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AN IMPROVED INEQUALITY FOR BALANCED INCOMPLETE BLOCK DESIGNS

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For a resolvable balanced incomplete block design (BIBD), Bose (1942) obtained an inequality $b \geqq v+r-1$. Stanton (1957) showed that this inequality is equivalent to an inequality $r \geqq \lambda+k$. The main purpose of this note is to improve Bose's inequality to $b \geqq 2v+r-2$ for a resolvable BIBD which is not affine resolvable.

1. Introduction and summary. In a balanced incomplete block design (BIBD) with parameters v, b, r, k and λ, we have the following relations:

$$vr = bk, \qquad \lambda(v-1) = r(k-1), \qquad b \geqq v. \tag{1.1}$$

If the blocks can be separated into r sets of n blocks each ($b = nr$) such that each set of n blocks forms a complete replication, the design is called resolvable. Moreover, if two blocks belonging to different sets have the same number of treatments in common, the design is called affine resolvable. Bose [1] proved that if a resolvable BIBD with parameters v, b, r, k and λ exists, then $b \geqq v+r-1$ and that the necessary and sufficient condition for a resolvable BIBD to be affine resolvable is $b = v+r-1$ and k/n an integer. Stanton [2] showed that Bose's inequality is equivalent to an inequality $r \geqq \lambda+k$. The purpose of this note is to improve these inequalities.

2. Theorem. *In a BIBD with parameters $v = nk$, b, r, k and λ, if $b > v+r-1$, then*

$$r \geqq \lambda+2k \tag{2.1}$$

and vice versa.

PROOF. From (1.1) and $b = vr/k = \{vr+k(v+r-1)-k(v+r-1)-r+r\}/k$, we have

$$b-(v+r-1) = (v-1)(r-\lambda-k)/k\,. \tag{2.2}$$

In (2.2), if $b > v+r-1$, then $(v-1)(r-\lambda-k)/k$ must be a positive integer. Since $v = nk$ implies $(v-1, k) = 1$, $(r-\lambda-k)/k$ must be a positive integer. Thus we obtain $(r-\lambda-k)/k \geqq 1$, i.e., $r \geqq \lambda+2k$. Conversely, if $r \geqq \lambda+2k$, then from (2.2) we have $b > v+r-1$.

Our theorem shows that the result due to Stanton [2] is improved to an inequality $r \geqq \lambda+2k$ for any BIBD with parameters $v = nk$, b, r, k and λ provided that

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$b > v+r-1$. From (1.1), since $(r-n\lambda)k = r-\lambda > 0$, we obtain $r-n\lambda \geqq 1$. Hence $r-\lambda \geqq k$. Thus the inequality $b \geqq v+r-1$ holds for any BIBD with $v = nk$.[1]

Multiplying (2.1) by n, we obtain $b \geqq n\lambda+2v$. On the other hand, our theorem also shows that if for a BIBD $v = nk$ and $b > v+r-1$, then $b = v+r-1+t(v-1_1)$ where t is a positive integer and hence $b \geqq 2v+r-2$. Since $r-\lambda = (r-n\lambda)k$ and $r \geqq \lambda+2k$ imply $r-2 \geqq n\lambda$, we have the following corollary from a necessary and sufficient condition for a resolvable BIBD to be affine resolvable.

COROLLARY. *For a resolvable* BIBD *which is not affine resolvable, an inequality*

$$b \geqq 2v+r-2$$

holds.

Note that since $v \geqq 2$ in a BIBD, Bose's inequality $b \geqq v+r-1$ can be replaced by a more stringent inequality $b \geqq 2v+r-2$ for a resolvable BIBD which is not affine resolvable.

EXAMPLE (i). Consider a resolvable BIBD with parameters $v = 15$, $b = 35$, $r = 7$, $k = 3$ and $\lambda = 1$ which is not affine resolvable. Then $b \geqq v+r-1$ implies $35 \geqq 21$ and $b \geqq 2v+r-2$ implies $35 \geqq 35$, i.e., the bound is attained by our inequality.

EXAMPLE (ii). Consider a resolvable BIBD with parameters $v = 28$, $b = 63$, $r = 9$, $k = 4$ and $\lambda = 1$ which is not affine resolvable. Then $b \geqq v+r-1$ implies $63 \geqq 36$ and $b \geqq 2v+r-2$ implies $63 \geqq 63$.

EXAMPLE (iii). Consider a resolvable geometrical BIBD with parameters $v = 16$, $b = 140$, $r = 35$, $k = 4$ and $\lambda = 7$ which is not affine resolvable. Then $b \geqq v+r-1$ implies $140 \geqq 50$ and $b \geqq 2v+r-2$ implies $140 \geqq 65$.

3. Acknowledgment. The author wishes to thank the referee for his valuable comments.

REFERENCES

[1] BOSE, R. C. (1942). A note on the resolvability of Balanced Incomplete Block Designs. *Sankhyā Ser. A* **6** 105–110.

[2] STANTON, R. G. (1957). A note on BIBDS. *Ann. Math. Statist.* **28** 1054–1055.

[1] *Note added in proof.* The author would like to remark that W. F. Mikhail (1960) gave an alternative proof of the result which the inequality $b \geqq v + r - 1$ holds for any BIBD with $v = nk$ in *Ann. Math. Statist.* **31** 520–522 (An inequality for balanced incomplete block des igns).