

A NOTE ON CERTAIN TYPES OF BIBD'S BALANCED FOR RESIDUAL EFFECTS

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1. Introduction. In a number of applications, experimental designs are required which possess some type of balance for certain residual (or "carry-over") effects. For example, designs may be required in which each variety occurs next to every other variety equally often within the blocks of the design. Occasionally (see, for example, Rees [3]) such designs are required for which the blocks are effectively circular; that is, in the block (1, 3, 7, 9), for example, variety 9 occurs next to both variety 7 and variety 1. The purpose of this note is to point out that two general families of balanced incomplete block designs (b.i.b.d.'s) due to Sprott [4] yield designs which have a number of properties with respect to balance for residual effects. These features of the designs might be useful in a situation where it was desirable to have a design which was a b.i.b.d., in addition to being balanced for certain residual effects.

2. B.i.b.d.'s and "neighbour designs". In the designs which follow, we consider the blocks to be "circular", in the sense noted above. First, consider designs which have the property that each variety occurs next to every other variety the same number of times, say λ' . (Rees [3] calls these "neighbour designs", for short). We will see that one of Sprott's [4] series yields b.i.b.d.'s which have this property. First, we note that if a b.i.b.d. has the "neighbour design" property, then its parameter set can be written as

$$(1) \quad v, b = \lambda'v(v-1)/2k, \quad r = \lambda'(v-1)/2, \quad k, \lambda = \lambda'(k-1)/2.$$

We consider the case where $\lambda' = 1$; the parameter set is then

$$(2) \quad v, b = v(v-1)/2k, \quad r = (v-1)/2, \quad k, \lambda = (k-1)/2.$$

(If a design with parameter set (2) exists for given v, k , then repeating this design λ' times gives a design (1) for the same v, k .) Now, Sprott's [4] Series B gives b.i.b.d.'s with parameter set

$$(3) \quad v = 2mk + 1, \quad b = mv, \quad r = mk, \quad k, \lambda = (k-1)/2$$

in the case where v is a prime power. The designs are formed by adding 1, 2, ..., $v-1$ (modulo v) to the m initial blocks $B_i(X^i, X^{i+2m}, \dots, X^{i+4\lambda m})$, $i = 0, 1, \dots, m-1$, where X is a primitive element in $GF(v)$.

If we consider the "backward" and "forward" differences between successive varieties in the initial blocks, we find that each nonzero field element occurs

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precisely once. For the backward and forward differences in B_i are $\pm(X^{i+2m} - X^i)$, \dots , $\pm(X^{i+4\lambda m} - X^{i+(4\lambda-2)m})$, $\pm(X^i - X^{i+4\lambda m})$. Since $X^{(4\lambda+2)m} = X^{2mk} = 1$, then $X^{2m} \neq 1$ if $k > 1$ and we can write $X^{2m} - 1 = X^s$. The backward and forward differences in B_i can then be written as $\pm X^{i+s}$, $\pm X^{i+s+2m}$, \dots , $\pm X^{i+s+4\lambda m}$. As i ranges over $0, 1, \dots, m-1$, these differences give each nonzero field element exactly once. Since the design is formed by developing the initial blocks modulo v , it is clear that if the blocks of the design are written in the order indicated above by the initial blocks, then each variety will occur next to each other variety precisely once. (Rees [3] notes that Bose's [1] series S_1 of symmetrical b.i.b.d.'s has this property; in fact, Bose's series is a special case of Sprott's Series B above.)

EXAMPLE. Consider the symmetric b.i.b.d. with $v = b = 11$, $r = k = 5$, $\lambda = 2$, formed by developing modulo 11 the initial block $(1, 4, 5, 9, 3)$. The backward and forward differences among successive varieties are ± 3 , ± 1 , ± 4 , ± 5 , ± 2 , so that the resultant design will be a "neighbour design".

3. B.i.b.d.'s balanced for one residual effect. Consider a design where each variety follows (or precedes) every other variety precisely λ'' times. (Such a design will also be a neighbour design with $\lambda' = 2\lambda''$.) In particular, consider the case $\lambda'' = 1$; if the design is a b.i.b.d., then its parameter set must be

$$(4) \quad v, b = v(v-1)/k, \quad r = v-1, \quad k, \lambda = k-1.$$

Sprott's [4] Series A provides solutions of the desired type here whenever $v = mk + 1$ is a prime power. The designs have parameters $v = mk + 1$, $b = mv$, $r = mk$, k , $\lambda = k - 1$, and are formed by developing (modulo v) the m initial blocks $(X^i, X^{i+m}, \dots, X^{i+(k-1)m})$, $i = 0, 1, \dots, m-1$, where X is a primitive element in $GF(v)$. Again, it is easily shown that among the successive forward (or backward) differences in the m blocks, each nonzero field element occurs exactly once, and hence the blocks lead to a design of the desired type.

4. Further remarks. Sprott's two series of designs above have other properties which might prove useful. It can be shown that each variety in the Series B designs not only occurs next to every other variety exactly once, but also occurs a distance of $2, \dots, (k-1)/2$ positions away from every other variety exactly once. Similarly, for Series A designs, each variety precedes (or follows) every other variety at a distance of $2, 3, \dots, (k-1)/2$ positions exactly once. It is possible to envisage situations where these structural properties might be desirable features of a design.

As a general comment, we note that in the case of designs with parameter sets (2) or (4) and not covered by Sprott's two general families, existing b.i.b.d.'s can often be arranged so as to give designs with a desired type of balance for residual effects. For example, a check of given solutions (see, for example, Takeuki [5]) to b.i.b.d.'s with parameters (2) in the range $v \leq 31$, $k \leq 15$, found that for all parameter sets which yielded designs except one ($v = 21$, $k = 7$), a "neighbour design" could be found in this way.

Finally, it seems worthwhile to re-emphasize that the blocks in the above design are effectively circular; if the blocks are arranged in a linear fashion so that all varieties but the first in each block are considered to follow the preceding variety, then we will not have the desired type of serial balance. Designs in which blocks are linear, and which have balance with respect to a single residual effect are of use in a number of areas (see, for example, Patterson and Lucas [2]). The "circular" designs considered here might be used in this latter situation by writing the blocks in linear order, and placing the final variety of each block also at the beginning of the block, but unless block sizes were large, this would likely be rejected as wasteful and inefficient.

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