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A PSEUDO-METRIC SPACE OF PROBABILITY MEASURES AND THE EXISTENCE OF MEASURABLE UTILITY

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1. Introduction. Derivations of measurable utility functions owe much to von Neumann-Morgenstern [6]. The axioms underlying these derivations apply to a preference pre-ordering $\succcurlyeq$ on a set $\mathscr{P}$ of probability measures on a set of consequences X. Fishburn, [3] and [4], has extended the coverage of these derivations to rather broad classes of both countably and finitely additive measures on X. In this paper, it is shown that measurable utility functions can be derived even if $\mathscr{P}$ is not assumed to be closed under countable convex combinations and the formation of conditional probabilities. This allows the possibility of unbounded utility which is ruled out in the Fishburn analysis unless all $P \in \mathscr{P}$ are simple.

2. Theory for simple distributions. The results of this section provide a basis for all that follows. We are given X, a set of consquences, $\mathscr{P}$, a set of probability measures on X, and $\succcurlyeq$, a binary relation on $\mathscr{P}$. We will make no notational distinction between the element $x \in X$ and the element $x \in \mathscr{P}$ which assigns probability 1 to $\{x\}$. The following axioms and theorem are found in ([6] Appendix 3, page 1054), and ([4] Chapter 8).

AXIOM 0. (a) $\mathscr{P}$ is *closed under finite convex combinations* (i.e., $P, Q \in \mathscr{P}$ and $\alpha \in [0, 1] \Rightarrow \alpha P + (1-\alpha)Q \in \mathscr{P}$) and, (b) $\mathscr{P}_s \subseteq \mathscr{P}$ where $\mathscr{P}_s$ is the set of all *simple distributions* on X (i.e. $P \in \mathscr{P}_s \Leftrightarrow \exists\, A \subseteq X$, A finite and $P(A) = 1$).

AXIOM 1. $\succcurlyeq$ is *complete* (i.e. $P, Q \in \mathscr{P} \Rightarrow P \succcurlyeq Q$ or $Q \succcurlyeq P$) and *transitive* (i.e. $P \succcurlyeq Q, Q \succcurlyeq R \Rightarrow P \succcurlyeq R$).

AXIOM 2. $[P, Q \in \mathscr{P}, \alpha \in (0, 1), P \succ Q] \Rightarrow \alpha P + (1-\alpha)R \succ \alpha Q + (1-\alpha)R, \forall R \in \mathscr{P}$, where $P \succ Q \Leftrightarrow P \succcurlyeq Q$ and not $Q \succcurlyeq P$.

AXIOM 3. $[P, Q, R \in \mathscr{P}, P \succ Q \succ R] \Rightarrow \alpha P + (1-\alpha)R \succ Q$ and $Q \succ \beta P + (1-\beta)R$ for some $\alpha, \beta \in (0, 1)$.

THEOREM 1. *Let $\mathscr{P} = \mathscr{P}_s$. Axiom 1, Axiom 2, and Axiom 3 hold $\Leftrightarrow \exists$ a real-valued function u from $\mathscr{P}$ such that, for all P, Q $\in \mathscr{P}$,*

(a) $u(P) \geqq u(Q) \Leftrightarrow P \succcurlyeq Q$
(b) $u(P) = \sum_{x \in X} u(x)P(x)$.

3. A metric space of measures. To extend the results in Theorem 1, we first construct a pseudo-metric space which is induced by $(\mathscr{P}, X \succcurlyeq)$. The following axiom makes this construction feasible.

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AXIOM 4. (a) $\mathscr{B}$ is a Boolean algebra on X such that, $\forall x \in X$, the sets $\{x\}$, $\{y \in X \mid y \succcurlyeq x\}$, and $\{y \in X \mid x \succcurlyeq y\}$ belong to $\mathscr{B}$. $\mathscr{P}$ is a set of measures on $\mathscr{B}$.

(b) $P \in \mathscr{P} \Rightarrow \exists x^*, x_* \in X$ such that $P\{x \in X \mid x > x^*\} = P\{x \in X \mid x_* > x\} = 0$.

We assume throughout the rest of this section that Axiom 0 to Axiom 4 are true. Let u be a function on $\mathscr{P}_s$ that satisfies the conditions of Theorem 1. For any $P \in \mathscr{P}$, the distribution function F_P, where $F_P\,(r) = P\{x \in X \mid u(x) \leqq r\}$ is well defined on $(-\infty, \infty)$, the real line. Let $D \equiv \{F_P \mid P \in \mathscr{P}, u \text{ given on } X\}$, and define the pseudo-metric ρ on the Cartesian product $D \otimes D$ by $\rho(F, \mathrm{G}) = \int |F(r) - G(r)|\, dr$ where integration is over the real-line unless otherwise indicated. If $u(X) = (-\infty, \infty)$, and $P \in \mathscr{P}$ implies P is countably additive, then Hacklemann ([5], page 24), shows that D is a separable metric space where $C \equiv \{F_P \in D \mid P \in \mathscr{P}_s\}$ is a countable dense subset of D. In our case, we need to modify the proof slightly in order to show that C is a dense subset of D.

THEOREM 2. [*Axioms* 0, 1, 2, 3, 4] $\Rightarrow$ *given* $\varepsilon > 0$, $F \in D$, *there is a* $G \in C$ *such that* $\rho\,(F, G) < \varepsilon$.

PROOF. For $F \in C$, let $G \equiv F$. For $F \in D - C$, by Axiom 4(b), let $a = u(x_*)$, $b = u(x^*)$. Let n satisfy $(b-a)/n < \varepsilon$, and $r_j = a + (j/n)(b-a)$ for $j = 0, 1, \cdots, n$. For each j such that $F(r_j) - F(r_{j-1}) > 0$, select x_j for which $r_{j-1} \leqq u(x_j) \leqq r_j$, where $r_0 = a$, $r_n = b$. Let $G(r) = \sum_{j=1}^n [F(r_j) - F(r_{j-1})] H_j(r)$ where $H_j(r) = 0$ for $r < u(x_j)$ and $H_j(r) = 1$ for $r \geqq u(x_j)$.
$G \in C$ and $\rho(F, G) = \sum_{j=1}^n \int_{r_{j-1}}^{r_j} |F(r) - G(r)|\, dr \leqq \sum_{j=1}^n [F(r_j) - F(r_{j-1})](r_j - r_{j-1}) \leqq (b-a)/n < \varepsilon$. $\square$

In the next section we use the dense subset C to extend u on $\mathscr{P}_s$ to a wider class of distributions.

4. Theory for non-simple measures. If $\mathscr{P}$ and $\succcurlyeq$ satisfy one additional axiom, then the function u from Theorem 1 can be extended to all of $\mathscr{P}$ and this extension is continuous on $\mathscr{P}$ with respect to ρ.

The axiom which, under Axioms 0, 1, 2, 3, and 4, is necessary and sufficient for the extension of u to be continuous with respect to ρ is:

AXIOM 5. Let u be any function on $\mathscr{P}_s$ which satisfies (a) and (b) of Theorem 1. Let T be the topology on $\mathscr{P}$ induced by the pseudo-metric ρ. That is, for all $k > 0$ and all $P^* \in \mathscr{P}$, $\{P \in \mathscr{P} \mid \rho(F_P, F_{P*}) < k\} \in T$. Then, $\{P \in \mathscr{P} \mid P > P^*\}$ and $\{P \in \mathscr{P} \mid P^* > P\} \in T$ for every $P^* \in \mathscr{P}$.

Several comments about this axiom should clarify the reasons for its use. Although it relies on the existence of a particular u on $\mathscr{P}_s$, Axioms 0, 1, 2, 3, and 4 will be assumed to hold simultaneously; hence, this creates no problem. Also, the arbitrary selection of one u, satisfying (a) and (b) of Theorem 1, from a class of such functions creates no problems since if u and v are both acceptable then $u(\boldsymbol{x}) = av(x) + b$, with $a > 0$. (See [4] Chapter 8.)

Axiom 5 is a necessary condition for the existence of an order preserving function (with respect to $\succcurlyeq$) on $\mathscr{P}$ which is continuous with respect to ρ [i.e., for an arbitrary sequence $\{P_k\} \subseteq \mathscr{P}$, $\rho(F_{P_k}, F_P) \to 0 \Rightarrow u(P_k) \to u(P)$].

THEOREM 3. *Let Axiom* 0 *and Axiom* 4 *hold. There is a real-valued u on* $\mathscr{P}$ *such that for all* $P, Q \in \mathscr{P}$,

(a) *u is continuous at P with respect to* ρ,

(b) $u(P) \geqq u(Q) \Leftrightarrow P \succcurlyeq Q$,

(c) $u(P) = \int r\, dF_P(r) = \int_X u(x)\, dP(x)$,

if and only if Axioms 1, 2, 3, *and* 5 *are true.*

PROOF. Necessity is straightforward. We show sufficiency. By Theorem 1, there is a real-valued function v on $\mathscr{P}_s$ such that (b) and (c) hold for all $P, Q \in \mathscr{P}_s$. For any $P \in \mathscr{P}$, let $u(P) = \inf\{v(Q) : Q \in \mathscr{P}_s \text{ and } Q \succcurlyeq P\} = \sup\{v(Q) : Q \in \mathscr{P}_s \text{ and } P \succcurlyeq Q\}$.

It is easy to show that the second equality holds, using Axiom 3. Also, since $P \in \mathscr{P}_s \Rightarrow u(P) = v(P)$, u is an extension of v. To show u satisfies (a) we can use a proof similar to Debreu's, ([1] page 59). To show that u satisfies (b) is straightforward.

To show that u satisfies (c), we know for $P \in \mathscr{P}_s$, $u(P) = v(P) = \sum_{x \in X} v(x)P(x) = \sum_{x \in X} u(x)P(x) = \int r\, dF_p$. Let $P \in \mathscr{P} - \mathscr{P}_s$. By Axiom 4(b), $\int |r|\, dF_p < \infty$. By Theorem 2, for all integer $k > 0$, there is $P_k \in \mathscr{P}_s$ such that $\rho(F_{P_k}, F_P) < 1/k$. Letting $k \to \infty$, we have, by (a), that $u(P) = \lim u(P_k) = \lim \int r\, dF_{P_k} = \int r\, dF_p$. □

One corollary to this theorem is of particular interest.

COROLLARY 3.1. [*Axioms* 0, 1, 2, 3, 4(a), 5] $\Rightarrow$ *if the function v on* $\mathscr{P}_s$ *implied by Theorem* 1 *is bounded, then there is a function u on* $\mathscr{P}$ *satisfying* (a), (b) *and* (c) *of Theorem* 3.

PROOF. v bounded $\Rightarrow \int |r|\, dF_P(r) < \infty\ \forall\, P \in \mathscr{P}$. Hence, Theorem 2 is applicable to all $\mathscr{P}$ even if 4(b) does not hold. □

To illustrate these results, we consider an example from Dubins and Savage ([2] pages 10–11) which is also found in ([4] Chapter 10). Let X be the positive integers and $\mathscr{P}$ be all measures on the set of all subsets of X. We consider three orderings on $\mathscr{P}$. For the first, let $v(x) = x$ and $P \succcurlyeq Q \Leftrightarrow \int v(x)\, dP(x) \geqq \int v(x)\, dQ(x)$. Let $P' \in \mathscr{P}$ be any diffuse measure on X, (i.e., P' assigns probability 1 to a denumerable subset of X and $P'\{x\} = 0$ for all $x \in X$). It is easy to show that P' violates Axiom 4(b). Also, $\int v(x)\, dP'(x) = \infty$ which indicates that $\succcurlyeq$ on $\mathscr{P}$ violates Axiom 3. In this case only $\mathscr{P}_s$ satisfies Axiom 4(b) and, therefore, if we consider only $\mathscr{P}_s$, there exists an expected-utility measure.

For the second case, let $v(x) = x/(1+x)$ and $P \succcurlyeq Q \Leftrightarrow \int v(x)\, dP(x) \geqq \int v(x)\, dQ(x)$. As in the previous example, if P is diffuse, P violates Axiom 4(b). However, $\int v(x)\, dP(x) = 1$ and Axiom 3 is not violated. In fact, the conditions of Corollary 3.1 are all satisfied and the function $u(p) = \int v(x)\, dP(x)$ satisfies (a)–(c) of Theorem 3.

The third ordering indicates that bounded utility on simple measures as well as Axioms 0 to 4 is not enough. That is, Axiom 5 is needed. Let $v(x) = x/(1+x)$, as in the previous case, but let $g(P) = \int v(x)\, dP(x) + \frac{1}{2} \lim_{\varepsilon \to 0} P(v(x) \geqq 1-\varepsilon)$, and $P \succcurlyeq Q \Leftrightarrow g(P) \geqq g(Q)$. Axiom 5 is violated by this ordering. Let $P^q \in \mathscr{P}$ where $P^q\{q\} = 1$ for $q \in X$. Let P^* be any diffuse measure. The measure $R^q = \frac{3}{4}P^1 + \frac{1}{4}p^q$ satisfies $R^q \to S = \frac{3}{4}P^1 + \frac{1}{4}P^*$ and $R^q \leqq \frac{7}{8}P^1 + \frac{1}{8}P^*$ for all q. But $S > \frac{7}{8}P^1 + P^*$. Thus, Axiom 5 does not hold. Also, there does not exist a function u with properties (a)–(c) of Theorem 3, even though Axioms 0 to 4 hold and v is bounded on $\mathscr{P}_s$.

As another illustration of the impact of Theorem 3 and its corollary, we consider their relationship to Theorem 10.1 of [4] Chapter 10]. (See also the general theorem in [3] page 1060.) This theorem is:

THEOREM 4. *If Axioms* 0, 1, 2, 3, and 4(a) *hold, if*

(B.1) *$\mathscr{P}$ is closed under countable convex combinations and under the formation of conditional probabilities, and if*

(B.2) $[A \in \mathscr{B}, P(A) = 1,\quad Q > x\ \forall x \in A \Rightarrow Q \succcurlyeq P]$ *and* $[A \in \mathscr{B},\ P(A) = 1,$ $x > Q\ \forall x \in A \Rightarrow P \succcurlyeq Q]$
then

(c) *there is a real-valued function u on X such that, $\forall P, Q \in \mathscr{P}$, $P \succcurlyeq Q \Leftrightarrow$ $\int u(x)\, dP(x) \geqq \int u(x)\, dQ(x)$ and*

(d) *u is bounded.*

Clearly the gain from the use of Axiom 5 and Axiom 4(b) in place of (B.1) and (B.2) is the removal of (d) as a necessary conclusion. Under Axioms 0, 1, 2, 3, and 4(a) which are common to both Theorem 3 and Theorem 4, it can be shown that [Axioms 4(b), 5 $\Rightarrow$ (B.2)] and [(B.1), (B.2) $\Rightarrow$ Axiom 5]. The last of these follows from the continuity of u (in Theorem 4) with respect to ρ. Let $\{P^k\} \subseteq \mathscr{P}$ be an arbitrary convergent sequence where $P^k \succcurlyeq Q$ for all k. *Since u is bounded*, $\rho(F_{P_k}, F_P) \to 0 \Rightarrow \int r\, dF_{P_k} \to \int r\, dF_P$. Since $\int r\, dF_{P_k} \geqq \int r\, dQ$ for all k, $\int r\, dF_P \geqq \int r\, dQ$. Similarly for a sequence $P^k \to P^0$, $P^k \leqq Q$, we have $Q \succcurlyeq P^0$. Thus u is continuous which implies Axiom 5.

Alternatively, neither [Axioms 5, 4(b) $\Rightarrow$ (B.1)] nor [(B.1), (B.2) $\Rightarrow$ Axiom 4(b)] need be true and there are examples to support this.

If we substitute, in Theorem 3 for Axiom 4(b), the requirement $[P \in \mathscr{P} \Rightarrow \int |r|\, dF_P < \infty]$, the conclusions still hold. Calling this Axiom 6, and the new theorem, Theorem 3′, it can be shown that [(B.1), (B.2) $\Rightarrow$ A.5, A.6]. Therefore, if one is willing to assume (i) whenever two measures imply almost the same distribution on the indifference classes of X, their utility is almost the same (Axiom 5), and (ii) combinations of preferences and sets of measures which imply infinite utility are not allowed (Axiom 4(b), or Axiom 6), Axioms (B.1) and (B.2) are unnecessary with the resultant gain that more classes of measures and preferences can be indexed by a measurable utility function which need not be bounded.

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REFERENCES

[1] DEBREU, G. (1959). *Theory of Value*. Wiley, New York.
[2] DUBINS, L. E. and SAVAGE, L. J. (1965). *How to Gamble if You Must*. McGraw-Hill, New York.
[3] FISHBURN, P. C. (1967). Bounded expected utility. *Ann. Math. Statist.* **38** 1054–1060.
[4] FISHBURN, P. C. (1970). *Utility Theory for Decision Making*. Wiley, New York.
[5] HACKLEMAN, R. P. (1967). Metric spaces of distribution functions and statistical information. Ph.D. dissertation, Carnegie Institute of Technology.
[6] von NEUMANN, J. and MORGENSTERN, O. (1953). *Theory of Games and Economic Behavior*. Princeton Univ. Press.