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A NOTE ON THE ARC-SINE LAW AND MARKOV RANDOM SETS

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0. Introduction. The purpose of this note is to point out an extension of a theorem of Dynkin [2], [3, page 447] concerning renewal processes to the context of Markov random sets (or, as we shall call them, *semilinear Markov processes*). Dynkin's result states that, if $X_1, X_2, \cdots$ is a "renewal sequence", i.e. a sequence of nonnegative, independent, identically distributed random variables, with partial sums S_n, and if x_t is defined by

$$x_t = t - \max\{S_n : S_n \leqq t\}, \qquad t \geqq 0, \tag{0.1}$$

then x_t/t has a nondegenerate limiting distribution as $t \to \infty$ iff $1-F(x) = x^{-\beta}L(x)$, where $F(x)$ is the common distribution function of the X_i, β is some number in $(0, 1)$, and $L(x)$ is slowly varying as $x \to \infty$.

Semilinear Markov processes arise when, in (0.1), we allow more general processes with stationary, independent increments. Specifically, let $T(s)$, $s \geqq 0$, be a subordinator (terminology is explained in Section 1) on a probability space $(\Omega, \mathscr{F}, P)$ having exponent $g(\lambda) = \lambda\alpha + \int_0^\infty (1-e^{-\lambda y})\mu(dy)$. Denote by $Q(\omega)$ the range of $T(s, \omega)$, $s \geqq 0$, and define the random function $\xi_t(\omega)$ by

$$\xi_t(\omega) = t - \sup\{u \leqq t : u \in Q(\omega)\}, \qquad t \geqq 0. \tag{0.2}$$

This is analogous to (0.1). Our extension of Dynkin's theorem may now be stated as follows.

0.3. THEOREM. *Let $h(x) = \mu(x, \infty]$. If $h(x) = x^{-\beta}L(x)$, where $0 < \beta < 1$ and $L(x)$ is of slow variation as $x \to \infty$, then ξ_t/t has a limiting distribution as $t \to \infty$, given by the measure*

$$v(dx) = \frac{\sin \pi\beta}{\pi} x^{-\beta}(1-x)^{\beta-1}\,dx \tag{0.4}$$

on $(0, 1)$.

The converse is a bit more delicate and is treated in Section 3. In particular, the possible degenerate limit laws are completely delineated, a result which appears to be new even in the case studied by Dynkin. The measure v is the "generalized arc-sine" distribution [3, page 446] which arises, among other places, in the theory of semi-stable Markov processes [8, page 68], [4, Section 4]. There are also some closely related results in [7], and in [9] see, especially, Theorem 8.1 (it is not known if our result follows therefrom; in any case the present method of proof should be of independent interest).

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One more remark is in order. In [4] it is shown that many of the properties of the renewal process x_t in (0.1) persist for the more general process ξ_t of (0.2). Indeed, if one replaces S_n by $T(s)$, and $1-F(x)$ by $h(x)$ in many of the "classical" renewal theorems, one obtains theorems on semi-linear Markov processes—several examples are given in Section 6.11 of [4]. The present work is offered in the same spirit.

1. Preliminaries. We collect here some material on subordinators and semilinear Markov processes which will be needed later. Information on subordinators may be found in [1, page 219], [3, Chapter X, Section 7], and [4, Section 3]; for semi-linear Markov processes, see [4] and [6]. Our notation for Markov processes is adapted from [1].

A *subordinator* is a random process $T(s)$, $s \geqq 0$, on some probability space $(\Omega, \mathscr{F}, P)$ such that

(i) $T(0) = 0$.

(ii) the trajectories are a.s. monotone increasing, right continuous.

(iii) the process has stationary, independent increments.

Let $\alpha \geqq 0$ be constant and let μ be a (Borel) measure on $(0, \infty]$ satisfying $\int_{(0,\infty]}(1-e^{-\lambda x})\mu(dx) < \infty$ for each $\lambda > 0$. Each subordinator has corresponding to it a function $g(\lambda) = \alpha\lambda + \int_{(0,\infty]}(1-e^{-\lambda x})\mu(dx)$, α, μ as described, called its *exponent*, having the property $E(e^{-\lambda T(s)}) = e^{-sg(\lambda)}$. Each such $g(\lambda)$ also gives rise to a $T(s)$, the correspondence being bi-unique (up to equivalence). If $\mu(\{\infty\}) > 0$, the process is killed at a finite time $\zeta : T(s, \omega) = \infty$ for $s \geqq \zeta(\omega)$. In this case ζ has an exponential distribution with parameter $\mu(\{\infty\})$. Otherwise $T(s) < \infty$ a.s. for all s. (The details concerning the "lifetime" ζ are in [1].)

The most primitive subordinator is constructed as follows (this example has been discussed by Doob): Let X_n be a renewal sequence with partial sums S_n and common df $F(x)$, all as in Section 0. Let $N(r)$, $r \geqq 0$, be a Poisson process with rate 1, independent of $\{X_n\}$. Then $T(r) = S_{N(r)}$ is a subordinator having $\alpha = 0$ and $h(x) = \mu(x, \infty] = 1-F(x)$. Notice that ξ_t in (0.2) coincides with x_t given in (0.1) in this case. We refer to this example as the "(classical) renewal case". If we allow $F(x)$ to be *defective* ($F(\infty) < 1$), T will have a finite lifetime.

NOTE. For an arbitrary subordinator, the parameters α, μ are called, respectively, the *rate of linear drift* and *Lévy measure*.

A *semilinear Markov process* (or *Markov random set*) is a Markov process $X = (x_t, \mathscr{M}_t, P^x)$ with state space $E = [0, a)$ for some $a \leqq \infty$, with the following property: for each $\omega \in \Omega$ there is a closed set $Z(\omega) \subset \mathbb{R}$ such that $Z(\omega) \cap (-\infty, 0]$ consists of exactly one point and

$$x_t(\omega) = t - \sup\{-\infty < u \leqq t : u \in Z(\omega)\}, \qquad t \geqq 0. \tag{1.1}$$

Notice that (1.1) is the same as (0.2) with Z instead of Q, at least when $0 \in Z(\omega)$. The following results are found in [4], [6].

1.2. THEOREM. (a) *Let T be a subordinator. The random function ξ_t defined in* (0.2) *is then strongly Markovian. Also, there exists a strongly Markov semilinear process $X = (x_t, \mathscr{M}_t, P^x)$ such that $\{x_t, P^0\}$ is equivalent to $\{\xi_t, P\}$.*

(b) *Given a strongly Markov semilinear process $X = (x_t, \mathscr{M}_t, P^x)$ on a measurable space $(\Omega, \mathscr{F})$, there exists a subordinator T on $(\Omega, \mathscr{F}, P^0)$ for which $Z(\omega) = \overline{Q(\omega)}$* a.s., *hence $\xi_t = x_t$ P^0*-a.s.

(c) *Let the subordinator T have exponent $g(\lambda) = \alpha\lambda + \int_{(0,\infty]}(1-e^{-\lambda x})\mu(dx)$. The transition function of the corresponding strongly Markov semilinear process X is given by*:

$$P_0(x, \Gamma) = \delta_x(\Gamma), \tag{1.2.1}$$

$$P_t(0, \Gamma) = I_\Gamma(0)\left(1-\int_{[0,t]} h(t-y)\, dm(y)\right) + \int_{[0,t]} I_\Gamma(t-y)h(t-y)\, dm(y), \qquad t > 0, \tag{1.2.2}$$

$$P_t(x, \Gamma) = \frac{h(x+t)}{h(x)} I_\Gamma(x+t) + \int_{[0,t]} P_{t-r}(0, \Gamma)\, d M_x(r), \qquad x > 0, \tag{1.2.3}$$

where $M_x(r) = 1-h(x+r)/h(x)$ and the measure m on $[0, \infty)$ *is determined by its Laplace–Stieltjes transform $\hat{m}(\lambda) = 1/g(\lambda)$, $\lambda > 0$.*

We shall often write $m(t)$ for $m([0, t])$ (likewise for other measures on $[0, \infty)$). In the renewal case, $m(t)$ turns out to be the so-called *renewal function*, i.e. the expected number of "renewals" during the time interval $[0, t]$ (cf. equation (6.3), page 182 of [3]). When $\alpha = 0$ and $h(0+) < \infty$, we are back in the renewal situation (see [4, Section 6.11]).

2. Proof of the theorem. We now proceed to prove Theorem 0.3. For the most part the proof follows Feller's treatment of Dynkin's theorem. However, since there is a bit of obscurity in Feller's proof, we shall repeat some of the details. Thus, suppose $h(x) = \mu(x, \infty] = x^{-\beta}L(x)$ as described. It is well known (see [3, page 273]) that, because of slow variation, $x^{-\beta}L(x) \to 0$ as $x \to \infty$, so $h(\infty) = \mu(\{\infty\}) = 0$. Moreover, $\int_0^\infty h(x)\, dx = \infty$ [3, page 272]. By Theorem 1.2(a) and (1.2.2) the problem is reduced to showing that $P_t(0, t\Gamma) \to \nu(\Gamma)$ as $t \to \infty$, for each subinterval Γ of $(0, 1)$, and that $P_t(0, \{0\}) \to 0$, since $P(\xi_t \in \Gamma) = P^0(x_t \in \Gamma) = P_t(0, \Gamma)$.

Let us prove the latter first. Using the ergodic properties of semilinear processes, namely [4, Theorem 6.14], we obtain, for every $x > 0$, $\lim_{t\to\infty} P_t(0, [0, x)) = \pi([0, x))$ where π is the measure on $[0, \infty)$ defined by

$$\pi(\Gamma) = \frac{\alpha}{\alpha + \int_0^\infty h(y)\, dy}\delta_0(\Gamma) + \frac{\int_\Gamma h(y)\, dy}{\alpha + \int_0^\infty h(y)\, dy}.$$

In the present case, because the denominators are infinite, $\pi = 0$, hence $\lim_{t\to\infty} P_t(0, \{0\}) = 0$. (When $\alpha > 0$, this follows also from a result of Kingman [5, Theorem 6].)

Now let $\Gamma = (x_1, x_2)$ be a subinterval of (0, 1). We have, by (1.2.2)

$$
\begin{aligned}
P(\xi_t/t \in \Gamma) &= P_t(0, t\Gamma) = \textstyle\int_0^t I_{(tx_1, tx_2)}(t-y)h(t-y)\, dm(y), \\
&= \textstyle\int_{t(1-x_2)}^{t(1-x_1)} h(t-y)\, dm(y), \qquad (2.1)\\
&= \textstyle\int_{1-x_2}^{1-x_1} h(t(1-y))\, dm(ty)
\end{aligned}
$$

(the endpoints in the last two integrals are excluded).

We now borrow a lemma from [3, page 446].

2.2. Lemma. *If $h(x) = x^{-\beta}L(x)$ as above, then*

$$m(t) \sim \frac{1}{\Gamma(1-\beta)\Gamma(1+\beta)} \frac{t^\beta}{L(t)} = \frac{\sin \pi\beta}{\pi\beta} \frac{t^\beta}{L(t)} \quad \textit{as} \quad t \to \infty,$$

and

$$h(t)m(t) \to \frac{\sin \pi\beta}{\pi\beta}.$$

Assuming the lemma for the moment, let us finish the proof of the theorem. By (2.1) and 2.2,

$$P(\xi_t/t \in \Gamma) \sim \frac{\sin \pi\beta}{\pi\beta} \int_{1-x_2}^{1-x_1} \frac{h(t(1-y))\, dm(ty)}{h(t)m(t)}. \qquad (2.3)$$

From 2.2 it follows that $dm(ty)/m(t)$ converges to the measure with density $\beta y^{\beta-1}$ on (0, 1); and $h(t(1-y))/h(t) \to (1-y)^{-\beta}$. Since $0 < x_1 < x_2 < 1$ the limit function $(1-y)^{-\beta}$ is bounded for the relevant values of y. Likewise the functions $h(t(1-y))/h(t) \leqq h(tx_1)/h(t) \leqq x_1{}^{-\beta}+1$ for all sufficiently large t. Hence the convergence of $h(t(1-y))/h(t)$ to $(1-y)^{-\beta}$ is uniform and we may conclude

$$P(\xi_t/t \in \Gamma) \to \frac{\sin \pi\beta}{\pi} \int_{1-x_2}^{1-x_1} y^{\beta-1}(1-y)^{-\beta}\, dy.$$

Thus, except for 2.2, Theorem 0.3 is proven.

Note. The deduction of the above uniform convergence is based on the following simple result: let $u_n(x)$ be a uniformly bounded sequence of monotone functions (all of the same sense), let $u(x)$ be continuous, and suppose $u_n(x) \to u(x)$ for each x. The convergence is then uniform on compact sets. An incorrect statement of this appears as problem 5, page 276 in [3], both the uniform boundedness and the condition of compactness being there omitted. The result fails under either of these omissions.

We note also that care must be exercised in the above argument to keep $1-y$ bounded away from zero, as it is possible to have $h(0+) = \infty$, a situation which does not arise in the classical renewal case.

PROOF OF LEMMA 2.2. Recall that $m(t)$ is the (distribution function of the) measure determined by the Laplace–Stieltjes transform $\hat{m}(\lambda) = 1/g(\lambda)$. This may be written in terms of h as follows:

$$\hat{m}(\lambda) = \frac{1}{\lambda\alpha + \lambda\hat{H}(\lambda)}$$

where H is the measure given by $H(x) = \int_0^x h(y)\,dy$ (or equivalently by $\hat{H}(\lambda) = \int_0^\infty e^{-\lambda y}h(y)\,dy$). By [3, page 423, Theorem 4] we have

$$\hat{H}(\lambda) \sim \frac{\Gamma(1-\beta)}{\lambda^{1-\beta}} L(1/\lambda) \quad \text{as} \quad \lambda \to 0,$$

thus

$$\hat{m}(\lambda) \sim \frac{1}{\lambda\alpha + \Gamma(1-\beta)\lambda^\beta L(1/\lambda)}, \qquad \lambda \to 0.$$

Now $\lambda^{1-\beta}/L(1/\lambda) \to 0$ as $\lambda \to 0$ since L is slowly varying, hence

$$\hat{m}(\lambda) \sim \frac{1}{\Gamma(1-\beta)\lambda^\beta L(1/\lambda)}.$$

The lemma now follows from Theorem 2 of [3, page 421], since $1/L$ is also slowly varying.

3. The converse result.

3.1. THEOREM. *Let T be a subordinator with α, μ, h, and ξ_t all as described in Section* 0. *Suppose that ξ_t/t has a limiting distribution G. Then G must be the* arc-sine *law v given in* (0.4) *for a suitable β and $h(x)$ must satisfy the condition in* 0.3, *or $G = \delta_1$ or δ_0 (δ_x = unit mass at x).*

Before proving this theorem let us remark that Feller [3, page 447] states that "it is easy to amend" the above arguments to get the converse of the theorem. Unfortunately, the present author has been unable to find such amendment, and has therefore adapted Dynkin's original method.

Let $v_t(s) = E(e^{-s\xi_t})$. Using (1.2.2) this may be expressed as

$$v_t(s) = 1 - \int_0^t h(t-y)\,dm(y) + \int_0^t e^{-s(t-y)}\,h(t-y)\,dm(y). \tag{3.2}$$

If we take Laplace transform in t we obtain

$$\begin{aligned}\int_0^\infty e^{-\lambda t}\,v_t(s)\,dt &= 1/\lambda - \tilde{h}(\lambda)\hat{m}(\lambda) + \tilde{h}(\lambda+s)\hat{m}(\lambda)\\ &= \alpha\hat{m}(\lambda) + \tilde{h}(\lambda+s)\hat{m}(\lambda),\end{aligned}$$

where $\tilde{h}(\lambda)$ denotes the (ordinary) Laplace transform of h. (Recall, as in Section 2, that $\hat{m}(\lambda) = (\lambda\alpha + \lambda\tilde{h}(\lambda))^{-1}$.)
Thus

$$\int_0^\infty e^{-\lambda t}\,v_t(s)\,dt = \frac{\hat{m}(\lambda)}{(\lambda+s)\hat{m}(\lambda+s)}. \tag{3.3}$$

We shall use (3.3) momentarily. Suppose now that a limiting distribution G exists. For each continuity point u of G we have $\lim_{t\to\infty} P(\xi_t/t \leqq u) = G(u)$. Then

$$\lim_{y\to\infty} P\left(\frac{\xi_{ty}}{y} \leqq u\right) = \lim_{y\to\infty} P\left(\frac{\xi_{ty}}{ty} \leqq \frac{u}{t}\right) = G(u/t)$$

for each pair of numbers u, t such that u/t is a continuity point of G. In terms of Laplace transforms the last relation reads

$$\lim_{y\to\infty} v_{ty}(s/y) = \int_0^\infty e^{-su}\, d_u G(u/t). \tag{3.4}$$

Since $v_{ty}(s/y) \leqq 1$ we may take (ordinary) transforms in (3.4), exchange integration with passage to the limit, and obtain thereby

$$\lim_{y\to\infty} \int_0^\infty e^{-\lambda t}\, v_{ty}(s/y)\, dt = \int_0^\infty e^{-\lambda t} \int_0^\infty e^{-su}\, d_u G(u/t)\, dt. \tag{3.5}$$

Comparison of (3.5) with (3.3) shows

$$\lim_{y\to\infty} \frac{1}{y} \frac{\hat{m}(\lambda/y)}{\left(\dfrac{\lambda+s}{y}\right)\hat{m}\left(\dfrac{\lambda+s}{y}\right)} = \int_0^\infty \int_0^\infty e^{-\lambda t - su}\, d_u G(u/t)\, dt; \tag{3.6}$$

in particular $\lim_{y\to\infty} \hat{m}((\lambda+s)/y)/\hat{m}(\lambda/y)$ exists for each λ, s. Taking $s = (k-1)\lambda$ and $z = y^{-1}$ we see $\lim_{z\to 0} \hat{m}(k\lambda z)/\hat{m}(\lambda z)$ exists for each k, λ, and finally that $\hat{m}(k\lambda)/\hat{m}(\lambda)$ approaches a limit as $\lambda \to 0$, for each k; in other words $\hat{m}$ is a function of regular variation [3] at zero. We may therefore write $\hat{m}(\lambda) = \lambda^{-\beta}K(1/\lambda)$ where $0 \leqq \beta < \infty$ and $K(x)$ is of slow variation as $x \to \infty$. In fact, since

$$\lambda\hat{m}(\lambda) = \frac{1}{\alpha+\tilde{h}(\lambda)} \to \frac{1}{\alpha+\eta} < \infty \qquad (\lambda \to 0),$$

where $\eta = \int_0^\infty h(y)\, dy$, we must have $0 \leqq \beta \leqq 1$. It now follows easily that the left member of (3.6) is equal to $((\lambda+s)/\lambda)^\beta(\lambda+s)^{-1}$. On the other hand, if $0 < \beta < 1$, it follows from standard formulas involving the beta and gamma functions that $\int_0^\infty e^{-\lambda t} \int_0^\infty e^{-sx}\, dv(x/t)\, dt = ((\lambda+s)/\lambda)^\beta(\lambda+s)^{-1}$; specifically see page 239 (Section 12.14) and page 261, no. 28 of [10]. By uniqueness of Laplace transforms we conclude $G = v$ if $0 < \beta < 1$.

It is easily verified that, if $\beta = 0$, then $G = \delta_1$, and, if $\beta = 1$, $G = \delta_0$.

Finally, suppose $0 < \beta < 1$, so $\hat{m}(\lambda) = \lambda^{-\beta}K(1/\lambda)$. Recalling the relation $\hat{m}(\lambda) = (\alpha\lambda+\lambda\tilde{h}(\lambda))^{-1}$ we have, after some easy maneuvers, $\tilde{h}(\lambda) \to \infty$ as $\lambda \to 0$ and $\tilde{h}(\lambda) \sim \lambda^{\beta-1}L(1/\lambda)$ where $L = 1/K$. It now follows from [3, page 422] that $h(x)$ satisfies the condition stated in 0.3. The proof is complete.

Each of the three types of limit law is possible, as the examples below demonstrate. The first example is admittedly pathological, and the second less so. Neither situation arises in the classical renewal case.

EXAMPLE 1. Consider the process "uniform motion to the right with velocity 1", discussed in [1, page 23]. The trajectories starting from $x \in \mathbb{R}_+ = [0, \infty)$ take the form $x_t = x+t$. Hence we have a semilinear Markov process with $h(x) \equiv 1$,

$\alpha = 0$ (see [4]). Clearly $Z(\omega) = \{-x\}P^x$-a.s. for $x \geqq 0$, and $P^0(x_t/t = 1) = 1$ for all t. Hence $G = \delta_1$.

EXAMPLE 2. Let $X = (x_t, \mathscr{M}_t, P^x)$ be a semilinear strongly Markov process corresponding via 1.2 to a subordinator T having drift α and Lévy measure μ such that, as $x \to \infty$, $h(x) = \mu(x, \infty] \to h(\infty) > 0$. In this case it is known that the set $Z(\omega)$ (cf. Section 1) is a.s. bounded [4, Theorem 6.10], and thus $\sigma(\omega) = \sup Z(\omega)$ is a well-defined finite valued random variable. Then, for any ε, $0 < \varepsilon < 1$,

$$P^0\left(\frac{x_t}{t} > 1-\varepsilon\right) = P^0\left(t < \sigma, \frac{x_t}{t} > 1-\varepsilon\right) + P^0\left(t \geqq \sigma, \frac{t-\sigma}{t} > 1-\varepsilon\right);$$

the first term on the right is dominated by $P^0(t < \sigma) \to 0$ and the second clearly tends to 1 as $t \to \infty$. Hence we again have $G = \delta_1$.

EXAMPLE 3. Suppose we have the same set-up as in Example 2, except that now assume there exists a number $a > 0$ such that $h(x) = 0$ for all $x \geqq a$. Using the results of [4] or [6] it is not difficult to see that $x_t \leqq a$ for all t, so that $G = \delta_0$.

It is natural to ask whether the converses of the results in Examples 2 and 3 are true. As for Example 3, consider:

EXAMPLE 4. Let $h(x) > 0$ for all $x > 0$, and suppose $\eta = \int_0^\infty h(y)\,dy < \infty$. ($h(x) = e^{-x}$ is a typical case.) Then $\hat{m}(\lambda) = (\lambda\alpha + \lambda\tilde{h}(\lambda))^{-1} \sim \lambda^{-1}(\alpha+\eta)^{-1}$. Hence $\beta = 1$ in the proof of 3.1, and $G = \delta_0$.

Finally, the converse of Example 2 is false, as is seen in

EXAMPLE 5. Let $L(x)$ be a function of slow variation at infinity and such that $L(x) \to 0$ as $x \to \infty$ (for instance $L(x) = 1/\log x$ will do the job). Let $h(x)$ be any decreasing function on $(0, \infty)$ which behaves like $L(x)$ as $x \to \infty$. By the lemma on page 422 of [3] we have $\tilde{h}(\lambda) \sim \lambda^{-1}L(1/\lambda)$ as $\lambda \to 0$. Let $\alpha = 0$. Then $\hat{m}(\lambda) = 1/\lambda\tilde{h}(\lambda) \sim 1/L(1/\lambda)$ so that $\beta = 0$ in the proof of 3.1. Therefore $G = \delta_1$.

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