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ABSTRACTS OF PAPERS

(*An abstract of a paper presented at the Western Regional meeting, Las Vegas, Nevada, March* 22–24, 1971. *Additional abstracts appeared in previous issues.*)

128-24. A nonlinear method of predicting and representing random variables. J. L. DENNY, University of Arizona.

We obtain several necessary and sufficient conditions for each nonnegative rv X to have a representation a.e. $\sum_1^n Y_i$, where each rv Y_i is nonnegative and measurable with respect to a fixed (in advance) sub-sigma-algebra. For example, one theorem characterizes those probabilities on R^2 which assign measure one to the graphs of two real-valued Borel functions. The conditions of the representation theorem are used to obtain sufficient conditions for convergence of a nonlinear method of prediction which depends on convergence properties of iterated conditional expectation operators. (Received February 15, 1971.)

(*Abstracts of papers presented at the Eastern Regional meeting, University Park, Pennsylvania, April* 21–23, 1971. *Additional abstracts appeared in earlier issues.*)

129-13. On some properties of a regular sequence of designs and asymptotically optimal design in time series. MARK C. K. YANG, University of Florida.

A regular sequence of designs generated by a probability density function is found to be useful in discrete time point design of a continuous time series. This paper generalizes Sacks and Ylvisaker's results in *Ann. Math. Statist.* [I (1966) page 66, II (1968) page 49, III (1970) page 2057] to two possible regression models $y(t) = \sum_{j=1}^n \alpha_j f_j(t) + N(t)$ and $y(t) = \sum_{j=1}^m \beta_j f_j(t) + N(t)$ $t \in [0, 1]$, where $N(t)$ is a stationary time series with mean function zero and a known covariance function. It can be shown that there exists a probability function h^* defined on [0, 1] such that the regular sequence of designs generated by h^* is asymptotically optimal for estimating both α's and β's. A nonlinear programming method of finding h^* is also obtained. (Received February 10, 1971.)

129-14. Convergence of sequences of regular functionals of empirical distributions to processes of Brownian motion. P. K. SEN, University of North Carolina.

For partial cumulative sums of independent and identically distributed random variables with zero mean and a finite (positive) variance, weak convergence to Brownian motion processes has been established by Donsker [see Billingsley: *Convergence of Probability Measures*, Wiley, New York (1968) 68]. The result is extended here to differentiable statistical functions of von Mises [*Ann. Math.*

Statist. (1947) **18** 309–348] and U-statistics of Hoeffding [*Ann. Math. Statist.* (1948) **17** 293–325], where the regular functional is assumed to be stationary of order zero. Few applications are briefly sketched. (Received February 11, 1971.)

129-15. Weak laws of large numbers in normed linear spaces. ROBERT LEE TAYLOR, University of South Carolina.

Some of the weak laws of large numbers for random variables can be extended to random elements (function-valued random variables) in separable normed linear spaces. For identically distributed random elements $\{V_n\}$ such that the Pettis integral EV_1 exists and $E\|V_1\| < \infty$, $\|n^{-1}\sum_{k=1}^n V_k - EV_1\| \to 0$ in probability if and only if $|n^{-1}\sum_{k=1}^n f(V_k) - Ef(V_1)| \to 0$ in probability for each continuous linear functional f. This weak law of large numbers can also be obtained by assuming only that the weak law of large numbers holds in each coordinate of a Schauder basis for the space. The condition of identically distributed random elements $\{V_n\}$ can not be relaxed by just assuming a bound on the moments of $\{\|V_n\|\}$, but these results can be extended to a class of random elements which need not be identically distributed. (Received February 11, 1971.)

129-16. A random-β extension of the Gauss–Markoff theorem (preliminary report). DAVID B. DUNCAN AND SUSAN D. HORN, The Johns Hopkins University.

A wide-sense random-β regression theory is presented based on the model $y - X\beta = \varepsilon \sim WS(0, \Sigma)$, (i.e., ε has any distribution with mean 0 and variance Σ) where X is known, etc., as usual but β is random and its prior mean $\mu = E(\beta)$ is included as a known subvector in y. Given this model, several natural extensions of fixed β regression theory follow including (i) $(b-\beta) \sim WS(0, V)$ where $b = Vg = (X'\Sigma^{-1}X)^{-1}X'\Sigma^{-1}y$ and (ii) a new Gauss–Markoff result: any vector estimator $\hat{\gamma}$ of any linear function $\gamma = C\beta$ of β is the unique MMSLE (minimum mean square linear estimator) of γ if (a) $E(\hat{\gamma} - \gamma) = 0$ and (b) $\hat{\gamma}$ is linear $(= Mg)$ in g. The stronger result: that $\hat{\gamma}$ is not only MVLU (minimum variance linear unbiased) but is also MMSL, follows from the natural inclusion of μ as well as the usual response observations in y. Application: it is indicated how the versatile linear dynamic recursive multivariate time series models of Kalman [ASME transactions, Part D, *J. Basic Engineering*) **82** (1960) 35–45], can be expressed in $y = X\beta + \varepsilon$ form and how Kalman recursive MMSL estimators may be derived by a new approach more natural to regression theory. (Received February 12, 1971).

129-17. Orthogonal least-squares estimation (preliminary report). PETER O. ANDERSON, The Ohio State University.

In certain cases one may be interested in the line which minimizes the sum of squared orthogonal distances from a set of bivariate data. The ideas involved are closely related to those of principal component analysis of a pair of variables. Our interest centers on the estimator, $\hat{\theta}$, of the "angle of rotation of the data," where $\tan \hat{\theta}$ is the slope of the orthogonal least-squares line. Asymptotic normality of $\hat{\theta}$ follows under quite general conditions. If bivariate normal data is assumed, $\hat{\theta}$ is the maximum likelihood estimator of θ, and the asymptotic variance of $\hat{\theta}$ is independent of θ. A Monte Carlo study shows a good approximation for small sample sizes. (Received February 12, 1971.)

129-18. Order of dependence in a normally distributed two-way series. P. K. BHATTACHARYA, University of Arizona.

In a normally distributed two-way series $\{x_{st}, s = 1, \cdots, S, t = 1, \cdots, T\}$ the order of dependence is defined to be the smallest $q \geqq 0$ so that the conditional distribution of x_{st} given $\{x_{s't'}, s' = 1, \cdots, S, t' = 1, \cdots, t-1\}$ depends only on $\{x_{s't'}, (s'-s)^2+(t'-t)^2 \leqq q^2\}$ and the conditional distribution of $U_{st} = x_{st} - E(x_{st}x_{s't'}, s' = 1, \cdots, S, t' = 1, \cdots, t-1)$ given $U_{s-1't}, \cdots, U_{1t}$ depends only on $U_{s-1,t}, \cdots, U_{s-[q],t}$. Under certain stationarity and symmetry conditions the joint density function of $\{x_{st}\}$, when S and T are large in comparison to the order of dependence, reduces to a simple form with a small number of parameters after some adjustments involving only those x_{st} for which at least one subscript is very small or very large. In this form a central role is played by the Kronecker products of some matrices derived from the $S \times S$ and $T \times T$ circulants that have been used by Anderson (*Proc. Symp. Time Series Analysis* (1962) 425–446) in his treatment of order of dependence in a Gaussian time series $\{x_t, t = 1, \cdots, T\}$. Maximum likelihood estimates of the parameters and likelihood ratio test criteria for certain hypotheses on order of dependence are derived. (Received February 12, 1971.)

129-19. The moment generating function of a bivariate noncentral chi-square distribution. MARAKATHA KRISHNAN, Temple University.

Let $X_j = (X_{1j}, X_{2j})$, $j = 1, \cdots, n$, be a set of n independent Gaussian two dimensional vectors with $E(X_j) = \mu_j = (\mu_{1j}, \mu_{2j})$, and a common covariance matrix $\Sigma = [\begin{smallmatrix}\sigma_1{}^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2{}^2\end{smallmatrix}]$. Let $P = \sum_{j=1}^n X_{1j}^2/\sigma_1{}^2$, $Q = \sum_{j=1}^n X_{2j}^2/\sigma_2^2$. Then the joint distribution of (P, Q) is the bivariate noncentral (biased) chi-square distribution with n degrees of freedom, correlation coefficient ρ, and noncentrality parameter $\lambda = \sum_{j=1}^n \mu_{1j}^2/\sigma_1{}^2 = \sum_{j=1}^n \mu_{2j}^2/\sigma_2{}^2$. The moment generating function of biv $(\chi^{12} \mid n, \rho, \lambda)$ is obtained in terms of Bessel and hypergeometric functions. The MGF of $(P+Q)$ is also derived. Additive properties are considered. It is demonstrated that the sample means $(\bar{x}_1, \bar{x}_2)$ are distributed independently of the sample variances $(s_1{}^2, s_2{}^2)$; a bivariated doubly noncentral t is defined. (Received February 16, 1971.)

129-20. Minimum variance unbiased estimation for truncated distributions with unknown truncation points. S. W. Joshi, University of Texas at Austin.

The minimum variance unbiased (MVU) estimators, and their variances, are obtained for functions of truncation points of discrete and continuous distributions truncated below and/or above assuming that there is no other unknown parameter. For the univariate power series distributions with the probability function of the form $a(x)\theta^x/f(\theta)$, $x = 0, 1, 2, \cdots$, truncated below at α, MVU estimators are obtained for functions of α and θ from the joint distribution of $X_{(1)}$ and ΣX_i. The results are applied to the geometric distribution with the probability function $q^{x-\alpha}p$, $x = \alpha, \alpha+1, \cdots$, with unknown α and p. (Received February 16, 1971.)

129-21. On criteria for choosing a regression equation for prediction in multivariate normal distributions. Stanley L. Sclove, Carnegie–Mellon University.

We have n observations on a multivariate normal random vector $(Y, X_1, X_2, \cdots, X_k)$. Later we shall be given additional observations on the X's and asked to predict the corresponding values of Y. Now our task is to use the data at hand to choose a predicting function for Y; this is to be a function of one, several, or all of the X's. This prediction problem is cast into a formal framework for the purpose of examining the reasonableness of the residual-mean-square criterion for the "best" regression equation. The mean squared error of prediction, where the mean is taken over the distribution of the n original observations as well as the additional ones, is not simply the residual mean square, MS(Res), but rather $[(n^2-n-2)/n(n-p-2)]$ MS(Res), which involves the number p of X's included in the regression equation. This suggests criteria other than minimum estimated residual mean square for choosing a regression equation for prediction. (Received February 16, 1971.)

129-22. Admissible estimators of θ^r in some extreme value densities. R. Singh, University of Saskatchewan.

Let X have extreme value density $f(x;\theta) = n\theta^{-n}x^{n-1}$, $0 \leqq x \leqq \theta$ and 0 otherwise. If the loss L_0 for taking T as an estimator of θ^r is $[(T-\theta^r)/\theta^r]^2$ then it is shown that $T(X) = [(n+2r)/(n+r)]X^r$ is an admissible minimax estimator of θ^r, $r > -n/2$. As a corollary to this we have the following: *If $X_1, \cdots, X_n$ is a random sample from a uniform* $(0, \theta)$ *density then* $[(n+2r)/(n+r)][\max(X_1, \cdots, X_n)]^r$, $r > -n/2$, *is an admissible minimax estimator of θ^r with respect to the loss L_0.* Similar results are proved for the extreme value densities $f(y;\lambda) = n\lambda^n y^{-n-1}$, $y \geqq \lambda$ and $=0$ otherwise; $f(z;\mu) = e^{-(z-\mu)}$, $z \geqq \mu$ and $=0$ otherwise. (Received February 16, 1971.)

129-23. Existence of a solution of a stochastic discrete Fredholm equation. CHRIS P. TSOKOS AND W. J. PADGETT, Virginia Polytechnic Institute and State University and University of South Carolina.

A stochastic discrete equation of the Fredholm type of the form $x_n(\omega) = h_n(\omega)+\sum_{j=1}^{\infty} C_{n,j}(\omega)f_j(x_j(\omega))$, $n = 1, 2, \cdots$, is studied, where $\omega \in \Omega$, the supporting set of a complete probability measure space (Ω, A, P). The Banach spaces of random functions used are defined, and a general theorem is proven that gives conditions under which a unique random solution of the equation exists. A random solution is defined to be a discrete parameter second order stochastic process $x_n(\omega)$, $n = 1, 2, \cdots$, which satisfies the equation P—almost everywhere. Several useful special cases of the main theorem are also given. (Received February 16, 1971.)

129-24. Random solution of a stochastic integral equation: almost sure and mean square convergence of successive approximations. W. J. PADGETT AND CHRIS P. TSOKOS, University of South Carolina and Virginia Polytechnic Institute and State University.

A nonlinear random integral equation of the Volterra type of the form $x(t; \omega) = h(t; \omega)+\int_0^t k(t, \Upsilon; \omega)f(\Upsilon, x(\Upsilon; \omega))\, d\Upsilon$, $t \geqq 0$, was considered by Tsokos (*Math. Systems Theory* **3** (1969) 222–231), where $\omega \in \Omega$, the underlying set of the probability measure space (Ω, A, P). He was concerned with the existence of a unique random solution to this equation, where a random solution is defined to be a second order stochastic process, $x(t; \omega)$, which satisfies the equation almost surely. The present paper shows that a sequence of successive approximations, $x_n(t; \omega)$, converges to the unique random solution at each $t \geqq 0$ with probability one and in mean square under the conditions of Tsokos' theorem. The rate of convergence of the successive approximations and a bound on the mean square error of approximation are also given. (Received February 16, 1971.)

129-25. Random perturbation of a stochastic integral equation of the Fredholm type. J. SUSAN MILTON AND CHRIS P. TSOKOS, Radford College and Virginia Polytechnic Institute and State University.

A study of a random or stochastic integral equation of the Fredholm type of the form $x(t; \omega) = h(t, x(t; \omega))+\int_0^{\infty} k_0(t, \tau; \omega)e(\tau, x(\tau; \omega))d\tau$, $t \geqq 0$ is presented where $\omega \in \Omega$ the supporting set of the probability measure space (Ω, A, μ). The existence and uniqueness of a random solution of the equation is considered by first investigating a stochastic integral equation of the mixed Volterra–Fredholm type of the form $x(t; \omega) = h(t, x(t; \omega))+\int_0^t k(t, \tau; \omega)f(\tau, x(\tau; \omega))d\tau$; $x(t; \omega) = h(t, x(t; \omega))+\int_0^{\infty} k_0(t, \tau; \omega)e(\tau, x(\tau; \omega))d\tau$, $t \geqq 0$. A random solution, $x(t; \omega)$ of an equation such as those above is defined to be a random function which satisfies the equation μ-a.e. Several theorems and special cases are presented which give conditions such that a random solution exists for each type of equation. (Received February 16, 1971.)

129-26. The effect of truncation on the size of the t-test. C. SCOTT AND A. K. EHSANES SALEH, Ottawa Post Office Department and Carleton University.

Student's t-statistic is often used to test the deviation of a sample mean from a hypothetical value of the population mean without regard to whether or not the underlying population is normal. The distribution of t for samples from non-normal populations has been studied by many workers including Rider (1929), Chung (1946), Bradley (1952), Hotelling (1961), Ali (1969). We examine the change in the size of the t-test when sampling from the symetrically truncated normal distribution and from the rectangular distribution. Values of the t-distribution for these nonnormal distributions have been computed for small samples using the Legendre–Gauss quadrature product formula to evaluate a multiple integral and for larger samples using an asymptotic expansion. For the normal distribution symmetrically truncated at least 2 standard deviations from the mean the size is not significantly affected. For truncation closer than 2 standard deviations to the mean and for the rectangular distribution the size of the .025 level test is not changed significantly for samples larger than 10. However, if the test is at the .005 level or smaller the size is affected significantly for samples as large as 30. We show that the distribution of t for the truncated normal distribution approaches the distribution of t for the rectangular distribution as the truncation point tends to the mean. (Received February 17, 1971.)

129-27. On the Bahadur representation of sample quantiles in some stationary multivariate autoregressive process. KALYAN DUTTA AND PRANAB KUMAR SEN, The University of North Carolina at Chapel Hill.

It is shown that Bahadur's (*Ann. Math. Statist.* (1966) **37** 577–580) almost sure (a.s.) asymptotic representation of a sample quantile for independent and identically distributed random variables holds under certain regularity conditions for a general class of stationary multivariate autoregressive processes. This yields the asymptotic (multi-) normality of the standardized forms of quantiles in autoregressive processes. Other useful applications will be considered in a subsequent paper. (Received February 17, 1971.)

129-28. Testing the independence of the components of a Gaussian complex matrix variate. A. K. GUPTA, University of Arizona.

Let $X_1 : p \times n$ and $X_2 : p \times n$ be real matrix random variables with covariance matrices $\sum_1 : p \times p$, a real symmetric positive definite matrix, and $\sum_2 : p \times p$, a real skew-symmetric matrix, respectively. Then Goodman [*Ann. Math. Statist.* **34** 152–176] has derived the complex Gaussian distribution of the complex matrix variate $Z = X_1 + iX_2$, ($i = \sqrt{-1}$), with covariance matrix $\sum = \sum_1 + i \sum_2$ which is hermitian positive definite. Khatri [*Ann. Math. Statist.* **36** 115–119] has derived the likelihood-ratio test for testing the independence of X_1 and X_2 sets of variates (i.e. testing the hypothesis $\sum_2 = 0$). In this paper we derive the distribution of the criterion, $U_{p,n,q}$, when the hypothesis is true, for $p = 4, 5, 6$ and 7 and prove:

THEOREM. *The distribution of* $U_{p,n,q}$ *is same as that of* $U_{p,n+t,q+t}$ *for all* $t \geqq 0$. Tables of correction factors for converting chi-square percentiles of a logarithmic function of $U_{p,n,q}$ are obtained for $p = 4$ and 5 and for selected values of $n-q$. The general form of the distribution for any p is also given. (Received February 17, 1971.)

129-29. On structural properties of a multivariate θ-generalized distribution. I. R. GOODMAN AND SAMUEL KOTZ, Temple University.

The authors discuss a general multivariate version of M. Subbotin's distribution (*Mathematicheskii Sbornik* **31** (1923)) given by the df of the form $C \cdot \exp |(x-\alpha)/\beta|^{\theta}$, which was more recently utilized by G. E. P. Box and G. C. Tiao [see, e.g. *Biometrika* **49** (1962)] as a prior distribution for Bayesian analyses. A canonical representation of the distribution and its characteristic functions are determined. Using the method of mixtures the marginals of the distribution and various other characteristics are explicitly derived. Regression models based on this class of distributions are investigated. (Received February 17, 1971.)

129-30. Bulk queueing with limited waiting space (preliminary report). TAPAN P. BAGCHI AND J. G. C. TEMPLETON, University of Toronto.

This paper considers a specific queueing phenomenon characterized by customers arriving in groups of random size, and by random sized batches served by a single server. The waiting line system is assumed to have a limited capacity, designated by a fixed maximum number of customers. Imbedded Markov chain formulations are obtained in two important cases, the $M^X/G^Y/1$, K and the $GI^X/M^Y/1$, K queueing systems. Integral equations in complex variables are derived for these cases, generalizing the integral equation of Cohen (1969) for the $M/G/1$, K queue. On the assumption of stationarity of input and service distributions, these equations are solved to yield analytic transient and stationary results. An outline of a new numerical approach in queueing theory is also presented which is based on the concept of sweeping probabilities in certain subspaces, rendering any specified form of non-stationarities in arrivals or services tractable. (Received February 18, 1971.)

129-31. Expanded Bayes decision making. M. C. CHEW, JR., A. RAO AND M. C. ZERHOUNI, Rensselaer Polytechnic Institute.

Most Bayesian models presented in the literature contain the underlying premise that the individual decision maker must have available a point estimate for the probability distribution of the states of nature. This a priori estimate of the distribution is obtained by experimentation or, when that is not permissible, by a subjective analysis of the problem. The decision maker may then study the sensitivity of the decision to errors in the a priori estimate. This paper presents an alternative method for the analysis of Bayes decisions. The decision maker need only specify a convex set of a priori probabilities. Mathematical programming techniques can then be used to identify strategies yielding the maximum and the minimum Bayes risks over the entire feasible set of a priori probabilities. The validity of using such a technique is discussed and illustrations given. (Received February 18, 1971.)

129-32. On the solution of linear stochastic pursuit-evasion games. WILLIAM G. NICHOLS AND CHRIS P. TSOKOS, Virginia Polytechnic Institute and State University.

This paper deals with pursuit-evasion games which are governed by linear stochastic differential games. Instead of the usual approach of adding Gaussian white process noise and/or Gaussian white noise to the observations, we make the system itself stochastic. Consider a pursuit-evasion game described by the following linear stochastic differential equation: (1) $d/(dt) \times (t; \omega) = A(\omega) \times (t; \omega) + Bu(t) - Cv(t)$, $t \geqq 0$ where $x(t; \omega) \varepsilon L_2(\Omega, A, \mu)$ is the unknown random vector and $A(\omega)$ is a matrix of measurable functions. (1) can easily be reduced to the stochastic integral equation: (2) $x(t; \omega) = e^{A(\omega)t} x_0(\omega) + \int_0^t e^{A(\omega)(t-\tau)} [Bu(\tau) - \in Cv(\tau)] d\tau$ where $x_0(\omega) = x(0, \omega)$ is the initial state. Necessary and sufficient conditions are given for the existence of a smallest max–min completion time for the game. (Received February 18, 1971.)

129-33. Nomograms for the design and A. R. L. of V-masks for cusum charts. AMRIT L. GOEL, Syracuse University.

Suppose that a process has a distribution $N(\mu_a \pm \delta\sigma, \sigma^2)$ with unknown $\delta(0 \leqq \delta)$ and known $\sigma^2(0 < \sigma^2)$. In order to maintain its mean at μ_a, sample averages x_j (sample size n) are obtained at regular time intervals and the sums $S_r = \sum_{j=1}^r (x_j - \mu_a)$ are plotted on a cumulative sum chart. A V-mask with half-angle ϕ and lead distance d is used to indicate changes in the mean. In this paper, we calculate the "exact" A.R.L.'s of such schemes by first using the method of Goel and Wu [*Technometrics* (1971)] and then employing Kemp's [*J. Roy. Statist. Soc. Ser. B* (1961) 249–253] relationships between two one-sided schemes and a V-mask. A nomogram based on the contours of L_a(A.R.L. when $\mu = \mu_a$) and L_r (A.R.L. when $\mu = \mu_{r_1}$ or μ_{r_2}), drawn in the d—v tan $\phi(v\sigma/n^{\frac{1}{2}}$ is the scale factor) plane is introduced and its use for obtaining n, d and tan ϕ is discussed. Nomograms are also presented for obtaining the A.R.L. curves for a wide range of values of d, tan ϕ and δ. (Received February 18, 1971.)

129-34. Minque estimation of variance and covariance components. C. RADHAKRISHNA RAO, Indian Statistical Institute.

Let us consider a linear model $Y = X\beta + U_1\xi_1 + \cdots + U_s\xi_s + U_{s+1}\xi_{s+1} + \cdots U_k\xi_k$ where X, U_i, $i = 1, \cdots, k$ are given matrices, β is a vector of unknown parameters, $\xi_1, \cdots, \xi_k$ are hypothetical vector variables such that $E(\xi_i) = 0$, $i = 1, \cdots, k$, $E(\xi_i\xi_i') = \Sigma$, $i = 1, \cdots, s(\Sigma$ is $q \times q$ matrix), $E(\xi_i\xi_i') = \sigma_i^2 I_{C_i}$, $i = s+1, \cdots, k$, Cov $(\xi_i\xi_j') = 0$ for all $i \neq j$. It may be seen that $\xi_1, \cdots, \xi_s$ are all q-vectors and ξ_i is a C_i-vector for $i = s+1, \cdots, k$. The problem is to estimate the unknowns β, Σ and σ_i^2, $i = s+1, \cdots, k$. For estimating any element of Σ or σ_i^2, we consider a quadratic function $Y'AY$ in Y such that $AX = 0$ and is unbiased. Under the condition $AX = 0$ $Y'AY$ can be written as a quadratic $\xi'B\xi = \xi'U'AU\xi$ in the hypothetical vector variable ξ defined by $\xi' = (\xi_1' \vdots \cdots \vdots \xi_k')$ where $U = (U_1 \vdots U_2 \vdots \cdots \vdots U_k)$. Let $\xi'C\xi$ be an estimator of the given parameter when the value of the hypothetical variable ξ corresponding to an observed Y is known. Then the matrix A is determined such that $||U'AU - C||$, the norm of $U'AU - C$, is a minimum subject to the condition $AX = 0$ and unbiasedness of $Y'AY$. The exact equations providing estimates of elements of Σ and σ_i^2 are obtained. The method is called MINQUE (Minimum Norm Quadratic Unbiased Estimation). The general linear model considered includes that of variance components, covariance components, heteroscedastic variances etc. as special cases. (Received February 18, 1971.)

129-35. Analyses of some mixed models for block and split-plot designs. R. P. BHARGAREA AND K. R. SHAH, OISE and University of Toronto and University of Waterloo.

In this paper we consider analysis of various models where the structure for the covariance matrix is intermediate between that of intra-class correlation form (i.e., one with equal variances and equal covariances) and completely arbitrary. These relate to Weiner process, Markoff process (and its generalisations) and others which arise naturally in experiments having split-plot structures. Some of the above covariance matrices have been found suitable in Studies of Growth-Curves, learning processes and other areas. (Received February 19, 1971.)

129-36. Minimax estimator for a cumulative distribution function. E. G. PHADIA, The Ohio State University.

Suppose a sample of size n from an unknown one-dimensional distribution function F is given and the problem is to estimate F. For the averaged squared error loss function a minimax procedure is obtained as an estimate of F which is essentially a step function. The method used is to construct a sequence of Bayes estimates $\hat{F}_k$ of F with respect to the priors which are constructed by using Dirichlet Process technique of Ferguson (Unpublished technical report, A

Bayesian analysis of some non-parametric problems (1969) Univ. of California, Los Angeles) and showing that Bayes risk of $\hat{F}_k$ converges to a constant which is the risk of the minimax estimate. (Received February 19, 1971.)

129-37. On the probabilities of moderate deviations of simple linear rank statistics. ROBERT P. CLICKNER AND J. SETHURAMAN, Florida State University.

Consider the statistic S_N (called a simple linear rank statistic) defined by $S_N = \sum_{i=1}^{n} a_N(R_i)$ where $n/N \to \lambda$, $0 < \lambda < 1$, $\sum_i a_N(i) = 0$, $\sum_i a_N{}^2(i) = N$ and $\mathbf{R}_N = (R_1, \cdots, R_N)$ is uniformly distributed over all permulations of $(1, \cdots, N)$. Under some further conditions on $a_N(1), \cdots, a_N(N)$ it is shown that, for $x > 0$, $\log P\{S_N > x[\lambda(1-\lambda) \log N]^{\frac{1}{2}}\} \sim -\frac{1}{2}x^2 \log N$. These are called probabilities of moderate deviations and were studied in some other situations by Rubin and Sethuraman [*Sankhya Ser. A* **27** (1965) 325–346]. (Received February 19, 1971.)

129-38. A test for structured covariance in the growth model. JACK C. LEE AND SEYMOUR GEISSER, SUNY at Buffalo.

The growth model is $E(Y_{p\times N}) = X_{p\times m}\tau_{m\times r}A_{r\times N}$ where X and A are known matrices of ranks $m < p$ and $r < N$ respectively; τ is unknown and the columns of Y are independent p-dimensional multinormal variates having unknown covariance matrix Σ. It was shown by Rao (Fifth Berkeley Symp. *Math. Statist. Prob.* (1967)) that $T_1 = (X'X)^{-1}X'YA'(AA')^{-1}$ is the least squares estimate of τ if and only if $\Sigma = X\Gamma X' + Z\theta Z' + \sigma^2 I$ where Γ, θ and σ^2 are arbitrary and Z is $p \times p - m$ such that $X'Z = 0$. Geisser (*Sankhyā Ser. A* **32** (1970)) noted that there is no loss in generality in taking $\Sigma = X\Gamma X' + Z\theta Z'$. In this paper a testing procedure for testing procedure for testing, $H_0 : \Sigma = X\Gamma X' + Z\theta Z'$ where Γ, θ are unspecified versus $H_1 : \Sigma$ arbitrary is obtained. The testing problem is shown to be equivalent to testing the independence of two sets of transformed variates. The test statistic for this problem is given by Anderson (*An Introduction to Multivariate Analysis* 241–243). Adapting this to the growth model leads to a $U_{m,p-m,N-(p-m)-1}$ distribution for the test statistic under H_0. (Received February 22, 1971.)

129-39. The curve décolletage and error-threshold loss in Bayesian estimation and testing for normal location parameters. JAMES M. DICKEY, State University of New York at Buffalo.

For estimation with an error-threshold loss function, the posterior expected utility is essentially approximately proportional to a posterior density ordinate. The posterior mode is an approximately optimal estimate. Given normal-linear-model data, a realistic family of posterior distributions for the unknown coefficients has densities proportional to products of two multivariate t densities (possibly bimodal). The curve *décolletage*, parameterized by λ, consists of the well-known closed-form minima of $Q_1 + \lambda Q_2$ where Q_1 and Q_2 are the quadratic forms in the two multivariate t densities. The posterior modes lie on the curve décolletage, and can easily be found by a one-dimensional search. If an additional lump of utility (a reward for simplicity) is attached to a "null" linear manifold, a theory of "tests" follows, yielding one Bayesian alternative to the F test. (Received February 22, 1971.)

129-40. Central limit theorems for subordinated processes. RICHARD SERFOZO, Syracuse University.

A process $\{X_t\}$ *is subordinated to* $\{Y_t\}$ if $X_t = Y(\tau_t)$ for some non-decreasing nonnegative real valued process $\{\tau_t\}$. Central limit theorems for $\{X_t\}$ are presented for the cases when (i) there exist constants $A > 0$ and B such that $(Y_t - At)/(Bt^{\frac{1}{2}})$ converges in distribution to a standardized normal random variable, and (ii) the process $\{Y_t\}$ has stationary independent increments and Y_1 belongs to the domain of attraction of a stable law. That is, conditions on $\{\tau_t\}$ are presented under which there exist constants a_t and b_t such that $a_t^{-1}X_t - b_t$ converges in distribution to some random variable as t tends to infinity. It is also shown that $\{X_t\}$ obeys a functional central limit theorem when both $\{Y_t\}$ and $\{\tau_t\}$ do. (Received February 22, 1971.)

129-41. On characterization of the exponential and related distributions. R. C. SRIVASTAVA AND Y. H. WANG, The Ohio State University.

Let $X_1, \cdots, X_n, n \geqq 2$ be a random sample from an exponential distribution with density $f(x, \theta, \sigma) = \{\sigma^{-1} \exp\{-\sigma^{-1}(x-\theta)\}\}$, $x \geqq \theta$, $\sigma > 0$; $f(x, \theta, \sigma) = 0$ elsewhere, and also let $Y_1 < \cdots < Y_n$ be the corresponding order statistics. Write $Z = \sum_{i=1}^{n} (Y_i - Y_1)$. Then it is well known that $E(Z \mid Y_1) = c$. In this paper, a characterization of the exponential distribution based on this property is obtained. It may be remarked here that Tanis (*Ann. Math. Statist.* **35** (1964) 220–236) obtained a characterization of the exponential distribution based on the independence of Z and Y_1. A shorter and simpler proof of Tanis' result is also given. Similar characterizations of the geometric, Pareto and distributions whose densities are of the form Ax^B are obtained. Finally the problem of characterizing distributions based on linear regression of Y_k on Y_m is considered. (Received February 22, 1971.)

129-42. A perturbed system of nonlinear differential equations related to the random behavior of cuscade processes. SIDNEY W. HINKLEY AND CHRIS P. TSOKOS. Virginia Polytechnic Institute and State University.

The paper is concerned with the existence and asymptotic character of the nonlinear boundary value problem (1) $dG/dt = f(t, G, F, |\alpha-\beta|)$ $|\alpha-\beta|\, dF/dt = g(t, G, F, |\alpha-\beta|)$; (2) $G(0, |\alpha-\beta|) = k_1$ $G(\infty, |\alpha-\beta|) = k_2$ as $|\alpha-\beta| \to 0+$. The discussion is related to the problem of particle-number fluctuations in the theory of cosmic radiation and G and F denote respectively the probability generating functions for the electron distribution in an electron-initiated and a photon-initiated shower. A solution of the system (1) satisfying the boundary conditions (2) is constructed so that specified limiting conditions are fulfilled. (Received February 25, 1971.)

129-43. On estimation of tail end probabilities of the sample mean for linear stochastic processes. KAMAL C. CHANDA, University of Florida, Gainesville.

Let $\{W_j^*; -\infty < j < \infty\}$ be a doubly infinite sequence of independent and identically distributed (i.i.d.) random variables (rv) which possess a moment generating function (m.g.f.) $M_{W^*}(t)$ over an open interval $\Omega = (-\omega, \omega)(\omega > 0)$ with $E\{W_i^*\} = 0$. Let $\{X_j; j \geqq 1\}$ be another sequence of rv's defined by $X_j = \sum g_v W_{j-v}^*, \sum_0^\infty g_v \neq 0$ (so far as the present investigation is concerned we can take $\sum_0^\infty g_v = 1$ without any loss of generality). We assume that there exists a positive constant $\rho_0 < 1$ such that $|g_v| < C\rho_0{}^v\ v \geqq 0$, C being a finite positive constant. Let $\Omega_0 = (-\omega A^{-1}, \omega A^{-1})$ where $A = \sum_{v=0}^\infty |g_v| > 1$. Further assume that $a > 0$ is such that if $M_W(t) = \exp(-at) M_{W^*}(t)$ then there exists a constant $\tau < \omega A^{-1}$ such that $\rho = M_W(t) = \inf_{\tau \in \Omega} M_W(t) < 1$ and $M_W'(\tau) = 0$. Then it is proved that $\tau > 0$ and for each $u = 0, 1, 2, \cdots$ there exists a bounded sequence of constants, $\beta_{u,1}, \beta_{u,2}, \ldots$, such that for any positive integer $r \geqq 3$, $P(\sum_{j=1}^n X_j/n \geqq a) = (\rho^n \lambda_n / \tau\sigma_n(2\pi)^{\frac{1}{2}}[\sum_{u=0}^{r-3} \beta_{u,n}/\sigma_n{}^u + O(\sigma_n{}^{-(r-2)})]$ where $\lambda_n > 0$, is a uniformly bounded function of n and τ and σ_n is a specified function of n and τ with the property that $\sigma_n > 0$ and $n^{-1}\sigma_n{}^2 \to$ a finite positive constant as $n \to \infty$. (Received March 4, 1971.)

(*Abstracts of papers presented at the Central Regional meeting, Columbia, Missouri, May* 5–7, 1971. *Additional abstracts appeared in earlier issues.*)

130-4. An inadmissible best invariant estimate of a two-dimensional location parameter. MARTIN FOX, Michigan State University.

Let Y be a real-valued random variable with density $g(y) = Ky^{-(5-\eta)}$ if $y > 1$, $= 0$ if $y \leqq 1$. Let X be a random variable taking values in two-space. Let the conditional distribution of X given $Y = y$ be uniform on the circle of radius y with center at θ. We wish to estimate θ with loss $||t||^2$ when t is the error. The best invariant estimate, X, is inadmissible. A better estimate is $X + Y\eta(X/Y)$ where $\eta(t) = -\varepsilon\delta(1-\varepsilon^2||t||^2)t$ if $||t|| < 1/\varepsilon$, $= 0$ otherwise. In this example, $E||X||^\alpha < \infty$ if, and only if, $\alpha < 4-\eta$. All other conditions in an unpublished paper of Brown are met. His moment condition in this case would be $E||X||^4 < \infty$. This example parallels Perng's (*Ann. Math. Statist.* **41** (1970) 1311–1321) which deals with the case of X and θ real. (Received February 12, 1971.)

130-5. Signed dependence. RICHARD DYKSTRA, WILLIAM A. THOMPSON AND JOHN E. HEWETT, University of Missouri.

Often, the probability of the simultaneous occurrence of dependent events can be well approximated by assuming them to be independent. Here, we discuss bounds on the error in using this procedure and conditions when under (or over) estimates occur. Bivariate dependence concepts treated by Lehmann are generalized to the multivariate case in such a way that relations valid for the bivariate case continue to hold. An inequality involving expectations of conditionally independent random variables is proven. Applications establish conservative bounds for limiting distributions of exchangeable random variables and error probabilities for simultaneous inference. (Received February 25, 1971.)

130-6. Type-I error and power of a test involving a Satterthwaite's approximate *F*-statistic. JAMES M. DAVENPORT AND J. T. WEBSTER, Southern Methodist University.

The problem of testing that the linear relation $\theta_3 = \theta_1 + \theta_2$ holds among the variances $\theta_i (i = 1, 2, 3)$ is considered by using the Satterthwaite approximate F-statistic $F_1' = v_3/(v_1+v_2)$, where the v_i are independent random variables such that $n_i v_i/\theta_i$ is distributed as a Chi-square variate with n_i degrees of freedom. The true type-I error (or average performance) and power function of the test based on Satterthwaite's criterion are calculated as functions of a nuisance parameter, $U = \theta_1/\theta_2$, n_1, n_2, n_3, and $\phi = \theta_1/(\theta_2+\theta_3)$ by a procedure given by Cochran (*Biometrics* **7** (1951) 17–32). The cases of interest are those combinations of n_1, n_2, and n_3 where the true type-I error is considered "close" enough to the nominal α for all possible values of the nuisance parameter U. Such "acceptable" cases are given and were found to be related to $|n_i - n_2|$. (Received March 1, 1971.)

130-7. The estimation of the probability of extinction and other parameters associated with branching processes. STEPHEN M. STIGLER, University of Wisconsin.

Maximum likelihood estimates of the probability of the eventual extinction of a (possibly age-dependent) branching process are given for both the parametric and nonparametric cases, and their asymptotic properties derived. It is found that the nonparametric estimate performs very well relative to the parametric estimate in a large variety of situations, and therefore would often be preferable to the parametric estimate in view of the latter's lack of robustness. (Received March 1, 1971.)

130-8. The $2^3 \times 3$ factorial design in three blocks. PETER W. M. JOHN, University of Kentucky.

Confounding in the $2^3 \times 3$ factorial with several replicates has been extensively discussed. In this paper a method is presented for dividing a single replicate into three blocks of eight points each. All the main effects and two factor interactions are estimable by intrablock contrasts, when the higher order interactions are suppressed. The individual blocks may also be considered as fractional factorials. They are shown to compare favorably for estimating main effects with the eight point fraction suggested by Webb. (Received March 2, 1971.)

130-9. Components of variance estimation for a split plot design. JEROME H. KLOTZ AND SOUTHWARD G. MORRIS, University of Wisconsin.

For a balanced two way layout split plot design let $X_{ijk} = \mu_{ij} + B_k + D_{ik} + e_{ijk}$, $i = 1, 2, \cdots I$, $j = 1, 2, \cdots J$, $k = 1, 2, \cdots K$, where $\mu_{ij} = \mu + \alpha_i + \beta_j + \gamma_{ij}$ ($\alpha\cdot = \beta\cdot = \gamma_{i\cdot} = \gamma_{\cdot j} = 0$) are fixed, and B_k, D_{ik}, e_{ijk} are random independent $N(0, \sigma_B^2)$, $N(0, \sigma_D^2)$, $N(0, \sigma_e^2)$. Restricted maximum likelihood estimators (Thompson, *Ann. Math. Statist.* (1962)) $\tilde{\sigma}_B^2, \tilde{\sigma}_D^2, \tilde{\sigma}_e^2$ are derived along with the maximum likelihood estimators $\hat{\sigma}_B^2, \hat{\sigma}_D^2, \hat{\sigma}_e^2$. For fixed K, and I, J large a lack of consistency for the maximum likelihood estimators is noted. On the other hand, the restricted maximum likelihood estimators are consistent and avoid problems of negative estimates of variance. We have $\tilde{\sigma}_e^2 = \min(S_e^2/(I(J-1)(K-1)), (S_e^2+S_D^2)/(IJ-1)(K-1), (S_e^2+S_D^2+S_B^2)/IJ(K-1))$ where $S_e^2 = \sum_i\sum_j\sum_k(X_{ijk} - X_{ij\cdot} - X_{i\cdot\cdot})^2$, $S_D^2 = J\sum_i\sum_k(X_{i\cdot k} - X_{i\cdot\cdot} - X_{\cdot\cdot k} + X_{\cdot\cdot\cdot})^2$, $S_B^2 = IJ\sum_k(X_{\cdot\cdot k} - X_{\cdot\cdot\cdot})^2$ and using methods similar to those of Stein (Klotz, Miton and Zacks *J. Amer. Statist. Assoc.* (1969)) it can be shown that the mean square error of $\tilde{\sigma}_e^2$ is uniformly smaller than that of the minimum variance unbiased estimator $S_e^2/(I(J-1)(K-1))$. (Received March 2, 1971.)

130-10. Characterization of self-decomposable probability measures on Banach spaces. ARUNOD KUMAR, Michigan State University.

Self-decomposable probability measures on a real separable Banach space E, are studied using extension of the definition of Loève, *Probability Theory* page 322. The analogue of Theorem, Loève, page 323, is proved for any real separable Banach space. For the space E_α, recently studied by J. Kuelbs and V. Mandrekar (to appear) the self-decomposable laws are characterized as infinitely divisible laws in terms of their Lévy–Khinchine representation. (Received March 3, 1971.)

130-11. Bayesian single sampling acceptance plans under a monotone cost structure. P. THYREGOD, University of Minnesota.

Let $X_1, X_2, \cdots, X_N$ be i.i.d. nonnegative random variables with density $f(x \mid \omega)$ satisfying certain monotonicity conditions. After observing a sample of the fixed size n we want to make one of two decisions, acceptance or rejectance, say, regarding the remaining $N-n$ variables. Let the expected cost per unit of sampling, acceptance and rejectance be $K_s(\omega)$, $K_a(\omega)$, $K_r(\omega)$, respectively and let ω be a random variable with distribution $W(\omega)$. A sampling plan is characterized by the sample size and the acceptance number, (n, c), with the rule that the lot is accepted if $\Sigma X_i \leqq c$. A sampling plan is optimal if it minimizes the expected cost. Assuming that $K_r(\omega)-K_a(\omega)$ is monotone we find that the optimal sample size and acceptance numbers are monotone functions of N. It is shown that the binomial, Poisson and exponential densities satisfy the conditions and it is indicated how the results may be used to tabulate the optimal sampling plans. (Received March 3, 1971.)

130-12. Optimum design of three way layouts without interactions. P. S. DWYER AND V. KUROTSCHKA, University of Michigan.

In a general three way layout without interactions: $Y_{ijk}l = a_i+b_j+c_k+Z_{ijk}l$, $i = 1, \cdots, I; j = 1, \cdots, J; k = 1, \cdots, K; l = 0, 1, \cdots, n_{ijk}$ a particular choice of the system $\{n_{ijk}\}$ of numbers of observations at the levels (i, j, k) of the three factors represent a specific (nonrandomized) design of the three way layout. Classes of uniformly optimal designs under different conditions on the availability of observations, i.e. in different sets of admitted designs, are characterized. In cases where uniformly optimal designs do not exist, D- and A-optimal designs are given. A particular result for instance is the following: If the numbers of observations are restricted by the systems $\{n_{i..}\}$, $\{n_{.j.}\}$, $\{n_{..k}\}$ of the total numbers of observations at each level of every factor, then a design concerned with inference about the first factor is uniformly optimal if and only if it satisfies the conditions $\{n_{ij.} = n_{i..}n_{.j.}/n\}$ and $\{n_{i.k} = n_{i..}n_{..k}/n\}$ for all i and j. The obtained results establish in particular optimality statements concerning incomplete blockdesigns such as presented by J. Kiefer in [*Ann. Math. Statist.* **29** (1958) 675–699] and extensions thereof which are given by V. Kurotschka [Symp. on Symmetric Functions in Statistics, Windsor, Ont., 1971]. (Received March 4, 1971.)

130-13. A generalization of Mercer's theorem and applications to reproducing kernels. WILLIAM TUCKER, Southern Methodist University.

A generalization of Mercer's Theorem (see Parzen, E., *Proc. Fourth Berkeley Symp. Math. Statist. Prob.* **1** 469–489) to unbounded sets in the real lines is obtained. This allows one to consider, with certain restrictions, prediction on infinite or semi-infinite intervals and employ the expansion afforded by Mercer's Theorem. Also the expansion is employed in obtaining iterative evaluations of reproducing kernel inner products. (Received March 4, 1971.)

130-14. A characterization of the distribution of three independent random variables with values in a locally compact Abelian group. PETER FLUSSER, Oklahoma State University.

For $k = 0, 1, 2$, let X_k be a random variable with values in the locally compact, second countable, Hausdorff Abelian group G_k. Let $G = G_0 \oplus G_1 \oplus G_2$, and let T

be a continuous homomorphism from G onto the locally compact, second countable, Hausdorff Abelian group H, satisfying certain conditions, among them that $T|_{G_1 \oplus G_2}$ be an isomorphism. Setting $X = [X_0, X_1, X_2]$ and $Y = T(X) = [Y_1, Y_2]$, it is shown that the joint distribution of $[Y_1, Y_2]$ determines the distributions of X_0, X_1 and X_2 up to a shift which depends on one parameter only. This theorem is a generalization of certain results of I. I. Kotlarski's [*Sankhyā* (1971)] and B. L. S. Prakasa Rao's [*Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* (1968)]. Other applications are also given. (Received March 4, 1971.)

130-15. On Strassen's version of the law of the iterated logarithm for Gaussian processes. HIROSHI OODAIRA, University of Minnesota and Yokohama National University.

Let $\{X(t, \omega), 0 \leqq t < \infty\}$ be a sample continuous Gaussian process with $X(0) \equiv 0$, mean zero and covariance $R(s, t)$. Put $\sigma^2(t) = R(t, t)$ and assume $\sigma^2(1) = 1$. Define, for each ω, a sequence of functions $\{f_n(t, \omega), n \geqq 3\}$ in $C[0, 1]$ by $f_n(t, \omega) = X(nt, \omega)/(2\sigma^2(n) \log \log n)^{\frac{1}{2}}$. Under some conditions on R it is shown that, for almost every ω, the set of limit points of $\{f_n(t, \omega)\}$ coincides with the unit ball of the reproducing kernel Hilbert space $H(R)$ with kernel $R(s, t)$, $0 \leqq s, t \leqq 1$. This generalizes a result of V. Strassen (*Z. Wahrscheinlichkeitstheorie und. Verw. Gebiete* **3** (1964) 211–226 for the Wiener process to a class of Gaussian processes. (Received March 4, 1971.)

130-16. Improving the estimate of the dispersion matrix of a multivariate normal population through the use of incomplete multivariate samples (preliminary report). J. N. SRIVASTAVA AND M. K. ZAATAR, Colorado State University.

Suppose there is a set of experimental units on each of which one or more out of a set of p responses $(1, 2, \cdots, p)$ could be measured. Let C_0 be the "cost" of one unit, and C_i the cost of measuring response i on any single unit. Suppose that the vector of the p responses on any unit has a known mean, and a not-completely-known dispersion matrix Σ. In this paper, for various values of the c's, we attempt to improve on the usual estimate of Σ by using the general incomplete multiresponse (GIM) design, under which we do not necessarily measure each response on each unit. (For similar studies on location parameters see, for example, Srivastava and McDonald, *Ann. Inst. Statist. Math.* (1969), 507–514.) We show that when (i) Σ is known except for a scalar multiple, and (ii) when $p = 2$ and Σ is unknown except for the correlation co-efficient, certain special subclasses of GIM models are optimal in the sense of maximizing the determinant of Fisher's information matrix. For $p = 2$ and Σ completely unknown, the maximum likelihood estimator of Σ and the corresponding variance matrix is obtained under certain subclasses of GIM models. Some similar studies for the above are also made for general p. (Received March 4, 1971.)

130-17. Interval estimation of functions of Bernoulli parameters with reliability applications. DAR-SHONG HWANG, University of Minnesota.

The reliability of series, parallel and other systems is expressible in terms of polynomials in Bernoulli parameters, p_i. For products of two or more parameters the Lehmann–Scheffé theory of exponential families applies when the components are sampled by the inverse binomial rule. For example if X, Y are the numbers of successes before r_1, r_2 failures on two populations of components, then the conditional distribution of X given $Y-X$ depends only on p_1p_2, the reliability of a series system. A second technique is "multiple stage compounding." It is known that if Y is Poisson, $EY = \lambda$, and if X given $Y = y$ is binomial (y, p), then X is Poisson, $EX = \lambda p$. This fact is exploited by taking for example a binomial observation X_1 from population one, where the sample size is a Poisson variate having known mean λ, and then taking a second binomial observation X_2 from population two, where the number of trials is the observed value of X_1. Then p_1p_2 can be estimated from the value of X_2, which is Poisson with mean λp_1p_2. Similar methods can be used to estimate any rational function of Bernoulli parameters. (Received March 4, 1971.)

130-18. Double sample tests for hypotheses about the mean of an exponential distribution. WILLIAM G. BULGREN AND JOHN E. HEWETT, University of Kansas and Oklahoma State University.

A double sample test is given for testing $H_0:\theta \geqq \theta_0$ vs. $H_A:\theta < \theta_0$ where θ is the unknown scale parameter of an exponential distribution. The test is applicable when the location parameter is either known or unknown if the test statistic is adjusted accordingly. Various comparisons are made between this test and the single sample test of Epstein (*Technometrics* **2** 435–446). (Received March 11, 1971.)

130-19. A characterization of the distributions of three independent random variables with values in a locally compact Abelian group. PETER FLUSSER, Oklahoma State University.

For $k = 0, 1, 2$, let X_k be a random variable with values in the locally compact, second countable, Hausdorff Abelian group G_k. Let $G = G_0 \oplus G_1 \oplus G_2$, and let T be a continuous homomorphism from G onto the locally compact, second countable, Hausdorff Abelian group H, satisfying certain conditions, among them that $T|_{G_1 \oplus G_2}$ be an isomorphism. Setting $X = [X_0, X_1, X_2]$ and $Y = T(X) = [Y_1, Y_2]$, it is shown that the joint distribution of $[Y_1, Y_2]$ determines the distributions of X_0, X_1 and X_2 up to a shift which depends on one parameter only. This theorem is a generalization of certain results of I. I. Kotlarski's [*Sankhyā*, to appear in 1971] and B. L. S. Prakasa Rao's [*Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* (1968)]. Other applications are also given. (Received March 16, 1971.)

(Abstracts of papers presented at the Annual meeting, Fort Collins, Colorado, August 23–26, 1971. *Additional abstracts will appear in future issues.)*

131-1. Two-sample ranking procedures for means of normal populations with unknown variances (preliminary report). EDWARD J. DUDEWICZ AND SIDDHARTHA R. DALAL, The University of Rochester.

Suppose an experimenter has k populations $\pi_1, \cdots, \pi_k$, where observations from π_i are normally distributed $N(\mu_i, \sigma_i^2)(1 \leqq i \leqq k)$ with $\mu_1, \cdots, \mu_k$, $\sigma_1^2, \cdots, \sigma_k^2$ all unknown. Further, suppose one has a goal (e.g. to select that population which has the largest mean $\mu_{[k]}$, where $\mu_{[1]} \leqq \cdots \leqq \mu_{[k]}$ denote the ordered means). We propose and study a two-stage Stein [*Ann. Math. Statist.* **16** (1945) 243–258] type procedure which takes an initial sample of size n_{0i} from π_i and then sets the total sample size n_i from π_i (which determines the second stage) so that $s_1^2/n_1 = \cdots = s_k^2/n_k$, where s_i^2 is the usual estimate of σ_i^2 based on n_{0i} observations ($i = 1, \cdots, k$). (A slight variant, with higher $P(CS)$ and essentially the same sample size, is recommended for actual use.) This procedure satisfies $P(CS) \geqq P^*$ whenever $\mu_{[k]} - \mu_{[k-1]} \geqq \delta^*$, and can be regarded as an analogue of Bechhofer's [*Ann. Math. Statist.* **25** (1954) 16–39] expedient of setting single-stage sample sizes so that $\sigma_1^2/n_1 = \cdots = \sigma_k^2/n_k$ when $\sigma_1^2, \cdots, \sigma_k^2$ are known. The probability integral which arises is an analogue (using k independent Student-t variates) of Bechhofer's [*loc. cit.*] basic integral (20); tables do not appear to be available, and their calculation is contemplated. This procedure solves a long-standing problem of great practical importance (e.g. see Kleijnen and Naylor [1969 *Business and Economic Statistics Section Proceedings of the American Statistical Association*, 605–615]). (Received February 18, 1971.)

131-2. A nonparametric test based on U-statistics for interaction in a two-way layout (preliminary report). ANIL P. GORE', University of Kentucky.

Assume the usual additive model, with equal number of observations per cell and that errors have a common continuous distribution F. Mehra and Sen [*Ann. Math. Statist.* **40** (1969) 658–664] and Mehra and Smith [*J. Amer. Statist. Assoc.* **65** (1970) 1283–1296] have proposed nonparametric tests for the hypothesis of no interaction. This paper offers an alternative test for the same problem. Let x_{ij} denote an observation in the (i, j)th cell (i.e. the cell in the ith row and jth column). Let $U_{i,i',j,j'}$ denote the proportion of contrasts $x_{ij} - x_{i'j} - x_{ij'} + x_{i'j'}$ that are positive, among all possible quadruplets of observations formed by taking one each from (i, j)th, (i', j)th, (i, j')th and (i', j')th cells. Let $W_{ij} = \sum_{i \neq i'} \sum_{j \neq j'} \times U_{i,i',j,j'}$. Then the test proposed rejects the hypothesis of no interaction if the statistic $T = \sum_{i=1}^{R} \sum_{j=1}^{C} (W_{ij} - (R-1)(C-1)/2)^2 / R^3 C^3 (\hat{\eta}(F) - \frac{1}{4})$ is too large, where R is the number of rows, C is the number of columns and $\hat{\eta}(F)$ is a consistent estimate of $P(Y_1 - Y_2 - Y_3 + Y_4 > 0, \;\; Y_1 - Y_5 - Y_6 + Y_7 > 0)$, where

Y_i, $i = 1, \cdots, 7$ are independent and identically distributed as F. It is shown that T has a limiting chi-square distribution with $(R-1)(C-1)$ degrees of freedom, if there is no interaction. (Received March 3, 1971.)

131-3. Restricted generalized inverse and the theory of minimum bias estimation in a Gauss–Markoff model. SUJIT KUMAR MITRA, Indian Statistical Institute, Calcutta.

This paper consists of two somewhat interconnected parts. Part 1 introduces a new class of g-inverse of matrices designated as the restricted g-inverse. Part 2 presents the theory of minimum bias estimation in a Gauss Markoff linear model. G is a restricted g-inverse of A if the solution $x = Gy$ of a consistent equation $Ax = y$ always belongs to a specified linear manifold, irrespective of y. Under n.s. conditions for existence the paper obtains the class of restricted g-inverse and the subclasses of restricted minimum norm g-inverse, restricted least squares g-inverse and restricted minimum norm least squares g-inverse. An approach to minimum bias estimation in a linear model was given in Rao and Mitra ((1970) I.S.I.-R.T.S. Tech. Report No. Math-Stat/36/70). An alternative approach would be to demand that the bias in an estimate is zero if the unknown parameter vector belongs to a specified linear manifold and to minimise the variance subject to this condition. This paper obtains n.s. conditions under which the two approaches lead to identical estimates and examines robustness of minimum bias estimators under specification errors in the linear model. The case when the dispersion matrix is singular is also discussed. An interesting byproduct is the characterization of pairs of norms leading to the same expression for Moore–Penrose inverse. (Received March 10, 1971.)

131-4. Limiting distributions of Kolmogorov–Smirnov type statistics under the alternative. M. RAGHAVACHARI, Carnegie–Mellon University.

Let $X_1, X_2, \cdots$ be a sequence of independent and identically distributed random variables with the common distribution being uniform on [0, 1]. Let $Y_1, Y_2, \cdots$ be a sequence of i.i.d. variables with continuous cdf $F(t)$ and with [0, 1] support. Let $F_n(t, \omega)$ denote the empirical distribution function based on $Y_1(\omega), \cdots, Y_n(\omega)$ and let $G_m(t, \omega)$ the empirical cdf pertaining to $X_1(\omega), \cdots, X_m(\omega)$. Let $\sup_{0 \leqq t \leqq 1} |F(t)-t| = \lambda$ and $D_n = \sup_{0 \leqq t \leqq 1} |F_n(t, \omega)-t|$. The limiting distribution of $n^{\frac{1}{2}}(D_n - \lambda)$ is obtained in this paper. The limiting distributions under the alternative of the corresponding one-sided statistic in the one-sample case and the corresponding Smirnov statistics in the two-sample case are also derived. The asymptotic distributions under the alternative of Kuiper's statistic are also obtained. (Received March 25, 1971.)

(ABSTRACTS SUBMITTED BY TITLE)

71T-31. On a class of nonparametric tests for scale and location parameters. YVES LEPAGE, Université de Montréal.

Two samples, X-sample and Y-sample of m and n independent observations from populations with continuous distribution functions, $F(x)$ and $G(y)$ respectively, are considered. A class of nonparametric test is proposed to test the hypothesis $G(x) = F(x)$ versus alternatives of the form $G(x) = F(ax+b)$ $(a > 0)$ with $a \neq 1$ or $b \neq 0$. Under the hypothesis, the exact distribution is given by a recursion formula and the asymptotic distribution are found. For a sequence of contiguous alternatives, the asymptotic distribution is given and, the asymptotically most powerful test for this sequence is found. The asymptotically maximin most powerful test for a certain class of contiguous alternatives is also found and finally, some asymptotic efficiencies relative to the maximin are evaluated. (Received February 16, 1971.)

71T-32. A two-stage procedure for estimating the difference between the mean vectors of two multivariate normal distributions. R. T. O'NEILL AND V. K. ROHATGI, The Catholic University of America.

Let $X_1, X_2, \cdots, X_n$ and $Y_1, Y_2, \cdots, Y_n$ be independent samples from two k-variate normal populations $N(\mu_i, \Sigma_i)$ $(i = 1, 2)$ where $\mu_i = (\mu_i(1), \cdots, \mu_i(k))'$ and $\Sigma_i = \sigma_i^2 I$, $\sigma_i^2 < \infty$, $i = 1, 2$ and I is the identity matrix. Estimate $\mu_1 - \mu_2$ by the unbiased estimator $\bar{X}_{n_1} - \bar{Y}_{n_2}$ using the loss function $L(n_1, n_2) = \sum_1^k |\bar{X}_{jn_1} - \bar{Y}_{jn_2} - (\mu_1(j) - \mu_2(j))|^s$ $s > 0$ where $\bar{X}_{jn_1} = n_1^{-1} \sum_1^{n_1} X_{jl}$, $\bar{Y}_{jn_2} = n_2^{-1} \sum_1^{n_2} Y_{jl}$, $j = 1, \cdots, k$, $\bar{X}_{n_1} = (\bar{X}_{1n}, \cdots, \bar{X}_{kn_1})'$, $\bar{Y}_{n_2} = (\bar{Y}_{1n_2}, \cdots, \bar{Y}_{kn_2})'$ and n_i indicate samples of size n_i from $N(\mu_i, \Sigma_i)$ respectively. Sampling is restricted by a bounded cost function $a_1 n_1 + a_2 n_2 \leqq A < \infty$. A modified version of the two-stage procedure used by Ghurye and Robbins [*Biometrika* **41** (1954) 146–152] for two univariate populations is shown to lead to an asymptotically efficient estimate of $\mu_1 - \mu_2$ in some sense. (Received February 16, 1971.)

71T-33. On sequential estimation of the mean vector of a multinormal population. V. K. ROHATGI AND R. T. O'NEILL, The Catholic University of America.

The problem of sequential point estimation of the mean vector of a k-variate normal distribution is considered. Precisely, consider the k-variate normal distribution $N(\mu, \Sigma)$ where $\mu = (\mu_1, \cdots, \mu_k)'$ and $\Sigma = \text{diag.}(\sigma_1^2, \cdots, \sigma_k^2)$, $0 < \sigma_i^2 < \infty$ and both μ, Σ unknown. The stopping rule for the point estimation of μ by $\bar{X}_n$ under the loss function $L(n) = \sum_1^k \lambda_i |\bar{X}_{in} - \mu_i|^s + n^t$, $s > 0$, $t > 0$, $\lambda_i \geqq 0$ where $\bar{X}_n = (\bar{X}_{1n}, \cdots, \bar{X}_{kn})'$ is obtained as $N = \inf\{n \geqq m \geqq 2 : n \geqq (\beta \sum_1^k \lambda_i s_{in}^s)^{2/(2t+s)}\}$ where $\beta = (s/2t)C(s)$, m is the starting sample size and $\bar{X}_{in} = n^{-1} \sum_1^n X_{ij}$, $s_{in}^s = (n-1)^{-1} \sum_1^n (X_{ij} - \bar{X}_{in})^2$. It is shown along the lines of Khan [*Sankhyā Ser. A* **30**

(1968) 331–334] that the sequential procedure defined by N is asymptotically efficient. In addition, it is shown that the regret in not knowing Σ is bounded in the sense of Starr and Woodroofe [*Proc. Nat. Acad. Sciences* **63** (1969) 285–288]. (Received February 16, 1971.)

71T-34. On a robustness problem in life testing (preliminary report). JOSEPH L. GASTWIRTH AND J. TERRY SMITH, The Johns Hopkins University and Harvard University.

In their recent paper (Technical Report No. 230, Department of Statistics, The University of Wisconsin (1970)), Bhattacharyya and Johnson illustrate the difficulty of obtaining a test which performs well for exponential scale change alternatives as well as for gamma and Weibull shape change alternatives. The present study shows that the appropriate maximin rank tests (see Gastwirth, *J. Amer. Statist. Assoc.* **61** (1966) 929–948) alleviate part of this difficulty. An asymptotic analysis corroborates the small-sample findings of Bhattacharyya and Johnson, and explains a discrepancy between their small-sample and asymptotic results. (Received February 22, 1971.)

71T-35. Two modifications of a rank test of Bhattacharyya and Johnson. HANS K. URY AND ALVIN D. WIGGINS, University of California, Berkeley and University of California, Davis.

Cronholm and Revusky (*Psychometrika* **30** (1965) 459–467) have proposed a rank test T_n based on a sequence of $n-1$ independent subexperiments for the comparison of a treatment effect with a control on a single group of n subjects when the treatment is irreversible but the control produces at most a transitory effect. For the case in which the observable responses are directly measurable, Bhattacharyya and Johnson (*J. Amer. Statist. Assoc.* **65** (1970) 1308–1319) give a modification $T_n{}^*$ which is more powerful than T_n. In the application of $T_n{}^*$, one is likely to compare the treatment response with several control responses of the same object, obtained in different subexperiments. Two equally powerful modifications of $T_n{}^*$ are proposed. The first of these ensures that all comparisons are carried out between responses of different subjects and maximizes the number of comparisons between responses obtained in the same subexperiment. The second modification requires only slightly more than half the number of subexperiments needed for $T_n{}^*$. (Received February 23, 1971.)

71T-36. Exact distributions of a few arbitrary roots of some complex random matrices. V. B. WAIKAR, T. C. CHANG AND P. R. KRISHNAIAH, Aerospace Research Laboratories, University of Cincinnati and Aerospace Research Laboratories.

Let $A = (r_{jk}+is_{jk})$ be a $p \times p$ random Hermitian matrix such that $r_{jk} \sim N(\mu_{jk}, 1)$ for $j < k$, $r_{jj} \sim N(\mu_{jj}, 2)$ and $s_{jk} \sim N(v_{jk}, 1)$ for $j < k$ where r_{jk}'s and

s_{jk}'s are independent and $E(A) = (\mu_{jk}+iv_{jk}) \neq 0$. The distribution problems concerning any few of the p unordered characteristic roots $l_1, l_2, \cdots, l_p$ of the matrix A and certain other random matrices were considered for the central case ($E(A) = 0$) in the literature (see Wigner, in *Statistical Theories of Spectra; Fluctuations* (1965), 446–461 and Mehta, *Random Matrices* (1967), 184–185). In this paper, the authors obtain exact expression for the joint density of any few (say, $l_1, \cdots, l_r$ $(1 \leqq r < p)$) unordered roots of A in the *noncentral* case. Further, the authors also obtain exact expressions for the joint density of any few unordered roots of certain other complex *noncentral* random matrices useful in Statistics, namely Wishart matrix, MANOVA matrix and canonical correlation matrix. (Received March 8, 1971.)

71T-37. Balanced optimal 2^m fractional factorial designs of resolution V, $m = 8$. DHARAM V. CHOPRA AND J. N. SRIVASTAVA, Wichita State University and Colorado State University.

A balanced 2^m fractional factorial design T of resolution V (T is called a design of resolution V if, assuming interactions of order higher than two zeros, we can estimate the general mean, the main effects and 2-factor interactions) is identical with a balanced array T with 2 symbols and strength 4 defined as follows: A balanced array T of strength 4, 2 symbols and with parameters $(m, N; \boldsymbol{\mu}')$ is a matrix $T(m \times N)$ whose elements are the two symbols 0 and 1 and is such that if T_0 is any $(4 \times N)$ submatrix of T and $\boldsymbol{\alpha}$ is any vector with 4 elements (out of which i elements are nonzeros, $i = 0, 1, 2, 3, 4$ and the remaining $(4-i)$ elements are zero) then $\boldsymbol{\alpha}$ occurs exactly μ_i times as a column of T_0 where the given nonnegative integers μ_i's are independent of the choice of T_0. The vector $\boldsymbol{\mu}' = (\mu_0, \mu_1, \mu_2, \mu_3, \mu_4)$ is called the index set of the array. T is said to be a balanced design if the covariance matrix of the estimates is symmetric w.r.t. the factor symbols. For a given N, the design presented in this paper is the one for which the trace of the covariance matrix is minimum. In this paper, for $m = 8$, and various practical values of N designs with the above properties are obtained and the parameters $\boldsymbol{\mu}' = (\mu_0, \mu_1, \mu_2, \mu_3, \mu_4)$ are $(7+x, 4, 1, 1, 4)$, $(7+y, 4, 1, 2, 8)$, $(8, 4, 1, 3, 12)$ and $(9, 4, 1, 3, 12)$, where x, y are integers satisfying $0 \leqq x \leqq 7$, $0 \leqq y \leqq 8$, the actual arrays along with other pertinent information will be presented in the paper. (Received March 3, 1971.)

71T-38. A class of distribution-free tests for multivariate interchangeability (preliminary report). ANIL P. GORE', University of Kentucky.

Suppose we have n observations on a p-variate random vector with continuous distribution $F(x_1, \cdots, x_p)$ and wish to test H_0: F is invariant under permutation of its arguments. Tests for this problem have been proposed by Sen [*Ann. Inst. Statist. Math.* **19** (1967) 451–472], Wormleighton [*Ann. Math. Statist.* **30** (1959) 1005–1017] and for its multidimensional extension by Sen [*Sankhyā Ser. A* **31**

(1969) 145–156] wherein he points out the use of these tests in the analysis of two-way layouts. This paper offers tests based on arbitrary linear combinations of within-vector comparisons, and in this sense is more in the spirit of Wormleighton's work than Sen's, who used between-vector comparisons also. We use simultaneous comparisons of one variable at a time with all others. Sen's tests are either permutational or involve estimation of nuisance parameters. Wormleighton's tests and ours are unconditionally distribution-free and do not involve nuisance parameters. Let U_{ij} denote the proportion of observations in which ith coordinate has rank j among the p coordinates of that observation. Let $U_i = \sum_{j=1}^{p} a_j U_{ij}$, a_j's not all equal. We reject H_0 if $S = n(p-1)\sum_{i=1}^{p}(U_i-\bar{a})^2/\sum_{i=1}^{p}(a_i-\bar{a})^2$ is too large, where $\bar{a} = \sum a_i/p$. As $n \to \infty$, S has a limiting χ^2 distribution with $p-1$ degrees of freedom under H_0. For $a_j = j$, S reduces to the Friedman statistic. Exact small sample significance points are obtained for the case $a_p = 1$, $a_i = 0, i \neq p$, for $p \leqq 5$ and $n \leqq 10$. (Received March 10, 1971.)

71T-39. A time series approach to the life table. EDWARD J. WEGMAN AND CRIS R. KUKUK, University of North Carolina.

This paper reviews some statistical techniques for estimating the life table. The population distribution may be regarded as a non-stationary stochastic process. In certain cases, it may be assumed that the population distribution is a non-stationary first order autoregressive scheme. The mean and covariance structure is described for such processes and some parameter estimates are given. The 1961 Indian Census data is examined in detail to illustrate the method. (Received March 11, 1971.)

71T-40. Power and robustness of two sample rank tests for scale with medians subtracted: some Monte Carlo results. MARK DISKIND AND FREDERICK SCHEUREN, The George Washington University.

Most two-sample rank tests for scale require that both samples come from distributions whose location parameters (e.g., population medians) are known or known to be equal. Numerous authors have analyzed the behavior of such tests when the sample medians are subtracted before the tests are made and have developed the regularity conditions necessary to assure that the modified test is asymptotically distribution-free. This paper presents some Monte Carlo results for small samples when the median is subtracted. The tests studied include among others those of Klotz, Ansari-Bradley, and Bell-Doksum. The distribution types examined are the Uniform, Normal, Double Exponential, and Cauchy. Eight sample sizes are considered ranging from $n = m = 6$ to $n = m = 20$. (Received March 2, 1971.)

71T-41. Sample size and Fisher's linkage problem. K. O. BOWMAN, AND L. R. SHENTON, Oak Ridge National Laboratory and University of Georgia.

In his classical text, *Statistical Methods for Research Workers*, Fisher considers estimators for θ when observations fall into four categories A, B, C, D, with probabilities $(2+\theta)/4$, $(1-\theta)/4$, $(1-\theta)/4$, $\theta/4$, and observed frequencies a, b, c, d, for a sample of n. The maximum likelihood estimator $\hat{\theta}$ is the nonnegative solution of $nx^2-(a-2b-2c-d)x-2d = 0$, and the paper gives the least sample size to force the distribution of $\hat{\theta}$ into near-normality ($\beta_1 < 0.1$, $\beta_2 = 3 \pm 0.1$). The approach is to determine higher order terms (as far as n^{-9}) in the first four moments to check by sample configuration methods where possible, and assess n so as to damp out all but the lower order terms. Denoting the least sample size N required for near-normality for $\hat{\theta}$ in the interval $\theta_0 \leqq \theta < \theta_1$ by $(\theta_0, \theta_1; N)$, we have evaluated the following: (0.1, 0.9; 200) (0.9, 0.95; 500), (0.95, 0.99; 2000), (0.99, 0.999; 20,000), (0.05, 0.1; 650), (0.04, 0.05; 800), (0.03, 0.04; 1150), (0.02, 0.03, 1850), (0.01, 0.02, 4000), (0.005, 0.01; 8000), (0.004, 0.005; 10,000), (0.003, 0.004; 13,000). (Received March 18, 1971.)

71T-42. Asymptotic normality of linear rank statistics for independence if the weight functions are simple step functions. F. H. RUYMGAART, Mathematisch Centrum.

For each n a random sample $(X_1, Y_1), \cdots, (X_n, Y_n)$ from a continuous bivariate distribution function (df) H is given. All samples are defined on a single probability space. The bivariate df H has marginal dfs F and G. The corresponding empirical dfs are denoted by H_n, F_n and G_n. This paper is a continuation of an earlier generalization of Theorem 1 of Bhuchongkul (*Ann. Math. Statist.* **35** (1964) 138–149) on asymptotic normality of linear rank statistics for testing the independence hypothesis $H = FG$, by Shorack, van Zwet and the author (submitted; see also *Ann. Math. Statist.* **41** (1970) Abstract 127–12). Asymptotic normality has been proved for suitably standardized statistics of the type $T_n = \iint J(F_n)K(G_n)dH_n$, where J and K are simple step functions on the unit interval. To keep touch with the previous result, the approach inspired by the method of Chernoff and Savage (*Ann. Math. Statist.* **29** (1958) 972–994) has been adjusted in order to handle the case in which the weight functions J and K are simple step functions. (Received March 23, 1971.)