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ASYMPTOTIC EFFICIENCY OF A CLASS OF ALIGNED RANK ORDER TESTS FOR MULTIRESPONSE EXPERIMENTS IN SOME INCOMPLETE BLOCK DESIGNS[1]

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1. Summary and introduction. Consider n replications of an incomplete block design D consisting of b blocks of constant size $k(\geqq 2)$ to which $v(\geqq k)$ treatments are applied in such a way that (i) no treatment occurs more than once in any block, (ii) the jth treatment occurs in $r_j(\leqq b)$ blocks, and (iii) the (j, j')th treatments occur together in $r_{jj'}$ (> 0) blocks, for $j \neq j' = 1, \cdots, v$. Let then S_i stand for the set of treatments occurring in the ith block, $i = 1, \cdots, b$. For the αth replicate, the response of the plot in the ith block and receiving the jth treatment is a stochastic p-vector $\mathbf{X}_{\alpha ij}$ and is expressed as

$$\mathbf{X}_{\alpha ij} = \boldsymbol{\mu}_\alpha + \boldsymbol{\beta}_{\alpha i} + \boldsymbol{\tau}_j + \boldsymbol{\varepsilon}_{\alpha ij}, \qquad j \in S_i,\ i = 1, \cdots, b,\ \alpha = 1, \cdots, n; \quad \sum \boldsymbol{\tau}_j = \mathbf{0}, \tag{1.1}$$

where the $\boldsymbol{\mu}_\alpha$ and the $\boldsymbol{\beta}_{\alpha i}$ are respectively the replicate and block effects (nuisance parameters in "fixed effect" model or spurious random variables in "mixed effect" model), $\boldsymbol{\tau}_1, \cdots, \boldsymbol{\tau}_v$ are the treatment effects (parameters of interest) and the $\boldsymbol{\varepsilon}_{\alpha ij}$ are the error vectors. It is assumed that $\{\boldsymbol{\varepsilon}_{\alpha ij}, j \in S_i\}$ have jointly a continuous cumulative distribution function (cdf) $G(\mathbf{x}_1, \cdots, \mathbf{x}_k)$ which is symmetric in its k vectors. This includes the conventional assumption of independence and identity of the cdf's of all the $N(= nbk)$ error vectors as a special case. We want to test the null hypothesis

$$H_0: \boldsymbol{\tau}_1 = \cdots = \boldsymbol{\tau}_v = \mathbf{0} \text{ vs } H_1: \boldsymbol{\tau}_j \neq \mathbf{0}, \qquad \text{for at least one } j(= 1, \cdots, v). \tag{1.2}$$

In the univariate case (i.e., $p = 1$), intra-block rank tests for this problem are due to Durbin (1951), Benard and Elteren (1953), and Bhapkar (1961), among others. For some special balanced designs, the studies made by Elteren and Noether (1959) and Bhapkar (1963) reveal the low (Pitman-) efficiency of these tests, particularly when k is small. In complete block designs, it is known [cf. Hodges and Lehmann (1962) and Sen (1968)] that the use of ranking after alignment increases the efficiency of the rank tests. The purpose of the present paper is to show that this merit of the ranking after alignment is preserved for a broad class of incomple block designs. In fact, certain bounds for the efficiency are derived which do not depend on the design D, i.e., on the particular values of $b, v, r_j, r_{jj'}$ and k.

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2. Preliminary notions. We define the aligned observations by

$$\mathbf{Y}_{\alpha ij} = \mathbf{X}_{\alpha ij} - k^{-1} \textstyle\sum_{l \in S_i} \mathbf{X}_{\alpha il}, \qquad j \in S_i,\ i = 1, \cdots, b,\ \alpha = 1, \cdots, n, \tag{2.1}$$

and let $R_{\alpha ij}^{(t)}$ be the rank of $Y_{\alpha ij}^{(t)}$ among the N aligned observations on the tth variate, for $t = 1, \cdots, p$. Also, let for each $i (= 1, \cdots, b)$,

$$\tau_{j,i} = \tau_j - k^{-1} \textstyle\sum_{l \in S_i} \tau_l \quad \text{and} \quad \mathbf{e}_{\alpha ij} = \boldsymbol{\varepsilon}_{\alpha ij} - k^{-1} \textstyle\sum_{l \in S_i} \boldsymbol{\varepsilon}_{\alpha il}, \qquad \alpha = 1, \cdots, n, j \in S_i. \tag{2.2}$$

Then, from (1.1), (2.1) and (2.2), we have

$$\mathbf{Y}_{\alpha ij} = \boldsymbol{\tau}_{j,i} + \mathbf{e}_{\alpha ij}, \qquad j \in S_i,\ i = 1, \cdots, b,\ \alpha = 1, \cdots, n. \tag{2.3}$$

Consider now p sequences of general rank scores

$$E_{N,s}^{(t)} = J_{N,t}(s/(N+1)), \qquad 1 \leqq s \leqq N, \text{ for } t = 1, \cdots, p, \tag{2.4}$$

where for each t, the following conditions [due to Chernoff and Savage (1958)] hold:

(2.5) (i) $\lim_{N \to \infty} J_{N,t}(u) = J_t(u)$ exists for all $0 < u < 1$ and is not a constant,

(ii) $J_t(u)$ is absolutely continuous with

$$\left|(d^r/du^r) J_t(u)\right| \leqq K[u(1-u)]^{-r-\frac{1}{2}+\delta}, \qquad \delta > 0, \text{ for } r = 0, 1, \text{ where } K < \infty, \tag{2.6}$$

$$\text{(iii)} \quad N^{-1} \textstyle\sum_{i=1}^{N} \left|J_{N,t}(i/(N+1)) - J_t(i/(N+1))\right| = o(N^{-\frac{1}{2}}). \tag{2.7}$$

For notational simplicity, we let for each $t (= 1, \cdots, p)$

$$\eta_{\alpha ij}^{(t)} = E_{N, R_{\alpha ij}^{(t)}}^{(t)}, \qquad j \in S_i,\ i = 1, \cdots, b,\ \alpha = 1, \cdots, n; \tag{2.8}$$

$$\eta_{\alpha i \cdot}^{(t)} = k^{-1} \textstyle\sum_{j \in S_i} \eta_{\alpha ij}^{(t)}, \qquad \eta_{\alpha \cdot\cdot}^{(t)} = b^{-1} \textstyle\sum_{i=1}^{b} \eta_{\alpha i \cdot}^{(t)} \quad \text{and} \quad \eta_{\cdots}^{(t)} = n^{-1} \textstyle\sum_{\alpha=1}^{n} \eta_{\alpha \cdot\cdot}^{(t)}. \tag{2.9}$$

The test statistic to be considered is a quadratic form in the following statistics.

$$T_{N,j}^{(t)} = \frac{1}{n} \sum_{\alpha=1}^{n} \textstyle\sum_{i \in P_j} \eta_{\alpha ij}^{(t)}, \tag{2.10}$$

(where $P_j = \{i : j \in S_i\}$), $j = 1, \cdots, v$, $t = 1, \cdots, p$.

To define the test statistic, we let $\mathbf{V}_N^{(i)} = ((v_{N,tt'}^{(i)}))$, $i = 1, 2$, where

$$v_{N,tt'}^{(1)} = \frac{1}{nbk} \sum_{\alpha=1}^{n} \sum_{i=1}^{b} \textstyle\sum_{j \in S_i} [\eta_{\alpha ij}^{(t)} - \eta_{\alpha i \cdot}^{(t)}][\eta_{\alpha ij}^{(t')} - \eta_{\alpha i \cdot}^{(t')}], \tag{2.11}$$

$$v_{N,tt'}^{(2)} = \frac{1}{nb} \sum_{\alpha=1}^{n} \sum_{i=1}^{b} [\eta_{\alpha i \cdot}^{(t)} - \eta_{\alpha \cdot\cdot}^{(t)}][\eta_{\alpha i \cdot}^{(t')} - \eta_{\alpha \cdot\cdot}^{(t')}], \qquad \text{for } t, t' = 1, \cdots, p. \tag{2.12}$$

Also, let $\mathbf{A}^{(i)} = ((a_{jj'}^{(i)}))$, $i = 1, 2$, where

(2.13) $$a_{jj'}^{(1)} = [kr_j\delta_{jj'} - r_{jj'}]/(k-1), \qquad a_{jj'}^{(2)} = [br_{jj'} - r_jr_{j'}]/(b-1), \qquad j,j' = 1, \cdots, v,$$

$\delta_{jj'}$ is the usual Kronecker delta and $r_{jj} = r_j$. Further, let

(2.14) $$\mathbf{W}_N = \mathbf{A}^{(1)} \otimes \mathbf{V}_N^{(1)} + \mathbf{A}^{(2)} \otimes \mathbf{V}_N^{(2)},$$

where $\otimes$ stands for the Kronecker product of two matrices. Finally, let $\mathbf{T}_N = (T_{N,1}^{(1)}, \cdots, T_{N,v}^{(1)}, \cdots, T_{N,1}^{(p)}, \cdots, T_{N,v}^{(p)})$, and let

(2.15) $$\boldsymbol{\eta} = (r_1\eta_{\cdots}^{(1)}, \cdots, r_v\eta_{\cdots}^{(1)}, \cdots, r_1\eta_{\cdots}^{(p)}, \cdots, r_v\eta_{\cdots}^{(p)}).$$

Then, our proposed (aligned) rank order test statistic is

(2.16) $$\mathscr{L}_N = (\mathbf{T}_N - \boldsymbol{\eta})\mathbf{W}_N^{-}(\mathbf{T}_N - \boldsymbol{\eta})',$$

where $\mathbf{W}_N^{-}$ is a generalized inverse of $\mathbf{W}_N$.

The construction of $\mathscr{L}_N$ is based on the permutational invariance structure of the joint distribution of the bk vectors $\{\mathbf{Y}_{\alpha ij}, j \in S_i, i = 1, \cdots, b\}$ (for each $\alpha = 1, \cdots, n$) under the group of $(k!)^b$ intra-block permutations of the b sets of k vectors as well as the $b!$ permutations of the b (block) pk-vectors. These $(b!(k!)^b)^n$ equally likely permutations generate a completely specified permutational (conditional) probability measure $\mathscr{P}_N$, for which $E_{\mathscr{P}_N}(\mathbf{T}_N) = \boldsymbol{\eta}$ and $\operatorname{Var}_{\mathscr{P}_N}(\mathbf{T}_N) = \mathbf{W}_N$, and hence, $\mathscr{L}_N$ is a quadratic form in $\mathbf{T}_N - E_{\mathscr{P}_N}(\mathbf{T}_N)$ with the discriminant $\mathbf{W}_N^{-}$, a generalized inverse of $\mathbf{W}_N$.

In the sequel, it will be assumed that $D \in \mathscr{D}$, where $\mathscr{D}$ is the class of all incomplete block designs for which

(2.17) $$\text{Rank } [\mathbf{A}^{(1)}] = v-1; \qquad \mathbf{A}^{(2)} \text{ and } b\mathbf{A}^{(1)} - (b-1)\mathbf{A}^{(2)} \text{ are nonnegative definite.}$$

It may be noted that (cf. Kempthorne ((1952), chapters 26 and 27) $\mathscr{D}$ includes the entire class of balanced, partially balanced and group divisible incomplete block designs.

3. Large sample properties of $\mathscr{L}_N$. For small sample sizes, the exact permutation distribution of $\mathscr{L}_N$ [under $\mathscr{P}_N$] can be used to construct a conditionally distribution-free test for H_0 in (1.2); the task becomes prohibitively laborious for large n. In this section, we briefly present the asymptotic results on $\mathscr{L}_N$. Since the proofs of these results are mostly lengthy and follow along the lines of the corresponding proofs [for the complete block cases] treated in Sen (1967), (1968), (1969), these are omitted.

THEOREM 3.1. *Under the conditions of Section* 2, *the permutational* (*conditional*) *distribution of* $\mathscr{L}_N$ [*under* $\mathscr{P}_N$] *converges, in probability, as* $n \to \infty$, *to the* χ^2-*distribution with* $p(v-1)$ *degrees of freedom* (df). *The unconditional distribution* (*under* H_0 *in* (1.2)) *depends on the parent* cdf, *but also converges asymptotically to the same limiting distribution.*

Thus, for large n, the critical values of $\mathscr{L}_N$ can be approximated by those of the chi-square distribution with $p(v-1)$ df. To study the asymptotic power properties of the test based on $\mathscr{L}_N$, we conceive of the following sequence $\{H_N\}$ of (Pitman-) alternatives, specified by

$$(3.1)\qquad H_n\colon \tau_j = \tau_{j,n} = n^{-\frac{1}{2}}\boldsymbol{\theta}_j;\ \boldsymbol{\theta}_j = (\theta_j^{(1)}, \cdots, \theta_j^{(p)}), \qquad j = 1, \cdots, v;\ \Sigma\boldsymbol{\theta}_j = \mathbf{0},$$

where the $\theta_j^{(t)}$ are real and finite. Also, corresponding to the pk-variate cdf $F(\mathbf{x}_1, \cdots, \mathbf{x}_k)$ of $[\mathbf{e}_{\alpha ij}, j \in S_i]$, the marginal cdf of $e_{\alpha ij}^{(t)}$ and the joint cdfs of $(e_{\alpha ij}^{(t)}, e_{\alpha ij}^{(t')})$ and $(e_{\alpha ij}^{(t)}, e_{\alpha ij'}^{(t')})$ are denoted by $F_{[t]}(x)$, $F_{[t,t']}^{(1)}(x, y)$ and $F_{[t,t']}^{(2)}(x, y)$ respectively; by the hypothesis of the symmetric structure of the joint cdf of $[\boldsymbol{\varepsilon}_{\alpha ij}, j \in S_i]$, these do not depend on i, j and $j'(\neq j)$. Let then, $\boldsymbol{v}^{(i)} = ((v_{tt'}^{(i)}))$, $i = 1, 2$, where

$$(3.2)\qquad v_{tt'}^{(i)} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} J_t[F_{[t]}(x)]J_{t'}[F_{[t']}(y)]\, dF_{[t,t']}^{(i)}(x, y) - \mu_t^*\mu_{t'}^*\ , \quad t, t' = 1, \cdots, p,$$

where $\mu_t^* = \int_0^1 J_t(u)\,du$ for $t = 1, \cdots, p$, and $F_{[t,t]}^{(1)}(x, y)$ is replaced by $F_{[t]}(x)$, $(i = 1, 2)$. We assume that $\boldsymbol{v}^{(1)}$ is positive definite, and let

$$(3.3)\qquad \mathbf{A} = [(k-1)/k]\mathbf{A}^{(1)} + [(b-1)/bk]\mathbf{A}^{(2)} \qquad \mathbf{B} = [(k-1)/k]\mathbf{A}^{(1)} - [(b-1)(k-1)/bk]\mathbf{A}^{(2)};$$

$$(3.4)\qquad \boldsymbol{\Omega} = \mathbf{A} \otimes \boldsymbol{v}^{(1)} - \mathbf{B} \otimes \boldsymbol{v}^{(2)}.$$

We also assume that $F_{[t]}$ is absolutely continuous with

$$(3.5)\qquad |(d/dx)J_t[F_{[t]}(x)]| \text{ is bounded as } x \to \pm\infty, \quad \text{for } t = 1, \cdots, p.$$

Let then

$$(3.6)\qquad B_t = \int_{-\infty}^{\infty} (d/dx)J_t[F_{[t]}(x)]\, dF_{[t]}(x), \qquad t = 1, \cdots, p,$$

$$(3.7)\qquad \xi_j^{(t)} = B_t\tilde{\theta}_j^{(t)};\ \tilde{\theta}_j^{(t)} = [r_j(k-1)/k]\theta_j^{(t)} - \textstyle\sum_{j'=1(\neq j)}^{v}(r_{jj'}/k)\theta_{j'}^{(t)} - r_j\tilde{\theta}_0^{(t)},$$

where

$$(3.8)\qquad \tilde{\theta}_0^{(t)} = (1/bk)\textstyle\sum_{i=1}^{b}\sum_{j\in S_i}[\theta_j^{(t)} - (1/k)\sum_{l\in S_i}\theta_l^{(t)}], \qquad t = 1, \cdots, p.$$

Finally, let

$$(3.9)\qquad \mathbf{K}_i = ((\kappa_{tt'}^{(i)}));\ \kappa_{tt'}^{(i)} = B_t v_{tt'}^{(i)} B_{t'}, \qquad t, t' = 1, \cdots, p;\ i = 1, 2,$$

$$(3.10)\qquad \boldsymbol{\Gamma} = \mathbf{A} \otimes \mathbf{K}_1 - \mathbf{B} \otimes \mathbf{K}_2,$$

and let $\boldsymbol{\xi} = (\xi_1^{(1)}, \cdots, \xi_v^{(1)}, \cdots, \xi_1^{(p)}, \cdots, \xi_v^{(p)})$, $\boldsymbol{\theta} = (\tilde{\theta}_1^{(1)}, \cdots, \tilde{\theta}_v^{(1)}, \cdots, \tilde{\theta}_1^{(p)}, \cdots, \tilde{\theta}_v^{(p)})$ and $\tilde{\boldsymbol{\theta}}^0 = (\tilde{\theta}_0^{(1)}, \cdots, \tilde{\theta}_0^{(1)}, \cdots, \tilde{\theta}_0^{(p)}, \cdots, \tilde{\theta}_0^{(p)})$. Then, we have the following theorem.

THEOREM 3.2. *Under $\{H_n\}$ in* (3.1) *and the conditions stated above, $\mathscr{L}_N$ has asymptotically a noncentral chi-square distribution with* $p(v-1)$ df *and noncentrality parameter*

$$(3.11)\qquad \Delta_{\mathscr{L}} = \boldsymbol{\xi}\,\boldsymbol{\Omega}^-\,\boldsymbol{\xi}' = (\tilde{\boldsymbol{\theta}} - \tilde{\boldsymbol{\theta}}^0)\boldsymbol{\Gamma}^-(\tilde{\boldsymbol{\theta}} - \tilde{\boldsymbol{\theta}}^0)',$$

where $\boldsymbol{\Omega}^-$ and $\boldsymbol{\Gamma}^-$ are generalized inverses of $\boldsymbol{\Omega}$ and $\boldsymbol{\Gamma}$ respectively.

4. Asymptotic efficiency of $\mathscr{L}_N$. In the parametric case (i.e., in normal theory models), for $p = 1$, the optimal invariant test is based on the variance-ratio ($\mathscr{F}$-) criterion comparing the mean square due to treatment (eliminating block effects) with the error mean square. For $p \geqq 2$, different generalizations of this test are (i) the normal-theory likelihood ratio (NTLR-) test based on the likelihood ratio criterion λ_N, (ii) the generalized (Hotelling–Lawley) T_0^2 statistic based on the trace of $\mathbf{S}_h\mathbf{S}_e^{-1}$, (where $\mathbf{S}_h$ and $\mathbf{S}_e$ are the sum of product matrices due to the hypothesis and error respectively), (iii) Roy's statistic based on the largest characteristic root of $\mathbf{S}_h\mathbf{S}_e^{-1}$, and others; the reader is referred to Anderson ((1958) chapter 8) for details. Unfortunately, none of these tests is uniformly (in the set of admissible parameters) better than the others, and hence, is uniformly best. However, it follows from the results of Wald (1943) that the NTLR-test is (asymptotically) the most stringent test and has the best average power over suitable ellipsoids in the parameter space. Also, it can be shown that T_0^2 and λ_N are asymptotically equivalent, in probability, and hence, enjoy the same properties. We shall compare the $\mathscr{L}_N$-test with the NTLR-test when the parent cdf is not necessarily normal. The distribution theory of $-2\log\lambda_N$ when the parent cdf is not necessarily normal has been studied in the context of the general multivariate linear hypothesis by Sen and Puri (1970). It follows from their results (particularly Theorem 2.2), that whenever $F(x)$ has finite moments up to the second order, and the dispersion matrix of $\mathbf{e}_{\alpha ij}$ (defined by (2.2)), denoted by $\Sigma = \Sigma(F)$, is finite and positive definite (pd), $-2\log\lambda_N$ has asymptotically a central chi-square distribution with $p(v-1)$ df under H_0 in (1.2), and under $\{H_n\}$ in (3.1), it has asymptotically a noncentral chi-square distribution with $p(v-1)$ df and the noncentrality parameter

$$\Delta_\lambda = (\tilde{\boldsymbol{\theta}}-\tilde{\boldsymbol{\theta}}^0)[\mathbf{A}^{(1)}\otimes\Sigma]^-(\tilde{\boldsymbol{\theta}}-\tilde{\boldsymbol{\theta}}^0)'. \tag{4.1}$$

Thus, in accordance with the usual definition of the asymptotic relative efficiency (ARE) (cf.[14]), the ARE of the $\mathscr{L}_N$-test with respect to the λ_N-test is

$$e(\mathscr{L};\lambda) = \Delta_{\mathscr{L}}/\Delta_\lambda = e(\mathbf{A}^{(1)}, \mathbf{A}^{(2)}, \Sigma, \mathbf{K}_1, \mathbf{K}_2, \boldsymbol{\theta}), \tag{4.2}$$

which not only depends on the design matrices $\mathbf{A}^{(1)}$ and $\mathbf{A}^{(2)}$, and the covariance matrices Σ, $\mathbf{K}_1$ and $\mathbf{K}_2$, but also on the shift $\boldsymbol{\theta}$. For the case of complete block designs, $\mathbf{A}^{(2)}$ is a null matrix, and hence it can be shown that (4.2) does not depend on $\mathbf{A}^{(1)}$ [cf. Sen (1968), (1969)]. We shall prove here some interesting inequalities on (4.2). For this, we denote the minimum and the maximum characteristic roots of $\Sigma\mathbf{K}_1^{-1}$ (i.e., $c_m(\Sigma\mathbf{K}_1^{-1})$ and $c_M(\Sigma\mathbf{K}_1^{-1})$) by $c_m(F, \mathbf{J})$ and $c_M(F, \mathbf{J})$ respectively, as both Σ and $\mathbf{K}_1$ depend on F, and in addition, $\mathbf{K}_1$ depends on the score functions $\mathbf{J}$. Side by side, we consider the multivariate one-sample location problem, treated in Sen and Puri (1967). The ARE of the rank order test proposed there [based on the same score functions] with respect to the normal-theory optimal invariant test based on the Hotelling T^2 statistic, in the notations of the current paper, is equal to

$$e(F, \mathbf{J}) = (\boldsymbol{\theta}\mathbf{K}_1^{-1}\boldsymbol{\theta}')/(\boldsymbol{\theta}\Sigma^{-1}\boldsymbol{\theta}'), \quad \text{where } \boldsymbol{\theta} = (\theta_1, \cdots, \theta_p). \tag{4.3}$$

By the Courant-theorem on the extrema of the ratio of two quadratic forms, we have then

$$(4.4)\qquad c_m(F, \mathbf{J}) = c_m(\Sigma \mathbf{K}_1^{-1}) \leqq e(F, \mathbf{J}) \leqq c_M(\Sigma \mathbf{K}_1^{-1}) = c_M(F, \mathbf{J}),$$

for all $\boldsymbol{\theta} \in R^p$.

In the following theorem, we show that (4.4) provides some design-free (i.e., valid for all $D \in \mathscr{D}$) lower bounds to the ARE in the current situation.

THEOREM 4.1. *Under the conditions of Sections* 2 *and* 3,

$$(4.5)\qquad \inf_\theta e(\mathbf{A}^{(1)}, \mathbf{A}^{(2)}, \Sigma, \mathbf{K}_1, \mathbf{K}_2, \boldsymbol{\theta}) \geqq c_m(F, \mathbf{J}),$$

$$(4.6)\qquad \sup_\theta e(\mathbf{A}^{(1)}, \mathbf{A}^{(2)}, \Sigma, \mathbf{K}_1, \mathbf{K}_2, \boldsymbol{\theta}) \geqq c_M(F, \mathbf{J}),$$

for all $D \in \mathscr{D}$, *where the equality sign holds* iff $J_t[F_{[t]}(x)]$ *is a linear function of* x, *with probability* 1, *for all* $t = 1, \cdots, p$.

PROOF. We obtain from (3.11), (4.1), (4.2) and the Courant-theorem [cf. (4.3)–(4.4)] that

$$(4.7)\quad \sup_\theta e_{\mathscr{L},\lambda} = c_M[\Gamma^-(\mathbf{A}^{(1)} \otimes \Sigma)] \quad \text{and} \quad \inf_\theta e_{\mathscr{L},\lambda} = c_m[\Gamma^-(\mathbf{A}^{(1)} \otimes \Sigma)].$$

Now, by (3.3) and (3.10), we have

$$(4.8)\quad c_M[\Gamma^-(\mathbf{A}^{(1)} \otimes \Sigma)] = 1/c_m[(\mathbf{A}^{(1)-} \otimes \Sigma^{-1})\Gamma]$$

$$= 1/c_m\Big[(\mathbf{A}^{(1)-} \otimes \Sigma^{-1})(\mathbf{A}^{(1)} \otimes \mathbf{K}_1 - \frac{1}{bk}(b\mathbf{A}^{(1)} - (b-1)\mathbf{A}^{(2)}) \otimes$$

$$[\mathbf{K}_1 + (k-1)\mathbf{K}_2])\Big]$$

$$= 1/c_m\Big[\mathbf{I}_v \otimes \Sigma^{-1}\mathbf{K}_1 - \frac{1}{bk}\mathbf{A}^{(1)-}[(b\mathbf{A}^{(1)} - (b-1)\mathbf{A}^{(2)})] \otimes$$

$$\Sigma^{-1}(\mathbf{K}_1 + (k-1)\mathbf{K}_2)\Big].$$

Similarly,

$$(4.9)\quad c_m[\Gamma^-(\mathbf{A}^{(1)} \otimes \Sigma)] = 1/c_M[(\mathbf{A}^{(1)-} \otimes \Sigma^{-1})\Gamma]$$

$$= 1/c_M\Big[\mathbf{I}_v \otimes \Sigma^{-1}\mathbf{K}_1 - \frac{1}{bk}[\mathbf{A}^{(1)-}(b\mathbf{A}^{(1)} - (b-1)\mathbf{A}^{(2)})] \otimes$$

$$\Sigma^{-1}(\mathbf{K}_1 + (k-1)\mathbf{K}_2)\Big].$$

Let us also prove the following lemma.

LEMMA 4.2. $\mathbf{K}_1$ *positive semi-definite* (p.s.d.) $\Rightarrow \mathbf{K}_1+(k-1)\mathbf{K}_2$ p.s.d. *Further, if* $\mathbf{K}_1$ *is* p.d. *and the distribution of* $[e_{\alpha ij}, j \in S_i]$ *is not contained in any* $p(k-1)-r$ $(r \geqq 1)$*-dimensional flat, then* $\mathbf{K}_1+(k-1)\mathbf{K}_2$ *is also* p.d., *unless* $J_t[F_{[t]}(x)]$ *is linear in* x, *with probability one, for all* $t = 1, \cdots, p$; *in the later case, it is a null matrix.*

PROOF. By (3.9), it suffices to prove the lemma for $\mathbf{v}^* = \mathbf{v}^{(1)}+(k-1)\mathbf{v}^{(2)}$. The first part of the lemma follows directly by noting that $\mathbf{v}^*$ is the dispersion matrix of $\mathbf{Z} = (Z_1, \cdots, Z_p)$, where $Z_t = \sum_{j \in S_i} J_t[F_{[t]}(e_{\alpha ij}^{(t)})]$, $t = 1, \cdots, p$, and that the dispersion matrix of $(J_t[F_{[t]}(e_{\alpha ij}^{(t)})]$, $t = 1, \cdots, p)$ is $\mathbf{v}^{(1)}$. The second part follows in the same way as in Lemma 4.5 of Sen (1968), after noting that $\sum_{j \in Si} J_t[F_{[t]}(e_{\alpha ij}^{(t)}) = 0$, with probability 1, for all $t = 1, \cdots, p$. □

Thus, from (2.17), (4.8), (4.9) and Lemma 4.2, it follows that for all $D \in \mathscr{D}$, $(1/bk)[b\mathbf{A}^{(1)}-(b-1)\mathbf{A}^{(2)})\mathbf{A}^{(1)-}] \otimes [\mathbf{\Sigma}^{-1}(\mathbf{K}_1+(k-1)\mathbf{K}_2)]$ is p.s.d. Hence,

$$c_M[\mathbf{\Gamma}^-(\mathbf{A}^{(1)} \otimes \mathbf{\Sigma})] \geqq 1/c_m[\mathbf{I}_v \otimes \mathbf{\Sigma}^{-1}\mathbf{K}_1] = c_M(\mathbf{\Sigma}\mathbf{K}_1{}^{-1}) = c_M(F, \mathbf{J}), \tag{4.10}$$

$$c_m[\mathbf{\Gamma}^-(\mathbf{A}^{(1)} \otimes \mathbf{\Sigma})] \geqq 1/c_M[\mathbf{I}_v \otimes \mathbf{\Sigma}^{-1}\mathbf{K}_1] = c_m(\mathbf{\Sigma}\mathbf{K}_1{}^{-1}) = c_m(F, \mathbf{J}), \tag{4.11}$$

where the equality signs hold iff $J_t[F_{[t]}(x)]$ is linear in x, with probability one, for all $t = 1, \cdots, p$. □

For special interest, we consider the case of $p = 1$ separately in the next section. For $p \geqq 2$, by virtue of (4.3), (4.4) and Theorem 4.1, the ARE results of Sen and Puri (1967) provide the corresponding lower bounds in our case; without going into the details, we refer to there for these bounds derived in various special cases. In particular, if F is multinormal and we use the normal scores $\mathscr{L}_N$-test, it follows that $\mathbf{K}_1+(k-1)\mathbf{K}_2 = \mathbf{0}$, and hence, this test and the NTLR λ_N-test are asymptotically power equivalent for the sequence of alternatives in (3.1). On the other hand, for multinormal F, the rank-sum $\mathscr{L}_N$-test not only entails some loss in ARE, but also, for $p \geqq 3$, can have ARE arbitrary close to 0, depending on the parent $\mathbf{\Sigma}(F)$. We have assumed so far that $k \geqq 2$. For $k = 2$, the design D reduces to the so-called paired comparisons design, for which rank tests in the multivariate case are studied in Sen and David (1968) and Shane and Puri (1969), among others. Our ARE results agree with theirs in this case.

5. ARE in the case of $p = 1$. Here the λ_N-test is equivalent to the variance-ratio ($\mathscr{F}$-) test, and also, upon defining $v_{11}^{(i)}$, $i = 1, 2$, and B_1 as in (3.2) and (3.6), we have $c_m(\mathbf{\Sigma}\mathbf{K}_1{}^{-1}) = c_M(\mathbf{\Sigma}\mathbf{K}_1{}^{-1}) = \sigma_{11}B_1{}^2/v_{11}^{(1)}$, where $\sigma_{11} = V(e_{\alpha ij}^{(1)})$. Hence, it follows that

$$e(\mathscr{L}; \mathscr{F}) \geqq B_1{}^2\sigma_{11}/v_{11}^{(1)}, \tag{5.1}$$

where the equality sign holds iff $v_{11}^{(2)}/v_{11}^{(1)} = -(k-1)^{-1}$. Now, (5.1) agrees with (4.14) of Sen (1968), representing the corresponding ARE in the complete block design. Hence, we have the following.

THEOREM 5.1. *The* ARE *results and bounds studied in Section* 4 *of Sen* (1968) *remain valid for the entire class* $\mathscr{D}$ *of incomplete block designs* D.

Thus, the normal scores $\mathscr{L}_N$-test has the ARE (with respect to the $\mathscr{F}$-test) bounded below by one, uniformly in the class of underlying distributions; the lower bound is attained iff $F_{[1]}$ is normal.

Finally, we compare the $\mathscr{L}_N$-test with the other intra-block rank tests available in the literature (cf. [2], [3], [4], [6], [7]). The ARE of the Brown–Mood median procedure with respect to the $\mathscr{F}$-test is studied by Bhapkar (1961), (1963) for various special types of incomplete block designs, and appears to be quite low, particularly for small k. The ARE of the rank-sum test by Durbin (1951) is studied by Elteren and Noether (1959), only for the balanced incomplete block design; extending their results to the general class of incomplete block designs considered here, it can be shown that the ARE of the rank-sum test by Benard and Elteren (1953) with respect to the $\mathscr{F}$-test is equal to

$$(5.2)\qquad e(BE;\mathscr{F}) = [12\sigma^2(1-\rho_\varepsilon)k/(k+1)]\,[\textstyle\int_{-\infty}^{\infty} g^*(x,x)\,dx]^2,$$

where $\sigma^2 = V(\varepsilon_{\alpha ij}^{(1)})$, $\rho_\varepsilon\sigma^2 = \mathrm{Cov}(\varepsilon_{\alpha ij}^{(1)}\,\varepsilon_{\alpha ij'}^{(1)})$ and $g^*(x,y)$ is the joint density of $(\varepsilon_{\alpha ij}^{(1)},\ \varepsilon_{\alpha ij'}^{(2)}), j, j' \in S_i$, whose marginal densities are both equal to $g(x)$. Thus, $\sigma_{11} = \{[(k-1)/k].\ \sigma^2(1-\rho_\varepsilon)\}$, and if the errors are mutually independent, then $\rho_\varepsilon = 0$ and $g^*(x,y) = g(x)g(y)$.

Consider now the particular rank-sum $\mathscr{L}_N$, based on the score function $J(u) = u$ $0 < u < 1$; this aligned rank statistic in a very simple type of incomplete block design appears to have been considered first by Hodges and Lehmann (1962). In this case, from (5.1), (5.2) and some standard computations [using $J(u) = u$], we obtain that

$$(5.3)\qquad e(BE;\mathscr{L}) \leqq [k^2/(k^2-1)]\,[\textstyle\int_{-\infty}^{\infty} g^*(x,x)\,dx]^2/[\textstyle\int_{-\infty}^{\infty} f^2(x)\,dx]^2,$$

where $f(x)$ is the marginal density of $e_{\alpha ij}^{(1)}$. When the errors are normally distributed, (5.3) reduces to $k/(k+1)(<1)$, for all $D \in \mathscr{D}$. Thus, the aligned rank-sum test is more (asymptotically) efficient than the intra-block rank-sum test when the underlying distribution is normal. However, it may be remarked that the intra-block rank-sum test does not require the assumption of additivity of the block and replicate effects which is implicit in our assumption that the cdf G of $[\varepsilon_{\alpha ij}, j \in S_i]$ does not depend on α and i. Thus, the comparatively low ARE of the Benard–Elteren test is counter-balanced by its greater scope of applicability.

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