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NOTE ON SOME FORMULAS FOR WEIGHTED SUMS OF ZONAL POLYNOMIALS[1]

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1. Introduction. The first purpose of this paper is to prove a stronger result than the previous lemma for the sum of zonal polynomials given in Sugiura and Fujikoshi [6], which played an important role in deriving the asymptotic expansions of the non-null distributions of the likelihood ratio criteria in multivariate analysis by these authors, Sugiura [5] and of the generalized variance by Fujikoshi [1].

THEOREM 1. *Let $C_\kappa(Z)$ be the zonal polynomial of degree k corresponding to a partition $\kappa = \{k_1, k_2, \cdots, k_p\}$ of k $(k_1 \geqq k_2 \geqq \cdots \geqq k_p \geqq 0)$ for a $p\times p$ positive definite matrix Z. Put*

$$a_1(\kappa) = \sum_{\alpha=1}^{p} k_\alpha(k_\alpha - \alpha) \tag{1.1}$$

$$a_2(\kappa) = \sum_{\alpha=1}^{p} k_\alpha(4k_\alpha^{\ 2} - 6\alpha k_\alpha + 3\alpha^2).$$

Then the following equalities hold:

$$\sum_{(\kappa)} a_1(\kappa)C_\kappa(Z) = k(k-1) \operatorname{tr} Z^2(\operatorname{tr} Z)^{k-2} \tag{1.2}$$

$$\sum_{(\kappa)} a_1(\kappa)^2C_\kappa(Z) = k(k-1)[\{\operatorname{tr} Z^2 + (\operatorname{tr} Z)^2\}(\operatorname{tr} Z)^{k-2} + 4(k-2) \operatorname{tr} Z^3(\operatorname{tr} Z)^{k-3} + (k-2)(k-3)(\operatorname{tr} Z^2)^2(\operatorname{tr} Z)^{k-4}] \tag{1.3}$$

$$\sum_{(\kappa)} a_2(\kappa)C_\kappa(Z) = k[(\operatorname{tr} Z)^k + 3(k-1)\{\operatorname{tr} Z^2 + (\operatorname{tr} Z)^2\}(\operatorname{tr} Z)^{k-2} + 4(k-1)(k-2)(k-3) \operatorname{tr} Z^3(\operatorname{tr} Z)^{k-3}] \tag{1.4}$$

where the symbol $\sum_{(\kappa)}$ means the sum of all possible partition κ of k.

Dividing by $k!$ on both sides of each of the equations (1.2), (1.3) and (1.4) and summing with respect to k from zero to infinity, we obtain the lemma given by Sugiura and Fujikoshi [6].

The second purpose of this paper is to give an alternative proof of the following theorem due to Fujikoshi [2], by using a differential equation for zonal polynomials obtained recently by James [4].

THEOREM 2. (Fujikoshi). *With the same notation as in Theorem* 1, *put*

$$(b)_\kappa = \prod_{\alpha=1}^{p} \left(b - \frac{\alpha-1}{2}\right)\left(b+1-\frac{\alpha-1}{2}\right)\cdots\left(b+k_\alpha-1-\frac{\alpha-1}{2}\right). \tag{1.5}$$

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Then the following equalities hold:

$$\sum_{k=0}^{\infty}\sum_{(\kappa)}\frac{(b)_\kappa a_1(\kappa)C_\kappa(Z)}{k!}=\frac{b}{2}|I-Z|^{-b}\{(2b+1)\operatorname{tr}W^2+(\operatorname{tr}W)^2\} \tag{1.6}$$

$$\begin{aligned}\sum_{k=0}^{\infty}\sum_{(\kappa)}\frac{(b)_\kappa a_1(\kappa)^2C_\kappa(Z)}{k!}&=\frac{b}{4}|I-Z|^{-b}\{(2b+1)(2b^2+b+2)(\operatorname{tr}W^2)^2\\&+2(2b^2+b+2)\operatorname{tr}W^2(\operatorname{tr}W)^2+2(8b^2+10b+5)\operatorname{tr}W^4\\&+8(2b+1)\operatorname{tr}W^3\operatorname{tr}W+b(\operatorname{tr}W)^4+8(2b^2+3b+2)\operatorname{tr}W^3\\&+12(2b+1)\operatorname{tr}W^2\operatorname{tr}W+4(\operatorname{tr}W)^3+2(2b+1)(\operatorname{tr}W)^2\\&+2(2b+3)\operatorname{tr}W^2\},\end{aligned} \tag{1.7}$$

where the positive definite matrix Z is assumed to have characteristic roots less than one, and $W = Z(I-Z)^{-1}$.

2. Proof of Theorem 1. Since the zonal polynomial $C_\kappa(Z)$ is a homogeneous symmetric polynomial of degree k with respect to the p characteristic roots of Z, it is sufficient to prove the equalities in Theorem 1 and Theorem 2, when Z is a diagonal matrix $Y = \operatorname{diag}(y_1, y_2, \cdots, y_p)$. Fujikoshi [2] has shown, in the proof of Theorem 2, that the following differential relations hold:

$$C_\kappa(Y)a_1(\kappa) = \operatorname{tr}(Y\partial)^2C_\kappa(\Sigma)\,|_{\Sigma=Y} \tag{2.1}$$

$$C_\kappa(Y)\{3a_1(\kappa)^2-a_2(\kappa)+k\} = [3\{\operatorname{tr}(Y\partial)^2\}^2+8\operatorname{tr}(Y\partial)^3]C_\kappa(\Sigma)\,|_{\Sigma=Y}, \tag{2.2}$$

where the symbol ∂ means a matrix of differential operators given by $\partial = (\frac{1}{2}(1+\delta_{ij})\partial/\partial\sigma_{ij})$, operating on a positive definite matrix $\Sigma = (\sigma_{ij})$, (δ_{ij} is a Kronecker delta). The proof of this formula is based on the asymptotic expression of the equality (1.22) in Sugiura and Fujikoshi [6]. By the formula $\sum_{(\kappa)} C_\kappa(\Sigma) = (\operatorname{tr}\Sigma)^k$ in James [3], we have from (2.1)

$$\textstyle\sum_{(\kappa)} a_1(\kappa)C_\kappa(Y) = \sum_{\alpha=1}^{p} y_\alpha{}^2(\partial^2/\partial\sigma_{\alpha\alpha}^2)(\operatorname{tr}\Sigma)^k\,|_{\Sigma=Y} = k(k-1)\operatorname{tr}Y^2(\operatorname{tr}Y)^{k-2} \tag{2.3}$$

$$\begin{aligned}\textstyle\sum_{(\kappa)} a_1(\kappa)^2C_\kappa(Y) &= \textstyle\sum_{(\kappa)}\operatorname{tr}(Y\partial)^2a_1(\kappa)C_\kappa(\Sigma)\,|_{\Sigma=Y}\\&=\operatorname{tr}(Y\partial)^2k(k-1)(\operatorname{tr}\Sigma^2)(\operatorname{tr}\Sigma)^{k-2}\,|_{\Sigma=Y}\\&=k(k-1)\{\textstyle\sum_{\alpha=1}^{p} y_\alpha{}^2(\partial^2/\partial\sigma_{\alpha\alpha}^2)+\frac{1}{2}\sum_{\alpha<\beta}y_\alpha y_\beta(\partial^2/\partial\sigma_{\alpha\beta}^2)\}\\&\quad\cdot\operatorname{tr}\Sigma^2(\operatorname{tr}\Sigma)^{k-2}\,|_{\Sigma=Y}\\&=k(k-1)\{2\operatorname{tr}Y^2(\operatorname{tr}\Sigma)^{k-2}+4(k-2)\operatorname{tr}Y^3(\operatorname{tr}\Sigma)^{k-3}\\&\quad+(k-2)(k-3)(\operatorname{tr}Y^2)^2(\operatorname{tr}\Sigma)^{k-4}\\&\quad+2\textstyle\sum_{\alpha<\beta}y_\alpha y_\beta(\operatorname{tr}\Sigma)^{k-2}\}\,|_{\Sigma=Y},\end{aligned} \tag{2.4}$$

which imply equalities (1.2) and (1.3) respectively. From (2.2) we have

$$\begin{aligned}\textstyle\sum_{(\kappa)} a_2(\kappa)C_\kappa(Y) &= 3\textstyle\sum_{(\kappa)} a_1(\kappa)^2 C_\kappa(Y)+k(\operatorname{tr} Y)^k-[3\{\textstyle\sum_{\alpha=1}^p y_\alpha{}^4(\partial^4/\partial\sigma_{\alpha\alpha}^4) \\ &\quad +2\textstyle\sum_{\alpha<\beta} y_\alpha{}^2 y_\beta{}^2(\partial^4/\partial\sigma_{\alpha\alpha}^2\partial_{\beta\beta}^2)\}+8\textstyle\sum_{\alpha=1}^p y_\alpha{}^3(\partial^3/\partial\sigma_{\alpha\alpha}^3)] \\ &\quad \cdot(\operatorname{tr}\Sigma)^k\,|_{\Sigma=Y},\end{aligned} \tag{2.5}$$

which yields (1.4) of Theorem 1.

It is interesting to note that the first two equations (1.2) and (1.3) in Theorem 1 can also be obtained from the following linear partial differential equation of second degree derived by James [4]:

$$\begin{aligned}&\textstyle\sum_{\alpha=1}^p y_\alpha{}^2(\partial^2/\partial y_\alpha{}^2)C_\kappa(Y)+\textstyle\sum_{\alpha\neq\beta} y_\alpha{}^2(y_\alpha-y_\beta)^{-1}(\partial/\partial y_\alpha)C_\kappa(Y) \\ &\quad =\{a_1(\kappa)+(p-1)k\}C_\kappa(Y).\end{aligned} \tag{2.6}$$

Summing both sides of the above formula with respect to κ for fixed k, yields equation (1.2). Operating $\sum_{(\kappa)} a_1(\kappa)$ on both sides of (2.6) yields equation (1.3).

3. Alternative proof of Theorem 2. Noting that

$$\sum_{k=0}^{\infty}\sum_{(\kappa)}\frac{(b)_\kappa C_\kappa(Z)}{k!}=|I-Z|^{-b}, \tag{3.1}$$

when all characteristic roots of positive definite matrix Z are less than one (James [3]), we can get from (2.6)

$$\begin{aligned}\sum_{k=0}^{\infty}\sum_{(\kappa)}\frac{(b)_\kappa a_1(\kappa)C_\kappa(Y)}{k!} &= \{\textstyle\sum_{\alpha=1}^p y_\alpha{}^2(\partial^2/\partial y_\alpha{}^2)+\textstyle\sum_{\alpha\neq\beta} y_\alpha{}^2(y_\alpha-y_\beta)^{-1}(\partial/\partial y_\alpha)\} \\ &\quad \cdot|I-Y|^{-b}-(p-1)(d/dt)|I-tY|^{-b}\,|_{t=1} \\ &= b|I-Y|^{-b}\{(b+1)\operatorname{tr} W^2+\textstyle\sum_{\alpha\neq\beta} y_\alpha{}^2(y_\alpha-y_\beta)^{-1} \\ &\quad \cdot(1-y_\alpha)^{-1}-(p-1)\operatorname{tr} W\}\end{aligned} \tag{3.2}$$

where $W = Y(I-Y)^{-1}$. The second term in the above equation can be simplified by noting $(I-Y)^{-1} = I+W$ as

$$\begin{aligned}\textstyle\sum_{\alpha\neq\beta} y_\alpha{}^2(y_\alpha-y_\beta)^{-1}(1-y_\alpha)^{-1} &= \textstyle\sum_{\alpha<\beta}(y_\alpha+y_\beta-y_\alpha y_\beta)(1-y_\alpha)^{-1}(1-y_\beta)^{-1} \\ &= \tfrac{1}{2}\{2\operatorname{tr} W\operatorname{tr}(I+W)-(\operatorname{tr} W)^2-2\operatorname{tr} W(I+W) \\ &\quad +\operatorname{tr} W^2\} \\ &= \tfrac{1}{2}\{2(p-1)\operatorname{tr} W+(\operatorname{tr} W)^2-\operatorname{tr} W^2\},\end{aligned} \tag{3.3}$$

which implies the first equation (1.6) in Theorem 2. From the differential equation (2.6), we have

$$\begin{aligned}\sum_{k=0}^{\infty}\sum_{(\kappa)}\frac{(b)_\kappa a_1(\kappa)^2 C_\kappa(Y)}{k!} &= \left\{\sum_{\alpha=1}^p y_\alpha{}^2\frac{\partial^2}{\partial y_\alpha{}^2}+\sum_{\alpha\neq\beta}\frac{y_\alpha{}^2}{y_\alpha-y_\beta}\frac{\partial}{\partial y_\alpha}\right\}f(Y)|I-Y|^{-b} \\ &\quad -(p-1)(d/dt)|I-tY|^{-b}f(tY)|_{t=1},\end{aligned} \tag{3.4}$$

where $f(Y) = (b/2)\{(2b+1)\,\mathrm{tr}\,W^2+(\mathrm{tr}\,W)^2\}$ with $W = Y(I-Y)^{-1}$. The first term in the right-hand side of (3.4) can be verified after some computation as

$$(3.5)\quad \tfrac{1}{2}b|I-Y|^{-b}\{b(b+1)(2b+1)(\mathrm{tr}\,W^2)^2+b(b+1)\,\mathrm{tr}\,W^2(\mathrm{tr}\,W)^2+8(b+1)^2\,\mathrm{tr}\,W^4$$
$$+4(b+1)\,\mathrm{tr}\,W\,\mathrm{tr}\,W^3+4(2b^2+5b+3)\,\mathrm{tr}\,W^3+4(b+1)\,\mathrm{tr}\,W\,\mathrm{tr}\,W^2$$
$$+4(b+1)\,\mathrm{tr}\,W^2\}.$$

The second term in (3.4) can be written as

$$(3.6)\quad \tfrac{1}{2}b|I-Y|^{-b}[2(2b+1)\textstyle\sum_{\alpha\neq\beta} y_\alpha^3(y_\alpha-y_\beta)^{-1}(1-y_\alpha)^{-3}$$
$$+2\mathrm{tr}\,W\textstyle\sum_{\alpha\neq\beta} y_\alpha^2(y_\alpha-y_\beta)^{-1}(1-y_\alpha)^{-2}+b\{(2b+1)\,\mathrm{tr}\,W^2$$
$$+(\mathrm{tr}\,W)^2\}\textstyle\sum_{\alpha\neq\beta} y_\alpha^2(y_\alpha-y_\beta)^{-1}(1-y_\alpha)^{-1}].$$

Noting that

$$(3.7)\quad \sum_{\alpha\neq\beta}\frac{y_\alpha^3}{(y_\alpha-y_\beta)(1-y_\alpha)^3} = \sum_{\alpha<\beta}\left\{\frac{y_\alpha^2}{(1-y_\alpha)^3(1-y_\beta)}+\frac{y_\alpha y_\beta}{(1-y_\alpha)^2(1-y_\beta)^2}\right.$$
$$\left.+\frac{y_\beta^2}{(1-y_\alpha)(1-y_\beta)^3}\right\}$$
$$= \tfrac{1}{2}\{-3\,\mathrm{tr}\,W^4+2\,\mathrm{tr}\,W^3\,\mathrm{tr}\,W+(\mathrm{tr}\,W^2)^2+2(p-3)\,\mathrm{tr}\,W^3+4\,\mathrm{tr}\,W^2\,\mathrm{tr}\,W$$
$$+(2p-3)\,\mathrm{tr}\,W^2+(\mathrm{tr}\,W)^2\}$$

$$(3.8)\quad \textstyle\sum_{\alpha\neq\beta} y_\alpha^2(y_\alpha-y_\beta)^{-1}(1-y_\alpha)^{-2} = \sum_{\alpha<\beta}\{y_\alpha(1-y_\alpha)^{-2}(1-y_\beta)^{-1}$$
$$+y_\beta(1-y_\alpha)^{-1}(1-y_\beta)^{-2}\}$$
$$= -\mathrm{tr}\,W^3+\mathrm{tr}\,W^2\,\mathrm{tr}\,W+(\mathrm{tr}\,W)^2$$
$$+(p-2)\,\mathrm{tr}\,W^2+(p-1)\,\mathrm{tr}\,W,$$

we can simplify the second term in (3.4) as

$$(3.9)\quad \tfrac{1}{2}b|I-Y|^{-b}[(b^2+2)\,\mathrm{tr}\,W^2(\mathrm{tr}\,W)^2+\tfrac{1}{2}b(\mathrm{tr}\,W)^4+(1-\tfrac{1}{2}b)(2b+1)(\mathrm{tr}\,W^2)^2$$
$$-3(2b+1)\,\mathrm{tr}\,W^4+4b\,\mathrm{tr}\,W^3\,\mathrm{tr}\,W$$
$$+\{8b+2+(2b^2+b+2)(p-1)\}\,\mathrm{tr}\,W^2\,\mathrm{tr}\,W+\{b(p-1)+2\}(\mathrm{tr}\,W)^3$$
$$+2(p-3)(2b+1)\,\mathrm{tr}\,W^3+(2b+1)(2p-3)\,\mathrm{tr}\,W^2+\{2b+1$$
$$+2(p-1)\}(\mathrm{tr}\,W)^2].$$

The third term in (3.4) can be written as

$$(3.10)\quad (d/dt)|I-tY|^{-b}f(tY)\,\Big|_{t=1} = \tfrac{1}{2}b|I-Y|^{-b}\{2(2b+1)\,\mathrm{tr}\,W^3$$
$$+(2b^2+b+2)\,\mathrm{tr}\,W^2\,\mathrm{tr}\,W+b(\mathrm{tr}\,W)^3+2(\mathrm{tr}\,W)^2+2(2b+1)\,\mathrm{tr}\,W^2\}.$$

Substituting (3.5), (3.9), and (3.10) for the right-hand side of (3.4), we can derive the second equality (1.7) in Theorem 2.

Fujikoshi [2] obtained a further formula concerning

$$\sum_{k=0}^{\infty} \sum_{(\kappa)} (b)_k \, a_2(\kappa) C_\kappa(Z)/k!,$$

based on the equality (2.2). It seems difficult, however, to give an alternative proof from the formula (2.6).

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