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A NOTE ON THE EXISTENCE OF QUANTITATIVE PROBABILITY

BY TERRENCE FINE
Cornell University

0. Introduction. A comparative probability relation $\succcurlyeq$ is an ordering of elements $A, B, \cdots$, of a field of events $\mathscr{F}$; "$A \succcurlyeq B$" is read "event A is as least as probable as event B." A function $P: \mathscr{F} \to R^1$ (strictly) agrees with $\succcurlyeq$ if $(\forall A, B)(A \succcurlyeq B \Leftrightarrow P(A) \geqq P(B))$; such functions will be called quantitative probabilities. After imposing some relatively uncontroversial restrictions on $\succcurlyeq$ we introduce an axiom that, as shown by Theorem 1, is a necessary and sufficient condition for the existence of a quantitative probability. We then consider the problem of the existence of an additive quantitative probability P (i.e., $A \cap B = \phi \Rightarrow P(A \cup B) = P(A) + P(B)$). In Theorem 2 we explicate the relation between additive quantitative probability and the assumption, entertained by Savage and others, that for all n there exist n-fold almost uniform partitions. We then observe that the hypothesis of almost uniform partitions leads to theories of additive quantitative probability that are uniformly weaker than that proposed by Luce. Finally, an axiom is introduced that is a necessary condition if there is to exist a countably additive quantitative probability. This leads to the observation that the axioms proposed by Villegas for the countably additive case yield a theory that is a special case of Luce's theory when we adjoin to Luce's axioms the axiom we have proposed.

1. A necessary and sufficient condition for the existence of quantitative probability. Following Savage and others we assume that a comparative probability relation $\succcurlyeq$ satisfies the following three axioms:

C1. $\succcurlyeq$ is a total order of the elements of $\mathscr{F}$.

C2. $(\forall A \in \mathscr{F})(A \succcurlyeq \varnothing)$, where $\varnothing$ denotes the null set.

C3. $A \succcurlyeq B, C \cap (A \cup B) = \varnothing \Leftrightarrow A \cup C \succcurlyeq B \cup C, C \cap (A \cup B) = \varnothing$.

The order topology $(\mathscr{F}, \mathscr{T})$ induced by $\succ$ ($A \succ B$ if $A \succcurlyeq B$ and false $B \succcurlyeq A$) is the collection $\mathscr{T}$ of open subsets of the space $\mathscr{F}$ having as a base all sets of the forms: $\{F: F \prec A\}$, $\{F: A \prec F\}$, $\{F: A \prec F \prec B\}$. We postulate

C4. $(\mathscr{F}, \mathscr{T})$ has a countable base.

The justification for C4 is contained in

THEOREM 1. *If* $\succcurlyeq$ *satisfies* C1, C2, C3, *then it admits of a* (*not necessarily additive*) *quantitative probability* P *if and only if* $\succcurlyeq$ *also satisfies* C4.

PROOF. First assume that $\exists P$ agreeing with $\succcurlyeq$. Let $P(\mathscr{F})$ be that subset of the reals that is the image of $\mathscr{F}$ under P. Choose a countable set $\{d_i\}$ that is a subset of

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$P(\mathscr{F})$ and is dense in $P(\mathscr{F})$. Corresponding to each d_i select an element $D_i \in P^{-1}(d_i)$ to form a collection $\mathscr{D} = \{D_i\}$. We assert that $\mathscr{B} = \{\{F:F \prec D_j\}, \{F:F \succ D_i\}, \{F:D_i \prec F \prec D_j\}; D_i, D_j \in \mathscr{D}\}$ forms a countable base for the order topology $(\mathscr{F}, \mathscr{T})$. Since $\mathscr{B}$ is a subset of the usual base for $(\mathscr{F}, \mathscr{T})$ we need only verify that any set in the usual base can be written as a union of sets in $\mathscr{B}$. Take, for example, the set $\{F:G \prec F \prec H\}$. Since $\{d_i\}$ is dense in $P(\mathscr{F})$ we can choose sequences $\{d_{i_j}\}$, $\{d_{k_j}\}$, such that $d_{i_j} \downarrow P(G)$, $d_{k_j} \uparrow P(H)$, $d_{i_j} \leqq P(H)$, $d_{k_j} \geqq P(G)$. We assert that $\{F:G < F < H\} = \bigcup_{j=1}^{\infty} \{F:D_{i_j} < F < D_{k_j}\}$. Clearly the left-hand side includes the right-hand side. Hence, assume $F' \in \{F:G < F < H\}$.

Note that $P(G) < P(F') < P(H)$.

Therefore $(\exists J)(\forall\, n > J)(d_{i_n} < P(F') < d_{k_n})$. Thus $F' \in \{F: D_{i_n} \prec F \prec D_{k_n}\} \subseteq \bigcup_{j=1}^{\infty} \{F:D_{i_j} \prec F \prec D_{k_j}\}$. The assertion is proven. The remaining cases of $\{F:F \prec A\}$, $\{F:A \prec F\}$, can be dealt with in a parallel manner. Thus, we have confirmed that if $\exists P$ agreeing with $\succcurlyeq$ then $(\mathscr{F}, \mathscr{T})$ has a countable base.

Now assume that $(\mathscr{F}, \mathscr{T})$ has a countable base

$$\mathscr{B} = \{\{F:F \prec D_j\}, \{F:F \succ D_i\}, \{F:D_i \prec F \prec D_j\}; D_i, D_j \in \mathscr{D}\}.$$

We first define P on $\mathscr{D}$ and then extend it to $\mathscr{F}$. Without loss of generality, assume that $D_1 = \varnothing$, $D_2 = \Omega = \bigcup_{F \in \mathscr{F}} F$, and $D_i \approx D_j \Rightarrow i = j$. Define $P(D_1) = 0$, $P(D_2) = 1$. Define $\{l_n\}$, $\{u_n\}$, for $n > 2$, by $1 \leqq l_n$, $u_n \leqq n$, $D_{l_n} \prec D_{n+1} \prec D_{u_n}$ and $(\nexists 1 \leqq k \leqq n) D_{l_n} \prec D_k \prec D_{n+1}$, $D_{n+1} \prec D_k \prec D_{u_n}$. That this definition is possible follows from C1, C2, and the easy consequence of C3 that $(\forall F)(F \preccurlyeq \Omega)$. Inductively define P on $\mathscr{D}$ through $P(D_{n+1}) = \frac{1}{2}(P(D_{l_n}) + P(D_{u_n}))$. It is immediate that P agrees with $\succcurlyeq$ on $\mathscr{D}$. Extend the definition of P to $\mathscr{F}$ as follows:

If $(\exists n)(F \approx D_n) \Rightarrow P(F) = P(D_n)$. Otherwise,

$$P(F) = \sup \{x: D_n \prec F, x = P(D_n)\}.$$

We must now verify that P, as defined above, agrees with $\succcurlyeq$ on $\mathscr{F}$. Assume that $P(H) > P(G)$. From the definition of P

$$P(H) = \sup \{x: D_i \prec H, x = P(D_i)\} > P(G) = \sup \{x: D_i \prec G, x = P(D_i)\}.$$

Hence,

$$\exists d_j \in \{x: D_i \prec H, x = P(D_i)\} - \{x: D_i \prec G, x = P(D_i)\},$$

and $\exists D_j \in P^{-1}(d_j)$. However, $H \succ D_j$ and $D_j \succcurlyeq G$, from which it is immediate that $H \succ G$. Thus, we have established that $G \succcurlyeq H \Rightarrow P(G) \geqq P(H)$.

To prove the converse implication, assume that $G \succ H$. If both G, H are equivalent to elements of $\mathscr{D}$ then from the definition of P it follows that $P(G) \succ P(H)$. Otherwise, at least one of G, H, say G, is not equivalent to any element of $\mathscr{D}$. Note that $\{F:F \prec G\}$ strictly contains $\{F:F \prec H\}$ and $(\exists i_j, k_j)$

$$\{F:F \prec G\} = \bigcup_{j=1}^{\infty} \{F:F \prec D_{i_j}\}$$

$$\{F:F \prec H\} = \bigcup_{j=1}^{\infty} \{F:F \prec D_{k_j}\}.$$

Hence, $(\exists D_{i_l})(H \preccurlyeq D_{i_l} \prec G)$. Note that $\{F: F \prec G\}$ strictly contains $\{F: F \prec D_{i_l}\}$, and repeat the above to conclude that $(\exists D_{i_n})(D_{i_l} \prec D_{i_n} \prec G)$. Thus, $H \preccurlyeq D_{i_l} \prec D_{i_n} \prec G$ and it follows that

$$P(H) = \sup \{x: D_i \prec H, x = P(D_i)\} \leqq P(D_{i_l}) <$$

$$P(D_{i_n}) \leqq \sup \{x: D_i \prec G, x = P(D_i)\} = P(G).$$

This verifies that $P(H) \geqq P(G) \Rightarrow H \succcurlyeq G$. Therefore, we have shown that P, as defined above, agrees with $\succcurlyeq$. □

An example of an ordering $\succcurlyeq$ satisfying C1, C2, C3 but not C4 can be developed from the lexicographic order. Let $\Omega = [0, 1]$, $\mathscr{F}$ be the Borel field of subsets of Ω, L Lebesgue measure, F a measure having a density f with respect to L given by

$$f(x) = x \qquad \text{if } x \in [0, 1].$$

Define an ordering $\succcurlyeq$ between elements G, H of $\mathscr{F}$ as follows:

$$G \succcurlyeq H \quad \text{if} \quad L(G) > L(H) \quad \text{or} \quad L(G) = L(H) \quad \text{and} \quad F(G) \geqq F(H).$$

It is easily verified that $\succcurlyeq$ satisfies C1, C2, C3. The proof that it violates C4 follows from a proof of Debreu that there is no utility function for commodities that are lexicographically preferenced.

Theorem 1 may prove to be of value in axiomatizations of $\succcurlyeq$ when the quotient space $\mathscr{F}/\approx$ is infinite; it is, of course, trivially satisfied when $\mathscr{F}/\approx$ is finite. For example, a somewhat more appealing sufficient condition for the existence of a quantitative probability P is supplied by the following corollary to Theorem 1.

COROLLARY. *If* $\succcurlyeq$ *satisfies* C1, C2, C3, *and* $(\exists\{D_i\})(\forall A < B)(\exists j)(A < D_j < B)$, *then there exists a (not necessarily additive) quantitative probability.*

[Note added in proof. It has been established, and will be proven in Fine (1972), that if $\succcurlyeq$ satisfies C1, C2, C3, and is monotonely continuous $(A_i \uparrow A, (\forall i)A_i \preccurlyeq B \Rightarrow A \preccurlyeq B)$ then there exists P agreeing with $\succcurlyeq$.]

2. Almost uniform partitions and the existence of finitely additive quantitative probability. Following Savage we define an n-fold almost uniform partition $\mathscr{P}_n = \{E_i^{(n)}\}$ of Ω to be a partition of Ω such that for all $k(1 \leqq k < n)$ the union of no k of the $\{E_i^{(n)}\}$ is more probable than the union of any $(k+1)$ of the $\{E_i^{(n)}\}$. Introduce axiom

C5. $(\forall n)(\exists \mathscr{P}_n)$.

Concerning C5, Savage has proven the following

THEOREM. *If* $\succcurlyeq$ *satisfies* C1, C2, C3, C5, *then there exists P additive and such that*

$$A \succcurlyeq B \Rightarrow P(A) \geqq P(B). \qquad (\textit{Almost agreement.})$$

Furthermore, $(\forall 0 \leqq \alpha \leqq 1)(\forall B)(\exists A \subseteq B)(P(A) = \alpha P(B))$.

A connection between agreeing and almost agreeing P is given by

THEOREM 2. *If* $\succcurlyeq$ *satisfies* C1, C2, C3, C5, *then there exists P additive and agreeing with* $\succcurlyeq$ *if and only if there exists any agreeing P′ (not necessarily additive.)*

PROOF. The "only if" part is trivial. To verify the "if" part assume P' agrees with $\succsim$. From Savage's theorem we know that there exists P additive and almost agreeing with $\succsim$. Either P agrees with $\succsim$, in which case there is nothing to be proven, or it does not. If P does not agree with $\succsim$ then there exist F, G such that $F \succ G$ but $P(F) = P(G)$. Either $P(F \cup G) < P(\Omega)$ or $P(F \cup G) = P(\Omega)$.

If $P(F \cup G) < P(\Omega)$ then $(\forall 0 < \alpha \leqq 1)(\exists H_\alpha \subseteq \overline{F \cup G})(P(H_\alpha) = \alpha P(\overline{F \cup G}) > 0)$.

We claim that for $\alpha \neq \beta$ $\{A : G \cup H_\alpha \prec A \prec F \cup H_\alpha\} \cap \{A : G \cup H_\beta \prec A \prec F \cup H_\beta\} = \varnothing$.
To verify this claim say $\alpha > \beta$, and hence,

$$P(G \cup H_\alpha) = P(G) + \alpha P(\overline{F \cup G}) > P(G) + \beta P(\overline{F \cup G}).$$
$$= P(F) + \beta P(\overline{F \cup G}) = P(F \cup H_\beta).$$

Since P almost agrees with $\succsim$, $G \cup H_\alpha \succ F \cup H_\beta$. Hence, in terms of the agreeing P' we find that $\alpha > \beta \Rightarrow P'(G \cup H_\beta) < P'(F \cup H_\beta) < P'(G \cup H_\alpha) < P'(F \cup H_\alpha)$. Corresponding to each α there is a non-void interval $(P'(G \cup H_\alpha), P'(F \cup H_\alpha))$ with disjoint intervals for unequal α. However, there are uncountably many values of α but only countably many disjoint, non-void, open intervals and we have reached a contradiction. Hence, if $P(F \cup G) < P(\Omega)$, then $P(F) = P(G) \Rightarrow F \approx G$.

To complete the proof we need to treat the case of $P(F \cup G) = P(\Omega)$, $F \succ G$, and $P(F) = P(G)$. This case can be dealt with by reducing it to the previous case as follows. Replace F, G by $A = F - (F \cap G)$, $B = G - (F \cap G)$, where now $P(A) = P(B)$ and $A \succ B$. If $P(A \cup B) < P(\Omega)$ then we have just proven that $A \approx B$ and this is a desired contradiction. If $P(A \cup B) = P(\Omega)$, then by Savage's theorem there exist $A' \subset A$, $B' \subset B$ such that $P(A') = P(B') = \frac{1}{2}P(A)$. Since $(A - A') \cup A' \succ (B - B') \cup B'$, at least one of $(A - A')$, A' is more probable than one of $(B - B')$, B', say $A' \succ B'$. However, $P(A' \cup B') = \frac{1}{2}P(A \cup B) < P(\Omega)$. Hence, by our preceding discussion $A' \approx B'$, and we again reach the desired contradiction.

Thus, we have shown that if $P(F) = P(G)$ then $F \approx G$; the almost agreeing, additive P in fact agrees with $\succsim$. ∎

COROLLARY. *If $\succ$ satisfies* C1, C2, C3, C4, C5, *then there exists an additive quantitative probability.*

PROOF. Immediate from Theorem 1, 2, and Savage's theorem. ∎

Theorem 2 in combination with Savage's theorem informs us as to the most that can be expected from an hypothesis about almost uniform partitions. Any agreeing additive P satisfying C5 must have the strong property that $(\forall B)(\forall 0 \leqq \alpha \leqq 1)(\exists A \subseteq B)(P(A) = \alpha P(B))$. While C4, C5, are implied by Savage's hypotheses that $\succsim$ be fine and tight, and, therefore, are at least as general, they in turn imply sufficient conditions for the existence of additive, quantitative probability proposed by Luce. Furthermore, since Luce's axioms hold in certain instances where $\mathscr{F}/\approx$ is finite, the converse implication does not hold. Thus we have shown that, despite the intuitive appeal of the almost uniform partition hypothesis, any theory based upon C5 will be but a special case of Luce's theory.

3. The existence of countably additive quantitative probability. A necessary condition for the existence of a countably additive quantitative probability is stated in axiom

C6. $(\forall\{B_i\})((B_i \supseteq B_{i+1}, \bigcap_{i=1}^{\infty} B_i = \varnothing) \Rightarrow \bigcap_{i=1}^{\infty} \{A: \varnothing < A \preccurlyeq B_i\} = \varnothing)$.

To substantiate the value of C6 we assert

THEOREM 3. *If P is a finitely additive quantitative probability, then P is countably additive if and only if $\succcurlyeq$ satisfies* C6.

PROOF. To verify "only if" assume P is countably additive and that $\{B_i\}$ is such that $B_i \supseteq B_{i+1}$, $\bigcap_i B_i = \varnothing$. If $\exists A \in \bigcap_i \{A: \varnothing < A \leqq B_i\}$, then $(\forall i)(P(B_i) \geqq P(A) > P(\varnothing))$. However, by the well-known continuity equivalence to countable additivity, $\lim_{i\to\infty} P(B_i) = 0$, and this contradicts $P(A) > 0$. Hence

$$\bigcap_i \{A: \varnothing < A \leqq B_i\} = \varnothing,$$

as claimed.

To verify "if" assume, to the contrary of our expectations, that $\exists\{B_i\}$, $B_i \supseteq B_{i+1}$, $\bigcap_{i=1}^{\infty} B_i = \varnothing$, $\bigcap_i \{A: \varnothing < A \leqq B_i\} = \varnothing$ yet $\lim_{i\to\infty} P(B_i) = \varepsilon > 0$. Hence there is no A for which $\varepsilon > P(A) > 0$; if there were such an A then $(\forall i)(\varnothing < A < B_i)$ in contradiction to $\bigcap_i \{A: \varnothing < A \leqq B_i\} = \varnothing$. It follows from $B_i \downarrow \varnothing$, $P(B_i) \downarrow \varepsilon$ that if for some n, $\delta \leqq \varepsilon$, $\varepsilon + \delta > P(B_n) > \varepsilon$ then $(\exists k > n)$ $(\delta \geqq P(B_n - B_k) > 0)$. This, however, contradicts $(\nexists A)(\varepsilon > P(A) > 0)$. Hence, either $P(B_i) \geqq 2\varepsilon$ or $P(B_i) = \varepsilon$, and it follows from $P(B_i) \downarrow \varepsilon$ that $(\exists j)(P(B_j) = \varepsilon)$. However, this shows that $(\forall i)(B_j \in \{A: \varnothing < A \leqq B_i\})$ in contradiction to $\bigcap_{i=1}^{\infty}\{A: \varnothing < A \leqq B_i\} = \varnothing$. The contradiction can only be avoided by rejecting the tentative hypothesis that $\varepsilon > 0$. Hence, P is continuous; and, therefore, countably additive. □

Villegas has proposed axioms for countable additivity that correspond to C1–C4, a strengthened form of C6, and the hypothesis that there are no atoms (i.e., $(\forall A > \varnothing)(\exists B \subset A)(\varnothing \prec B \prec A)$. It follows from Villegas' Theorem 5 that his axioms imply C1–C6. Hence, combining our remarks at the end of Section 2 with Theorem 3, we see that Villegas' theory of countable additivity is a special case of the theory we have if we adjoin C6 to Luce's axioms for finite additivity.

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