The Annals of Mathematical Statistics
1971, Vol. 42, No. 4, 1452–1454

FINDING BEST TESTS APPROXIMATELY FOR TESTING HYPOTHESES ABOUT A RANDOM PARAMETER[1]

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In an earlier paper the author proved the existence of a best test for testing hypotheses about a random parameter with unknown distribution. This paper gives a result which helps one find the best test approximately for several of the examples considered in the previous paper.

1. Introduction. Let X be a real-valued random variable with a family of possible distributions indexed by $\lambda \in \Omega$, a set of real numbers. For each λ, let f_λ denote the density of X with respect to a measure μ, where μ is either Lebesgue measure or counting measure on the positive integers. Assume that the family f_λ has strict monotone likelihood ratio property in x, i.e., for $\lambda_1 < \lambda_2, f_{\lambda_2}(x)/f_{\lambda_1}(x)$ is a strictly increasing function of x, and for each $\lambda, f_\lambda(x) > 0$ for all x in the space of X. In the discrete case we assume that the space of X is either the set $\{0, 1, \cdots, N\}$ for some positive integer N or the set of positive integers. λ is a realization of a random variable Λ with a family of possible a priori distributions $\mathscr{G} = \{g_\theta : \underline{\theta} < \theta < \bar{\theta}\}$ where g_θ is a density with respect to some σ-finite measure v on Ω and $-\infty \leqq \underline{\theta} < \bar{\theta} \leqq +\infty$.

Consider the problem of observing X and then testing $H: \lambda \leqq \lambda_0$ against $K: \lambda > \lambda_0$ where both H and K are composite hypotheses.

Analogous to the type I and type II errors of the Neyman–Pearson theory are

$$\text{type (i) error: } \Lambda > \lambda_0 \text{ is decided and } \Lambda \leqq \lambda_0 \text{ occurs,}$$
$$\text{type (ii) error: } \Lambda \leqq \lambda_0 \text{ is decided and } \Lambda > \lambda_0 \text{ occurs.}$$

Analogous to the problem of finding uniformly most powerful level α tests is the problem

$$\text{subject to: } P_\theta \text{ (type (i) error)} \leqq \alpha \text{ for } \theta \in (\underline{\theta}, \bar{\theta})$$
$$\text{minimize } \quad P_\theta \text{ (type (ii) error) uniformly for } \theta \in (\underline{\theta}, \bar{\theta}).$$

A test which achieves this is called a uniformly most powerful (UMP) level α test relative to $\mathscr{G}$. UMP tests for this problem can be found as follows. (See Meeden (1970) for details.)

For each $\theta' \in (\underline{\theta}, \bar{\theta})$ there exist constants $\gamma(\theta')$ and $c(\theta')$ and a test function δ_θ which is of the form

$$(1) \qquad \begin{aligned} \delta_{\theta'}(x) &= 1 && \text{for} \quad x > c(\theta'), \\ &= \gamma(\theta') && \text{for} \quad x = c(\theta'), \\ &= 0 && \text{for} \quad x < c(\theta'), \end{aligned}$$

Received November 9, 1970.

[1] This research was supported in part by National Science Foundation Grant GP-7363.

such that $\delta_{\theta'}$ is a most powerful test at level α relative to $\mathscr{G}' = \{g_{\theta'}\}$, that is, where $g_{\theta'}$ is the known a priori distribution. (In the case where X has a continuous distribution we take $\gamma(\theta') = 1$.) If there exists a $\theta^* \in (\underline{\theta}, \bar{\theta})$ such that

$$(2) \qquad \delta_{\theta^*}(x) = \inf_{\theta \in (\underline{\theta}, \bar{\theta})} \delta_\theta(x) \quad \text{for all} \quad x,$$

then δ_{θ^*} is a UMP level α test relative to $\mathscr{G}$. δ_{θ^*} satisfies (2) if and only if the function

$$(3) \qquad \psi(\theta) = c(\theta) + 1 - \gamma(\theta)$$

defined on $(\underline{\theta}, \bar{\theta})$ has a maximum at θ^*. The purpose of this note is to prove that, under certain conditions, ψ is maximized at exactly one point $\theta_M \in (\underline{\theta}, \bar{\theta})$ and that ψ is non-decreasing over $(\underline{\theta}, \theta_M]$ and non-increasing over $[\theta_M, \bar{\theta})$.

Section 3 of Meeden (1970) deals with several examples which are special cases of the problem treated here. In the earlier paper only the existence of a test satisfying (2) was proved. This best test can be found approximately as follows. For a given θ it is possible to calculate $\psi(\theta)$ approximately (in one case exactly) without too much difficulty. By doing this for various values of θ the maximum of ψ can be found approximately and the UMP level α test relative to $\mathscr{G}$ corresponds to this maximum.

2. To avoid trivial cases we assume $0 < \alpha < 1$ and that there exists a θ' for which $\psi(\theta') > \underline{x} = \inf_x\{x : f_\lambda(x) > 0\}$ and hence

$$P_{\theta'}(\text{type (i) error of } \delta_{\theta}') = \int \int_{\{\lambda \leqq \lambda_0\}} \delta_{\theta'}(x) f_\lambda(x) g_\theta(\lambda) \, dv \, d\mu = \alpha$$

where the integral involving X is over the entire space of X. We need two additional assumptions:

(4) • (a) If Φ is a bounded measurable function defined on Ω with $\Phi(\lambda) < 0$ for $\lambda < \lambda_1$ and $\lambda > \lambda_2$, where $\lambda_1 < \lambda_2$, then $E_\theta \Phi(\Lambda) < 0$ for θ sufficiently close to $\underline{\theta}$ and $\bar{\theta}$.

(b) $g_\theta(\lambda)$ is Pólya type ∞ and $g_\theta(\lambda)$ can be differentiated two times with respect to θ for all λ. If Φ is a bounded measurable function on Ω then $u(\theta) = E_\theta \Phi(\Lambda)$ can be differentiated two times with respect to θ inside the integral sign.

Next a lemma will be proved from which the main result follows easily.

LEMMA. *If δ is a test of form* (1) *with $\mu(x : \delta(x) > 0) > 0$ and $\mu(x : \delta(x) < 1) > 0$ then $F(\theta, \delta) = P_\theta\{$type* (i) *error of $\delta\}$ is maximized at exactly one point $\theta_m \in (\underline{\theta}, \bar{\theta})$ and $F(\theta, \delta)$ is strictly increasing over $(\underline{\theta}, \theta_m)$ and strictly decreasing over $(\theta_m, \bar{\theta})$.*

PROOF. Let $h(\lambda) = E(\delta(X)/\lambda)$ or 0 as $\lambda \leqq \lambda_0$ or $\lambda > \lambda_0$. h is strictly increasing on $\{\lambda : \lambda \leqq \lambda_0\}$ since X has the strict monotone likelihood ratio property. If c is chosen such that $\inf_{\lambda \leqq \lambda_0} h(\lambda) < c < \sup_\theta F(\theta, \delta) = c_0$ then $h(\lambda) - c$, as a function on Ω, has two sign changes.

If $F(\theta, \delta)$ does not have a unique maximum then there exist $\theta_1 < \theta_2$ such ttha $F(\theta_1, \delta) = F(\theta_2, \delta) = c_0$ since by Assumption (4.a) the sup is attained. Let

$u_c(\theta) = F(\theta, \delta) - c = \int_\Omega [h(\lambda) - c] f_\theta(\lambda) dv$. By the choice of $c, u_c(\theta_1) > 0$ and $u_c(\theta_2) > 0$ and by Assumption (4.a), $u_c(\theta)$ is negative for θ sufficiently close to $\underline{\theta}$ or $\bar{\theta}$. By Assumption (4.b) we may use Theorem 3 of Karlin (1957) which implies u_c has at most two sign changes on $(\underline{\theta}, \bar{\theta})$. Hence there exist $\theta_1{}^* \leqq \theta_1 < \theta_2 \leqq \theta_2{}^*$($\theta_1{}^*$ and $\theta_2{}^*$ depending on c) such that $u_c(\theta) \leqq 0$ or $\geqq 0$ as $\theta \notin [\theta_1{}^*, \theta_2{}^*]$ or $\theta \in [\theta_1{}^*, \theta_2{}^*]$. For each θ, $u_c(\theta)$ decreases as c increases and $\lim_{c \uparrow c_0} u_c(\theta) \geqq 0$ for $\theta \in [\theta_1, \theta_2]$. But for each θ, by the Lebesgue dominated convergence theorem and the choice of c_0, it follows that $\lim_{c \uparrow c_0} u_c(\theta) = u_{c_0}(\theta) \leqq 0$. So $u_{c_0}(\theta) = 0$ for $\theta \in [\theta_1, \theta_2]$, which is impossible by Theorem 3 of Karlin (1957), and $F(\theta, \delta)$ has a unique maximum, θ_m.

The proof that $F(\theta, \delta)$ is strictly increasing for $\theta < \theta_m$ and strictly decreasing for $\theta > \theta_m$ follows easily from Theorem 3 of Karlin and will be omitted.

THEOREM. *The function* ψ, *defined by* (2) *for* $\theta \in (\underline{\theta}, \bar{\theta})$, *is maximized at exactly one point* $\theta_M \in (\underline{\theta}, \bar{\theta})$. *There exists a number* $\bar{\theta}'$, *such that* $\theta_M < \bar{\theta}' \leqq \bar{\theta}$, *and* $\psi(\theta) > \underline{x}$ *for* $\theta \in (\underline{\theta}, \bar{\theta}')$ *and* $\psi(\theta) = \underline{x}$ *for* $\theta \notin (\underline{\theta}, \bar{\theta}')$ *where* $\underline{x} = \inf\{x : f_\lambda(x) > 0\}$. ψ *is strictly increasing over* $(\underline{\theta}, \theta_M)$ *and strictly decreasing over* $(\theta_M, \bar{\theta}')$.

PROOF. The proof that ψ is continuous is straightforward and will be omitted. The $\sup_{\theta \in (\underline{\theta}, \bar{\theta})} \psi(\theta)$ is finite. To see this, note that for each θ, $\delta_\theta(x) \leqq \delta'(x)$ for all x, where considering λ a fixed but unknown parameter, δ' is the uniformly most powerful level α test of $\lambda \leqq \lambda_0$ against $\lambda > \lambda_0$. By the Lemma the sup is attained in the interval. If ψ does not have a unique maximum then there exist numbers $\theta_1 < \theta_2$ such that $\psi(\theta_1) = \psi(\theta_2) = \sup_\theta \psi(\theta)$. Then $\delta_{\theta_1} = \delta_{\theta_2}$ and $F(\theta_1, \delta_{\theta_1}) = F(\theta_2, \delta_{\theta_1}) = \alpha$ and by the Lemma $F(\theta, \delta_{\theta_1}) > \alpha$ for $\theta \in (\theta_1, \theta_2)$. But δ_{θ_1} is the UMP level α test relative to the family $\mathscr{G}$ and $F(\theta, \delta_{\theta_1}) \leqq \alpha$ for all θ, which is a contradiction.

Let θ_M denote the unique maximum of ψ. Since for the test δ, which is one for all x, P_θ(type (i) error of δ) is a non-increasing function of θ there exists a number $\bar{\theta}'$ such that $\theta_M < \bar{\theta}' \leqq \bar{\theta}$ and $\psi(\theta) > \underline{x}$ for $\theta \in (\underline{\theta}, \bar{\theta}')$ and $\psi(\theta) = \underline{x}$ for $\theta \notin (\underline{\theta}, \bar{\theta}')$. To prove that ψ is strictly increasing on $(\underline{\theta}, \theta_M)$ it is enough to show that the following two cases are impossible:

CASE (a). ψ is constant on some sub-interval of $(\underline{\theta}, \theta_M)$.

CASE (b). There exist $\theta_i \in (\underline{\theta}, \theta_M)$ for $i = 1, 2$, and 3 such that

$$\theta_1 < \theta_2 < \theta_3 \text{ and } \psi(\theta_1) = \psi(\theta_3) > \psi(\theta_2).$$

That Case (a) is not possible follows from the Lemma. If Case (b) holds then $\delta_{\theta_1} = \delta_{\theta_3}$ and $F(\theta_1, \delta_{\theta_1}) = F(\theta_3, \delta_{\theta_1}) = \alpha$, and by the Lemma $F(\theta, \delta_{\theta_1}) > \alpha$ for $\theta \in (\theta_1, \theta_3)$. $\psi(\theta_1) > \psi(\theta_2)$ implies that $\delta_{\theta_2}(x) \geqq \delta_{\theta_1}(x)$ for all x and hence $F(\theta, \delta_{\theta_2}) \geqq F(\theta, \delta_{\theta_1})$ for all θ, which is a contradiction since $\alpha = F(\theta_2, \delta_{\theta_2})$. The proof that ψ is strictly decreasing on $(\theta_M, \bar{\theta}')$ is similar.

REFERENCES

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