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LOCAL PROPERTIES OF THE AUTOREGRESSIVE SERIES

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0. Summary. Let us have a normal stationary autoregressive series $\{X_t\}_{-\infty}^{\infty}$ of the nth order with $EX_t = 0$. Denote b the vector of autoregressive parameters. In this paper the Radon–Nikodym derivative dP_b/dP is studied, where P_b is the probability measure corresponding to the finite part (of length N) of the autoregressive series and $P = P_0$, i.e., P corresponds to the case, when X_t are independent normal random variables. The function dP_b/dP may be expanded in the power series of components of vector b. If the norm $||b||$ is small, then the absolute term and the linear terms are most important. These terms are given in the paper and they are used for an approximation of the probability $P_b(A)$, where A is a Borel set in the N-dimensional Euclidean space R_N. The probability that a normal stationary autoregressive series does not exceed a constant barrier is analysed as an example. A second example is devoted to the properties of the sign-test when the observations are dependent and may be described by the autoregressive model.

1. Introduction. Let $a_0, a_1, \cdots, a_n$ ($a_n > 0$, $a_0 \neq 0$) be real constants such that the equation $\sum_{k=0}^{n} a_k \lambda^k = 0$ has all roots in absolute value smaller than 1. Let $\{Y_t\}_{-\infty}^{\infty}$ be a sequence of independent random variables with $N(0, 1)$. The sequence $\{X_t\}_{-\infty}^{\infty}$ given by recurrent formula

$$\sum_{i=0}^{n} a_{n-i} X_{t-i} = Y_t, \qquad -\infty < t < \infty, \tag{1}$$

is called the autoregressive series of the nth order. Under above assumptions $\{X_t\}_{-\infty}^{\infty}$ is stationary. Let us have a finite part of this series, say $X_1, \cdots, X_N$. The vector $X = (X_1, \cdots, X_N)'$ has normal distribution. (Prime denotes the transposition.) Put

$$a = a_n, \qquad b_i = -a_{n-i}/a_n \qquad (1 \leqq i \leqq n),\ b_0 = -1. \tag{2}$$

Relation (1) may be written in the form

$$X_t = \sum_{i=1}^{n} b_i X_{t-i} + a^{-1} Y_t, \qquad -\infty < t < \infty, \tag{3}$$

and so $b = (b_1, \cdots, b_n)'$ is called the vector of autoregressive parameters. Its norm

$$||b|| = (\sum_{i=1}^{n} b_i^2)^{\frac{1}{2}}.$$

We shall study the properties of the probability measure corresponding to the random vector X, if $||b||$ is small. Such properties may be called "local" with respect to an analogy with the local properties of rank tests (e.g. local power) which are contained in [3], for example. The present author used a similar technique in [1] in connection with properties of Kolmogorov–Smirnov tests.

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2. The probability density of the finite part of the normal stationary autoregressive series.

LEMMA 1. *If $N \geqq 2n$, then the probability density of $X = (X_1, \cdots, X_N)'$ is*

$$p(x) = (2\pi)^{-N/2} a_n^{N-n} |E|^{\frac{1}{2}} \exp\{-\tfrac{1}{2}x'Hx\}, \tag{4}$$

where

$$x = (x_1, \cdots, x_N)', \qquad E = \|e_{jk}\|_{j,k=1}^{n}, \qquad H = \|h_{ts}\|_{t,s=1}^{N},$$

$$e_{jk} = \sum_{i=0}^{\min(j-1,k-1,n-|j-k|)} a_{n-i}a_{n-i-|j-k|} - \sum_{i=n-\max(j,k)+1}^{n-|j-k|} a_{n-i}a_{n-i-|j-k|},$$

$$h_{ts} = \sum_{i=0}^{f(t,s)} a_{n-i}a_{n-i-|t-s|}, \qquad f(t,s) = \min(t-1, s-1, n-|t-s|, N-t, N-s),$$

$$|E| = \det E.$$

The matrices E and H are regular.

PROOF. Lemma 1 is a slightly modified theorem from Hájek's paper [2], where the probability density of the vector $(X_0, X_1, \cdots, X_N)'$ is given with a more explicit formula for $f(t, s)$.

LEMMA 2. *The probability density* (4) *may be rewritten in the form*

$$p(x) = (2\pi)^{-N/2} a^N |E^*|^{\frac{1}{2}} \exp\{-\tfrac{1}{2}a^2 \textstyle\sum_{i=0}^{n}\sum_{j=0}^{n} q_{ij} b_i b_j\}, \tag{5}$$

where

$$q_{ij} = \sum_{t=\min(i,j)+1}^{N-\max(i,j)} x_t x_{t+|i-j|}, \qquad 0 \leqq i,j \leqq n, \tag{6}$$

$$|E^*| = \det E^*, \qquad E^* = \|e_{jk}^*\|_{j,k=1}^{n} = a^{-2}E, \qquad \textit{i.e.},$$

$$e_{jk}^* = \sum_{i=0}^{\min(j-1,k-1,n-|j-k|)} b_i b_{i+|j-k|} - \sum_{i=n-\max(j,k)+1}^{n-|j-k|} b_i b_{i+|j-k|}. \tag{7}$$

The parameters $a, b_0, \cdots, b_n$ are given in (2).

This lemma with the following proof belongs to Professor Hájek. He gave it in his lectures on stationary processes in 1967–1968 at Charles University.

PROOF. The relation $a_n^{N-n}|E|^{\frac{1}{2}} = a^N|E^*|^{\frac{1}{2}}$ obviously holds. The elements e_{jk}^* follow from the Lemma 1 with respect to (2). Further

$$h_{ts} = \sum_{i=0}^{f(t,s)} a_{n-i}a_{n-i-|t-s|} = a^2 \sum_{i=0}^{f(t,s)} b_i b_{i+|t-s|},$$

and

$$\sum_{t=1}^{N}\sum_{s=1}^{N} h_{ts}x_t x_s = a^2 \sum_{t=1}^{N}\sum_{s=1}^{N}\sum_{i=0}^{f(t,s)} b_i b_{i+|t-s|} x_t x_s \tag{8}$$

$$= a^2 \sum_{t=1}^{N}\sum_{i=0}^{f(t,t)} b_i^2 x_t^2 + 2a^2 \sum\sum_{1 \leqq t < s \leqq N} \sum_{i=0}^{f(t,s)} b_i b_{i+s-t} x_t x_s.$$

Consider the first sum. From $0 \leqq i \leqq f(t,t) = \min(t-1, N-t, n)$ it follows $0 \leqq i \leqq n$, $i+1 \leqq t \leqq N-i$, and therefore

$$a^2 \sum_{t=1}^{N}\sum_{i=0}^{f(t,t)} b_i^2 x_t^2 = a^2 \sum_{i=0}^{n} b_i^2 \sum_{t=i+1}^{N-i} x_t^2 = a^2 \sum_{i=0}^{n} b_i^2 q_{ii}.$$

As for the second sum in (8), we introduce $j = i+s-t$. We have $s = t+j-i$ and from $s > t$ it follows $j > i$. Thus $0 \leqq i \leqq f(t, s) = \min(t-1, n-j+i, N-t-j+i)$. From here we obtain $i+1 \leqq t \leqq N-j$, $0 \leqq i < n$. The relation $i \leqq n-j+i$ implies $j \leqq n$. Together

$$\sum\sum_{1 \leqq t < s \leqq N} \sum_{i=0}^{f(t,s)} b_i b_{i+s-t} x_t x_s = \sum\sum_{0 \leqq i < j \leqq n} b_i b_j \sum_{t=i+1}^{N-j} x_t x_{t+j-i}$$
$$= \sum\sum_{0 \leqq i < j \leqq n} b_i b_j q_{ij}.$$

The proof is finished.

3. Radon–Nikodym derivative.

THEOREM 3. *Put $a = 1$ and denote the probability measure corresponding to the density* (5) *by P_b. Further put $P_0 = P$. Then $P_b \ll P$ and*

$$dP_b/dP = 1 + \sum_{i=1}^{n} q_{0i} b_i + o(b), \tag{9}$$

where $\lim_{b \to 0} o(b)/\|b\| = 0$ and q_{0i} are given in (6), *i.e.,*

$$q_{0i} = \sum_{t=1}^{N-i} x_t x_{t+i}. \tag{10}$$

PROOF. Normal distribution with the density $p(x)$ is regular according to Lemma 1. The probability measure P corresponds to the N-dimensional normal distribution $N_N(0, I)$ with zero mean values and unit covariance matrix. This normal distribution is regular, too. Thus the measures P_b and P are equivalent and $P_b \ll P$ obviously holds.

Denoting μ the Lebegue's measure in space R_N, we may write density $p(x)$ in (5) in the form $dP_b/d\mu$. The well-known property of the Radon–Nikodym derivatives is

$$\frac{dP_b}{dP} = \frac{dP_b}{d\mu} \Big/ \frac{dP}{d\mu}. \tag{11}$$

For $a = 1$ we have

$$dP/d\mu = (2\pi)^{-N/2} \exp\{-\tfrac{1}{2}\sum_{i=1}^{N} x_i^2\} = (2\pi)^{-N/2} \exp\{-\tfrac{1}{2} q_{00}\}$$

and with respect to (11) we obtain

$$dP_b/dP = |E^*|^{\frac{1}{2}} \exp\{-\tfrac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} q_{ij} b_i b_j + \sum_{i=1}^{n} q_{0i} b_i\} \tag{12}$$

because of $q_{0i} = q_{i0}$, $1 \leqq i \leqq n$.

The elements e_{jk}^* are functions of $b_1, \cdots, b_n$ and we denote occasionally $e_{jk}^* = e_{jk}^*(b_1, \cdots, b_n)$. Analogously $E^* = E^*(b_1, \cdots, b_n)$ etc. According to (7) we get

$$e_{jj}^*(b_1, \cdots, b_n) = 1 + \sum_{i=1}^{j-1} b_i^2 - \sum_{i=n-j+1}^{n} b_i^2 \qquad \text{for} \quad 1 \leqq j \leqq n$$

and, therefore,

$$\left.\frac{\partial e_{jj}^*(b_1, \cdots, b_n)}{\partial b_s}\right|_{b=0} = 0 \qquad \text{for} \quad 1 \leqq s \leqq n. \tag{13}$$

Further we see that

$$E^*(0,0,\cdots,0) = I, \tag{14}$$

where I is the unit matrix. Now, we want to prove

$$\left.\frac{\partial\,|E^*(b_1,\cdots,b_n)|}{\partial b_s}\right|_{b=0} = 0 \qquad \text{for} \quad 1 \leqq s \leqq n. \tag{15}$$

From the calculus we know that

$$\frac{\partial\,|E^*(b_1,\cdots,b_n)|}{\partial b_s} = \sum_{k=1}^{n} |D_k(b_1,\cdots,b_n)|, \tag{16}$$

where

$$D_k(b_1,\cdots,b_n) = \|d_{ij}^{(k)}(b_1,\cdots,b_n)\|_{i,j=1}^{n},$$

$$d_{ij}^{(k)}(b_1,\cdots,b_n) = e_{ij}^*(b_1,\cdots,b_n) \qquad \text{for} \quad i \neq k,\ 1 \leqq j \leqq n,$$

$$d_{kj}^{(k)}(b_1,\cdots,b_n) = \frac{\partial}{\partial b_s} e_{kj}^*(b_1,\cdots,b_n) \qquad \text{for} \quad 1 \leqq j \leqq n.$$

From the definition of $d_{ij}^{(k)}(b_1,\cdots,b_n)$ it follows with respect to (14) that $d_{ik}^{(k)}(0,0,\cdots,0) = 0$ for $i \neq k$. From (13) we have $d_{kk}^{(k)}(0,0,\cdots,0) = 0$. Then all the elements of the kth column of $D_k(0,0,\cdots,0)$ are zeros and therefore

$$|D_k(0,0,\cdots,0)| = 0 \qquad \text{for} \quad 1 \leqq k \leqq n.$$

Now, the relation (16) implies (15). Denote

$$dP_b/dP = f(b_1,\cdots,b_n). \tag{17}$$

Taylor formula gives

$$f(b_1,\cdots,b_n) = f(0,\cdots,0) + \sum_{s=1}^{n} \left\{ \left.\frac{\partial f(b_1,\cdots,b_n)}{\partial b_s}\right|_{b=0} \right\} b_s + o(b). \tag{18}$$

Using (15) we easily get

$$\left.\frac{\partial f(b_1,\cdots,b_n)}{\partial b_s}\right|_{b=0} = q_{0s} \qquad \text{for} \quad 1 \leqq s \leqq n.$$

From here it follows (9).

THEOREM 4. *Denote B_N the system of Borel sets in R_N. Then*

$$P_b(A) = P(A) + \textstyle\sum_{i=1}^{n} b_i \int_A q_{0i}\, dP + o(b) \tag{19}$$

holds for any $A \in B_N$, where $\lim_{b\to 0} o(b)/\|b\| = 0$.

PROOF. From the Taylor formula we have

$$f(b)=f(0)+\left(b_1\frac{\partial}{\partial b_1}+\cdots+b_n\frac{\partial}{\partial b_n}\right)f(0)+Z,$$

where

$$Z=\tfrac{1}{2}\left(b_1\frac{\partial}{\partial b_1}+\cdots+b_n\frac{\partial}{\partial b_n}\right)^2 f(\Theta b), \qquad \Theta\in(0,1)$$

and $f(b)=f(b_1,\cdots,b_n)$, $f(0)=f(0,\cdots,0)$. In Theorem 3 it has been proved that

$$\left(b_1\frac{\partial}{\partial b_1}+\cdots+b_n\frac{\partial}{\partial b_n}\right)f(0)=\sum_{i=1}^{n} q_{0i}b_i .$$

We are obliged to analyze the remainder term Z only. With respect to (12) we write $f(b)=|E^*|^{\frac{1}{2}}g(b)$, where

$$g(b)=g(b_1,\cdots,b_n)=\exp\{-\tfrac{1}{2}\textstyle\sum_{i=1}^{n}\sum_{j=1}^{n}q_{ij}b_ib_j+\sum_{i=1}^{n}q_{0i}b_i\}.$$

We easily get

$$\frac{\partial^2 f}{\partial b_i\partial b_j}=\tfrac{1}{2}|E^*|^{-\frac{1}{2}}g\frac{\partial^2|E^*|}{\partial b_i\partial b_j}-\tfrac{1}{4}|E^*|^{-\frac{3}{2}}g\frac{\partial|E^*|}{\partial b_i}\frac{\partial|E^*|}{\partial b_j}$$

$$+\tfrac{1}{2}|E^*|^{-\frac{1}{2}}\frac{\partial|E^*|}{\partial b_i}\frac{\partial g}{\partial b_j}+\tfrac{1}{2}|E^*|^{-\frac{1}{2}}\frac{\partial|E^*|}{\partial b_j}\frac{\partial g}{\partial b_i}+|E^*|^{\frac{1}{2}}\frac{\partial^2 g}{\partial b_i\partial b_j}$$

for $1\leqq i,j\leqq n$, where $f=f(b)$, $|E^*|=|E^*(b)|$, $g=g(b)$. Obviously $|E^*|$, $\partial|E^*|/\partial b_i$ and $\partial^2|E^*|/\partial b_i\partial b_j$ $(1\leqq i,j\leqq n)$ are polynomials in $b_1,\cdots,b_n$. From the proof of Theorem 3 we see that $|E^*(0)|=1$. Thus for every i,j $(1\leqq i,\ j\leqq n)$ the $\varepsilon>0$ and finite constants K_1, K_2, K_3, K_4 exist such that for $\|b\|\leqq\varepsilon$

$$\left|\frac{\partial^2 f(b)}{\partial b_i\partial b_j}\right|\leqq K_1g(b)+K_2\left|\frac{\partial g(b)}{\partial b_j}\right|+K_3\left|\frac{\partial g(b)}{\partial b_i}\right|+K_4\left|\frac{\partial^2 g(b)}{\partial b_i\partial b_j}\right|$$

holds. Using the Schwarz inequality we get from (6)

$$|q_{ij}|\leqq\textstyle\sum_{t=1}^{N}x_t^{\,2} \qquad \text{for}\quad 0\leqq i,j\leqq n. \tag{20}$$

Further we derive

$$\frac{\partial g(b)}{\partial b_r}=\left(-\sum_{i=1}^{n}q_{ri}b_i+q_{0r}\right)g(b) \qquad \text{for}\quad 1\leqq r\leqq n,$$

$$\frac{\partial^2 g(b)}{\partial b_r\partial b_s}=-q_{rs}g(b)+\left(-\sum_{i=1}^{n}q_{ri}b_i+q_{0r}\right)\left(-\sum_{i=1}^{n}q_{si}b_i+q_{0s}\right)g(b) \quad \text{for}\quad 1\leqq r,s\leqq n.$$

Suppose $\varepsilon < 1/8n$. It implies $|b_i| < 1/8n$ for $1 \leqq i \leqq n$. For $\|b\| \leqq \varepsilon$ we have with respect to (20) $g(b) \leqq \exp\{\frac{1}{4}\sum_{t=1}^N x_t^2\}$ and

$$\left|\frac{\partial g(b)}{\partial b_r}\right| \leqq 2\sum_{t=1}^N x_t^2 \exp\left\{\tfrac{1}{4}\sum_{t=1}^N x_t^2\right\} \qquad \text{for} \quad 1 \leqq r \leqq n,$$

$$\left|\frac{\partial^2 g(b)}{\partial b_r \partial b_s}\right| \leqq \left[\sum_{t=1}^N x_t^2 + 4\left(\sum_{t=1}^N x_t^2\right)^2\right] \exp\left\{\tfrac{1}{4}\sum_{t=1}^N x_t^2\right\} \qquad \text{for} \quad 1 \leqq r, s \leqq n.$$

From $0 < \Theta < 1$ it follows that $\|\Theta b\| < \|b\|$ and thus for $\|b\| \leqq \varepsilon$ we get

$$|Z| \leqq \tfrac{1}{2}\sum_{i=1}^n \sum_{j=1}^n |b_i b_j| \left|\frac{\partial^2 f(\beta)}{\partial \beta_i \partial \beta_j}\right|_{\beta=\Theta b} \leqq \|b\|^2 R,$$

where

$$R = \frac{n}{2}\left\{K_1 + 2(K_2+K_3)\sum_{t=1}^N x_t^2 + K_4\left[\sum_{t=1}^N x_t^2 + 4\left(\sum_{t=1}^N x_t^2\right)^2\right]\right\}\exp\left\{\tfrac{1}{4}\sum_{t=1}^N x_t^2\right\}.$$

For any set $A \in B_N$

$$0 \leqq \int_A R\,dP \leqq \int_{R_N} R(2\pi)^{-N/2} \exp\{-\tfrac{1}{2}\textstyle\sum_{t=1}^N x_t^2\}\,dx_1 \cdots dx_N < \infty$$

obviously holds. Now, the assertion of Theorem 4 follows from the Radon–Nikodym theorem.

Unfortunately, the bound for Z given above is rather large for practical purposes.

4. Examples. We choose some special sets $A \in B_N$, which occur in classical statistical problems.

EXAMPLE 5. Let c be a real number. Put $A = \{x_1 \leqq c, \cdots, x_N \leqq c\}$. Then $P_b(A)$ is the probability that normal stationary autoregressive series $X_1, \cdots, X_N$ with zero mean values does not exceed the value c (does not exceed the barrier c). Denote $\varphi(x) = (2\pi)^{-\frac{1}{2}}\exp\{-\frac{1}{2}x^2\}$, $\Phi(x) = \int_{-\infty}^x \varphi(u)\,du$. We have

$$P(A) = (2\pi)^{-N/2}\int_{-\infty}^c \cdots \int_{-\infty}^c \exp\{-\tfrac{1}{2}\textstyle\sum_{j=1}^N x_j^2\}\,dx_1 \cdots dx_N = \Phi^N(c)$$

and

$$\int_A q_{0i}\,dP = (2\pi)^{-N/2}\sum_{t=1}^{N-i}\int_{-\infty}^c \cdots \int_{-\infty}^c x_t x_{t+i} \exp\{-\tfrac{1}{2}\textstyle\sum_{j=1}^N x_j^2\}\,dx_1 \cdots dx_N$$
$$= (N-i)(2\pi)^{-1}\exp\{-c^2\}\Phi^{N-2}(c).$$

From Theorem 4 we obtain

$$P_b(A) = \Phi^N(c) + (2\pi)^{-1}\exp\{-c^2\}\Phi^{N-2}(c)\textstyle\sum_{i=1}^n (N-i)b_i + o(b).$$

EXAMPLE 6. Let the set A consist of all the points $(x_1, \cdots, x_N)$, which have exactly k components positive. It corresponds to the study of properties of the sign test in the case, when the observations with zero mean values are dependent and they form the normal stationary autoregressive series with "small" autoregressive parameters. It is well known that the critical region of this test is given as the unification of sets A for some k's.

What result may be expected? Suppose for simplicity that $X_1, \cdots, X_N$ is the autoregressive series of the first order generated by $X_t = bX_{t-1} + Y_t$ (in this case we put briefly $b_1 = b$). If b is near to 1, the members $X_1, \cdots, X_N$ are highly correlated. With a large probability all the members have the same sign. If b is near to -1, then with a large probability the members change the sign regularly. The number of positive members is expected to be near to $N/2$. We also shall see similar properties for small b.

Come back to the normal stationary autoregressive series of nth order. The set A may be written in the form

$$A = \bigcup_{(i)} A_{i_1, \cdots i_k},$$

where $i_1, \cdots, i_k$ $(i_1 < i_2 < \cdots < i_k)$ is a subset of $\{1, 2, \cdots, N\}$ and $A_{i_1, \cdots, i_k} = \{(x_1, \cdots, x_N): x_t > 0 \text{ for } t \in \{i_1, \cdots, i_k\},\ x_t \leqq 0 \text{ for } t \notin \{i_1, \cdots, i_k\}\}$. The symbol $\bigcup_{(i)} A_{i_1, \cdots, i_k}$ means the union of all the sets $A_{i_1, \cdots, i_k}$. Their total number makes $\binom{N}{k}$ and the union is obviously disjoint. The analogous meaning will have the symbol $\sum_{(i)} p_{i_1, \cdots, i_k}$ as the sum of numbers. We have

$$\int_A q_{0j}\, dP = \sum_{t=1}^{N-j} \sum_{(i)} \int_{A_{i_1}, \cdots, i_k} x_t x_{t+j}\, dP.$$

When t and j are fixed, then there are exactly $\binom{N-2}{k-2}$ sets $A_{i_1, \cdots, i_k}$ such that $x_t > 0$, $x_{t+j} > 0$, and exactly $\binom{N-2}{k}$ sets such that $x_t \leqq 0$, $x_{t+j} \leqq 0$. In other cases, the total number of which makes

$$\binom{N}{k} - \binom{N-2}{k-2} - \binom{N-2}{k} = 2\binom{N-2}{k-1},$$

we have either $x_t > 0$, $x_{t+j} \leqq 0$, or $x_t \leqq 0$, $x_{t+j} > 0$.
With respect to the formulas

$$\int_{-\infty}^{0} \varphi(x)\, dx = \int_0^\infty \varphi(x)\, dx = \tfrac{1}{2},$$

$$\int_{-\infty}^{0} \int_{-\infty}^{0} xy\varphi(x)\varphi(y)\, dx\, dy = \int_0^\infty \int_0^\infty xy\varphi(x)\varphi(y)\, dx\, dy$$
$$= -\int_0^\infty \int_{-\infty}^{0} xy\varphi(x)\varphi(y)\, dx\, dy = (2\pi)^{-1}$$

we obtain

$$\sum_{(i)} \int_{A_{i_1}, \cdots, i_k} x_t x_{t+j}\, dP = 2^{2-N}(2\pi)^{-1}\left[\binom{N-2}{k-2} + \binom{N-2}{k} - 2\binom{N-2}{k-1}\right]$$
$$= 2^{1-N}\pi^{-1}\left[\binom{N}{k} - 4\binom{N-2}{k-1}\right].$$

Obviously $P(A) = \binom{N}{k}2^{-N}$ holds and therefore we get according to Theorem 4

$$P_b(A) = \binom{N}{k}2^{-N} + \sum_{j=1}^{n} (N-j)2^{1-N}\pi^{-1}\left[\binom{N}{k} - 4\binom{N-2}{k-1}\right]b_j + o(b).$$

Especially for the autoregressive series of the first order we have

$$P_b(A) = \binom{N}{k}2^{-N} + (N-1)2^{1-N}\pi^{-1}\left[\binom{N}{k} - 4\binom{N-2}{k-1}\right]b + o(b).$$

It may be easily derived that $\binom{N}{k} - 4\binom{N-2}{k-1} > 0$ for $k < \frac{1}{2}(N - N^{\frac{1}{2}})$ and for $k > \frac{1}{2}(N + N^{\frac{1}{2}})$; $\binom{N}{k} - 4\binom{N-2}{k-1} < 0$ for $\frac{1}{2}(N - N^{\frac{1}{2}}) < k < \frac{1}{2}(N + N^{\frac{1}{2}})$. Let b be a sufficiently small positive number. Then $P_b(A) > P(A)$, if k is near to zero or to N, whereas $P_b(A) < P(A)$, if k is near to $N/2$. For negative b, which is sufficiently small in absolute value, the contrary is true.

REFERENCES

[1] ANDĚL J. (1967). Local asymptotic power and efficiency of tests of Kolmogorov–Smirnov type. *Ann. Math. Statist.* **38** 1705–1725.

[2] HÁJEK, J. (1962). On linear statistical problems in stochastic processes. *Czechoslovak Math. J.* **12** 404–444.

[3] HÁJEK, J. and ŠIDÁK, Z. (1967). *Theory of Rank Tests.* Academia, Prague.