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CONSTANT COEFFICIENT LINEAR DIFFERENTIAL EQUATIONS DRIVEN BY WHITE NOISE

BY R. V. ERICKSON

Michigan State University

1. Introduction. Consider the process w_t defined by the linear stochastic differential equation

$$\dot{w}_t = Aw_t + B\dot{\beta}_t,\ w_0 = c, \tag{1}$$

where $A \in R^{n\times n}$, $B \in R^{n\times p}$ ($R^{k\times l}$ = real $k \times l$ matrices, $R^n = R^{n\times 1}$) and β_t is the standard p-dimensional Brownian motion. The purpose of this note is to analyze the trajectories of this process, to find conditions for transience of w_t in terms of the eigenvalues of (a matrix related to) A, and to give the general form of the invariant densities for the process.

These results can be given in abbreviated form because of the groundwork laid by Dym [1] who has considered all of the above problems for the special case where w_t is a solution of

$$(D^n - a_1 D^{n-1} - \cdots - a_n)w_t = \dot{\beta}_t,\ w_0^{(k-1)} = c_k, \qquad k = 1, \cdots, n, \tag{2}$$

and β_t is one-dimensional Brownian motion.

Zakai and Snyders [4] give three equivalent necessary and sufficient conditions for the existence of a stationary probability measure for solutions of (1); two of these conditions are implicit in our final theorem.

2. The trajectories of w_t remain on an m-flat. The formal solution to (1) is given by

$$w_t = \int_0^t e^{(t-s)A}\, B\, d\beta_s + e^{tA}c \tag{3}$$

and is known to be a diffusion (see, e.g., Dynkin [2]). From the properties of stochastic integrals and Brownian motion it follows that w_t is Gaussian with mean

$$E^c(w_t) = e^{tA}c \tag{4}$$

and covariance

$$R_t = E^c((w_t - e^{tA}c)(w_t - e^{tA}c)^*) = \int_0^t e^{sA}BB^*\, e^{sA^*}\, ds, \tag{5}$$

where C^* is the transpose of the matrix C.

The first fact to be noted is that R_t is nonnegative definite and not necessarily positive definite, for if $v \in R^n$ is orthogonal to the subspace

$$[A, B] := \text{span}_{R^n}\{A^{k-1}B\varepsilon_i \mid k = 1, \cdots, n,\ i = 1, \cdots, p,\ \varepsilon_i = (\delta_{i1}, \cdots, \delta_{ip})^*\},$$

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then $v^*R_tv \equiv 0$. Actually, for $t > 0$, R_t has constant rank equal to the dimension of $[A, B]$, for in the notation of the next paragraph, $MR_tM^{-1} = \begin{pmatrix} S_t & 0 \\ 0 & 0 \end{pmatrix}$ where $S_t = \int_0^t e^{sC_1}D_1D_1^* e^{sC_1^*} ds \in R^{m \times m}$ is positive definite, $m = \dim [A, B]$.

To study the trajectories of w_t, choose a basis $\mathbf{u}_1, \cdots, \mathbf{u}_m$ for $[A, B]$ and a basis $\mathbf{v}_1, \cdots, \mathbf{v}_k$ for $[A, B]^\perp$, form the nonsingular matrix $M^* = (\mathbf{u}_1, \cdots, \mathbf{u}_m, \mathbf{v}_1, \cdots, \mathbf{v}_k)$, let $C = MAM^{-1}$ and $D = MB$, and solve

$$\dot{x}_t = Cx_t + D\dot{\beta}_t, \quad x_0 = b, \tag{6}$$

so that $x_t = Mw_t$. Clearly $D = \begin{pmatrix} D_1 \\ 0 \end{pmatrix}$, $D_1 \in R^{m \times p}$. Also, $C = \begin{pmatrix} C_1 & C_2 \\ C_3 & C_4 \end{pmatrix}$, $C_1 \in R^{m \times m}$ and $C_3 = 0$, for row $(m+i)$ of $C^{j+1}D = \mathbf{v}_i^*A^{j+1}B = 0$, $i = 1, \cdots, k$, so that $C_3C_1{}^jD_1 = 0$, $j = 1, \cdots, n$. Since the columns of $C_1{}^jD_1 \in R^{m \times p}$ (embedded in R^n) span $[C, D]$ which has dimension m, we conclude that $C_3 = 0$. Hence, if $x_0 = b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$,

$$\begin{aligned} x_t &= \int_0^t \begin{pmatrix} e^{(t-s)C_1}D_1 \\ 0 \end{pmatrix} d\beta_s + e^{tC}b \\ &= \begin{pmatrix} \int_0^t e^{(t-s)C_1}D_1 d\beta_s + e^{tC_1}b_1 \\ 0 \end{pmatrix} + e^{tC}\begin{pmatrix} 0 \\ b_2 \end{pmatrix} \\ &= \begin{pmatrix} y_t \\ 0 \end{pmatrix} + \begin{pmatrix} z_t^1 \\ z_t^2 \end{pmatrix} \end{aligned} \tag{7}$$

where $z_t{}^2 = e^{tC_4}b_2$ and $z_t{}^1 = \int_0^t e^{(t-s)C_1}\, C_2\, e^{sC_4}\, b_2\, ds$ are purely deterministic, and y_t is the Gaussian diffusion satisfying

$$\dot{y}_t = C_1y_t + D_1\dot{\beta}_t, \quad y_0 = b_1 \in R^m \tag{8}$$

with mean $E^{b_1}y_t = e^{tC_1}b_1$ and covariance

$$S_t = E^{b_1}((y_t - e^{tC_1}b_1)(y_t - e^{tC_1}b_1)^*) = \int_0^t e^{sC_1}D_1D_1^*\, e^{sC_1^*}\, ds$$

which is positive definite. So y_t has transition density

$$p(t, a, b) = [(2\pi)^m|S_t|]^{-\frac{1}{2}} \exp\{-\tfrac{1}{2} < b - e^{tC_1}a,\ S_t{}^{-1}(b - e^{tC_1}a) > \}, \tag{9}$$

$< , >$ = inner product in R^m, $| \ |$ = determinant.

The above decomposition shows that when $x_0 = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ then x_t stays in the m-flat $H_t{}^{b_2} = \{(a^*, b_2{}^*\, e^{tC_4^*})^* \mid a \in R^m\}$ with probability one. Since $w_t = M^{-1}x_t$, it is clear that w_t enjoys a similar property.

3. Transience and recurrence. Since w_t is transient or recurrent iff $x_t = Mw_t$ is, we restrict attention to x_t.

Now x_t is transient (by definition) iff the probability of hitting the nonempty interior of a compact set, in a finite time after time t, goes to zero as t increases, for each starting point, in other words, iff $P^b(\lim_{t \uparrow \infty} x_t = \infty) = 1$ for each starting point b (∞ = one point compactification of R^n, so $\lim x_t = \infty$ means x_t eventually remains outside any given compact.) Since $x_t = \begin{pmatrix} y_t \\ 0 \end{pmatrix} + z_t$ and $z_t \equiv 0$ if $x_0 = \begin{pmatrix} b_1 \\ 0 \end{pmatrix}$, x_t transient implies y_t transient. Certainly if y_t is transient and $z_t{}^1$ is bounded, when

z_t^2 is bounded, then x_t is transient. But if $C = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$ and $D = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ then y_t satisfies $\dot{y}_t = y_t + \dot{\beta}_t$ (in R^1) and is therefore transient by Dym's theorem; but x_t is not, for $x_t^* = (\beta_{\frac{1}{2}(e^{2t}-1)}, 0) + e^{-t} x_0^*$ when $x_0^* = (-\frac{1}{2}, 1)$.

Recurrence of x_t is impossible unless $x_t \equiv y_t$ (i.e. $[A, B] = R^n$), for x_t never hits open sets off the m-flat $H_t^0 = \{(a^*, 0) \mid a \in R^m\}$ if $x_0 = (b_1^*, 0^*)^*$.

Let us confine our attention to the process y_t. Then y_t is a Gaussian diffusion with transition density given by (9) and it clearly satisfies all the properties $P1$ to $P7$ listed by Dym. Thus it is transient (recurrent) iff the average sojourn time in every compact (nonempty open) set is finite (infinite). The characterization of recurrence proved by Dym continues to hold:

THEOREM. *The process y_t is either transient or recurrent, and the following are equivalent*:

(a) *y_t is recurrent*,

(b) *the matrix C_1 is type* I (*below*),

(c) $\int_1^\infty |S_t|^{-\frac{1}{2}}\, dt = +\infty$.

But the definition of type I matrix must be modified slightly.

Let $M \in R^{n \times n}$ have complex Jordan form diag $(J_1, \cdots, J_p)$, where $J_i = \lambda_i I + N_i$ and N_i has 1 in positions $(k+1, k)$ and 0 elsewhere, and Re $\lambda_i \geqq$ Re λ_{i+1}.

DEFINITION. The matrix M is *type* I if

(a) Re $\lambda_1 < 0$, or

(b) Re $\lambda_2 < 0$ and $J_1 = 0 \in R^{1 \times 1}$, or

(c) Re $\lambda_3 < 0$ and $J_1 = -J_2 = (-1)^{\frac{1}{2}}\beta$, $\beta \in R^{1 \times 1}$.

This differs from the definition given by Dym only in allowing $\beta = 0$ in (c), and the proof of the above theorem is now essentially that used by Dym. Notice that the matrix $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ is *not* type I, while $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ is.

THEOREM. (a) *The process x_t is recurrent* iff *$x_t \equiv y_t$ and $C_1 = C$ is type I.* (b) *If x_t is transient, then y_t is transient and C_1 is not type I.* (c) *If C_1 is not type I then y_t is transient, and x_t will also be transient if $z_t = e^{tC}\begin{pmatrix} 0 \\ b_2 \end{pmatrix}$ is bounded for each b_2 such that $z_t^2 = e^{tC_4} b_2$ is bounded.*

4. Invariant densities. A Markov process with transition density $p(t, a, b)$ is said to have an *invariant density* f if $f: R^n \to [0, \infty)$ is Borel measurable, not identically zero and $\int f(a) p(t, a, b)\, da = f(b)$ for all $b \in R^n$ and $t > 0$.

Let $\pi: R^n \to R^m$ be the projection sending $x = (y^*, z^*)^*$ to $y \in R^m$ and let $\pi'(x) = z$. Clearly if g is an invariant density for y_t (with transition density $p(t, a, b)$ given by (9)), then $f(a) = \delta_{0,\pi'(a)}[g(\pi(a))]$ is an invariant density for x_t, since x_t has transition density $\delta_{0,\pi'(b)} p(t, a, b)$ if $\pi'(a) = 0$. (Here $\delta_{a,b}$ is 0 if $a \neq b$ and 1 if $a = b$.) Hence to study the existence of invariant densities it suffices to consider only the process y_t.

It is easily seen that most of the results of Section 7 of Dym's paper remain true for our process y_t with proper modifications. Replace his "a_1" by "tr C_1." His

Theorem 7.1 holds save for the statement that $Q_{ij} = 0$ if $i+j$ is odd (e.g., let $C_1 = \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}$ and $D_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, and see that

$$S_t = \tfrac{1}{4}\begin{pmatrix} 2(1-e^{-2t}) & 1-e^{-2t}-2t\,e^{-2t} \\ 1-e^{-2t}-2t\,e^{-2t} & 2(1-e^{-2t}) \end{pmatrix} \to \tfrac{1}{4}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

as $t \uparrow \infty$). Theorems 7.2 and 7.3 hold, and Corollary 1 becomes (by direct calculation):

THEOREM. *For symmetric $Q \in R^{m \times m}$ and $v \in R^m$, if $g(x)$ is defined as*

$$\exp\{-< x, Qx > - < v, x >\}$$

then g is an invariant density for y_t iff

(a) $2QD_1D_1{}^*Q + QC_1 + (QC_1)^* = 0$

(b) $v^*(2D_1D_1{}^*Q + C_1) = 0$

(c) $2\,tr\,(C_1 + D_1D_1{}^*Q) = < v, D_1D_1{}^*v >$.

Let $sp(A) = \{\lambda \mid \lambda$ is an eigenvalue of $A\}$ and let $\Lambda = \{z \mid \text{Re } z < 0\}$, $\bar{\Lambda} = \{z \mid \text{Re } z \leqq 0\}$. We have finally

THEOREM. *If y_t is recurrent, then, up to multiplication by a constant, y_t has a unique invariant density g. Further, $\int g(a)da < \infty$* iff *$sp(C_1) \subset \Lambda$, and in this case g has the form $c \exp\{-< x, Qx >\}$, where $Q = \frac{1}{2}\lim_{t\uparrow\infty} S_t^{-1}$.*

PROOF. Dym's Lemma 3.1 shows that, given a compact K and $\varepsilon, T > 0$, there exist constants c_1 and c_2 so that $P^a(\|y_t - a\| > \varepsilon) \leqq c_1\,e^{-c_2/t}$ for all $a \in K$ and $t \in (0, T)$ for just introduce polar coordinates and integrate the given estimate and note that $|S_t| \sim t^N$, as $t \downarrow 0$, for some integer $N > 0$. Since y_t is a strong Feller process with a positive transition density and is assumed recurrent, the uniqueness statement follows from the results of Khasminskii [3]. Dym's Theorem 7.1 shows that if $spC_1 \subset \Lambda$ (which obtains when y_t is recurrent) then y_t has an invariant density $\Psi(x) = \exp\{-< x, Qx >\}$, where $Q = \frac{1}{2}\lim_{t\uparrow\infty} S_t^{-1}$. So $\int_{R^n} \Psi(x) < \infty$ if Q is positive definite. If $sp\ C_1 \subset \Lambda$, then $S_\infty = \lim_{t\uparrow\infty} S_t$ exists and is positive definite, which implies $Q = S_\infty{}^{-1}/2$ has this property. Conversely, if $sp\ C_1 \not\subset \Lambda$, then the only nonnegative integrable function f such that $\int f(a)p(t, a, b)da \equiv f(b)$ is $f \equiv 0$, using the argument of Dym's Theorem 7.5, and this contradicts the first assertion of the theorem.

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