition 5. Let $N = 2N_1N_2/(N_1+N_2)$ and let N_i, $T_N \to \infty$ such that $T_{N_i}/N_i \to 0$. Then, if $N_i/N_j > c > 0$, $i \neq j$, and if $\alpha > 0$*

(i) $\lim P[\sup_{0\leqq\lambda\leqq\pi} N^{\frac{1}{2}} |C_{XYN_1}^{(1)}(\lambda) - C_{XYN_2}^{(2)}(\lambda)| \leqq \alpha(H_{T_{N_1}}^{c(1)}(\pi) + H_{T_{N_2}}^{c(2)}(\pi))^{\frac{1}{2}}] = \Delta(\alpha)$,

(ii) $\lim P[\sup_{0\leqq\lambda\leqq\pi} N^{\frac{1}{2}} |Q_{XYN_1}^{(1)}(\lambda) - Q_{XYN_2}^{(2)}(\lambda)| \leqq \alpha(H_{T_{N_1}}^{q(1)}(\pi) + H_{T_{N_2}}^{q(2)}(\pi))^{\frac{1}{2}}] = \Delta(\alpha)$.

PROOF. The facts that

$$\lim_{N_i, T_N\to\infty; T_{N_i}/N_i\to 0} \frac{N_1N_2}{N_1+N_2} \operatorname{Cov}[(C_{XYN_1}^{(1)}(\lambda) - C_{XYN_2}^{(2)}(\lambda)), (C_{XYN_1}^{(1)}(\mu) - C_{XYN_2}^{(2)}(\mu))]$$
$$= H_{12}^{c}[\min(\lambda, \mu)]$$

and

$$H_{T_{N_1}}^{c(1)}(\pi) + H_{T_{N_2}}^{c(2)}(\pi) \to_p 2H_{12}^{c}(\pi)$$

imply (i) and (ii) is proved similarly.

Suppose one were to consider multiple series (each with N observations taken) and were to look for a more general statement of Theorem 5.2 or of Theorem 5 [6], page 198. Then, assuming $\sum c_i = 0$ and each series to have a common but unknown spectral distribution function, one can show that $L_N(\lambda) = N^{\frac{1}{2}}\mathbf{c}'\mathbf{F}_N(\lambda)\mathbf{c}$. Also in the notation of Lemma 2.2.2 it can be shown that $H_L(\pi) = \mathbf{c}'\mathbf{R}^2\mathbf{c}$ where $\mathbf{R}^2$ has as its i, jth component, $(2\pi)^{-1}\sum_{-\infty}^{\infty} R_{ij}(v)$, and that $H_L(\pi)$ can be consistently estimated by $H_{LT_N}(\pi) = \mathbf{c}'\mathbf{R}^2_{T_N}\mathbf{c}$ where the i, jth component of $\mathbf{R}^2_{T_N}$ is given by $(2\pi)^{-1}\sum_{|v|<T_N} R^2_{ijN}(v)$ where $N, T_N \to \infty$ and $T_N/N \to 0$. Thus, under the above assumption, expressions such as (5.1) remain valid with $H_L(\pi)$ replaced by $H_{LT_N}(\pi)$.

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