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THE SERIAL CORRELATION COEFFICIENTS OF WAITING TIMES IN THE STATIONARY GI/M/1 QUEUE[1]

BY A. G. PAKES
Monash University

1. Introduction. The serial correlation coefficients $\{r_n\}$ of a stationary sequence of waiting times in the GI/G/1 queueing system have recently been studied by Daley (1968a) and Blomqvist (1968, 1969). From a practical point of view, knowledge of the properties of $\{r_n\}$ is useful for obtaining the variance of the mean of a sample of waiting times, and thus for obtaining some idea of required sample sizes for estimation and simulation. For example, Blomqvist (1968) has defined, for a stable GI/G/1 system with zero initial waiting time, an estimator for the expected stationary waiting time which is based on a sample of successive waiting times. He shows that the mean square error of this estimator can be expressed in terms of Σr_n. Blomqvist (1969) has given heavy traffic approximations for Σr_n and r_n. The special case of the stationary M/G/1 queue has been treated in some detail by Daley (1968a) and Blomqvist (1967).

In this paper we consider the stationary GI/M/1 queue, thus complementing the work of Daley and Blomqvist, and also of Daley (1968b) and Pakes (1971) where a discussion is given of the serial correlation coefficients of a stationary sequence of queue lengths embedded at the epochs of arrival of successive customers. In Section 3 we evaluate $\{r_n\}$ for the stationary GI/M/1 queue and in Section 4 we discuss heavy traffic approximations.

A quantity related to waiting time is the sojourn, or waiting plus service, time of a customer. In Section 5 we consider $\{\tau_n\}$, the serial correlation coefficients of a stationary sequence of sojourn times in the GI/G/1 queue. Using the results and methods of Section 3, we evaluate $\{\tau_n\}$ for the stationary GI/M/1 queue and thus show the equality of this sequence and the sequence of correlation coefficients of a stationary sequence of queue lengths embedded at arrival epochs.

2. Notation. We consider a GI/G/1 queue where the nth arriving customer is denoted by $C_n (n = 0, 1, \cdots)$, T_n is the interarrival time of C_n and C_{n+1}, and S_n, W_n and $V_n = W_n + S_n$ are the service, waiting and sojourn times of C_n, respectively. For $n = 0, 1, \cdots$, we let $A(x) = \Pr\{T_n \leqq x\}$ and $B(x) = \Pr\{S_n \leqq x\}$ $(x \geqq 0)$ with $A(0+) = B(0+) = 0$. We assume that $\{S_n\}$ and $\{T_n\}$ are independent sequences of mutually independent random variables and we put $U_n = S_n - T_n$ with $U(x) = \Pr(U_n \leqq x)$ $(-\infty < x < \infty)$. We denote the moments of the inter-arrival times by $\lambda_r = E(T_n^r)$ and of the service times by $\mu_r = E(S_n^r)$ $(r = 1, 2, \cdots)$, when they exist. We always assume $\lambda_1, \mu_1 < \infty$.

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The distribution functions of the sojourn times and waiting times will be denoted by $F_n(x) = \Pr(V_n \leqq x)$ and $G_n(x) = \Pr\{W_n \leqq x\}$, respectively. When $E(U_n) < 0$ it is well known that $F(x) = \lim_{n\to\infty} F_n(x)$ and $G(x) = \lim_{n\to\infty} G_n(x)$ exist and are proper distribution functions. Moreover, they define stationary distributions of the sojourn time and waiting time processes respectively. When they exist, we denote the moments of these distributions by $v_r = \int_0^\infty x^r dF(x)$ and $w_r = \int_0^\infty x^r dG(x)$ $(r = 1, 2, \cdots)$. Given $\lambda_1 < \infty$, a necessary and sufficient condition for the finiteness of v_r and w_r is that $\mu_{r+1} < \infty$. Letting $\mu_3 < \infty$, we define $\sigma_v^2 = v_2 - v_1^2$ and $\sigma_w^2 = w_2 - w_1^2$. When $E(U_n) < 0$, we define $\tau_n = [\mathrm{Cov}(V_0, V_n)]/\sigma_v^2$ and $r_n = [\mathrm{Cov}(W_0, W_n)]/\sigma_w^2 (n = 0, 1, \cdots)$ to be the serial correlation coefficients of a stationary sequence of sojourn times and waiting times, respectively. Daley (1968a) has shown that in a stationary GI/G/1 system, $\{r_n\}$ is monotone non-increasing with limit zero and $\sum_{n=0}^\infty r_n < \infty$ iff $\mu_4 < \infty$.

3. The serial correlation coefficients of waiting times in GI/M/1. For the GI/M/1 queue we have $B(x) = 1 - e^{-\mu x} (\mu > 0)$, so that $\mu_1 = 1/\mu$. When $\mu_1 - \lambda_1 < 0$ it is well known (e.g. Takacs (1962)) that the stationary (and limiting) waiting time distribution function is given by $G(x) = 1 - \xi e^{-vx}$ where $\xi = 1 - v/\mu$, v is the unique positive solution of $\mu - v - \mu\alpha(v) = 0$, and $\alpha(\cdot)$ is the Laplace–Stieltjes transform of $A(\cdot)$. With this notation we prove the following theorem.

THEOREM 1. *The generating function* $R(t) = \sum_{n=0}^\infty r_n t^n (|t| < 1)$ *of the serial correlation coefficients* $\{r_n\}$ *of the waiting times in the stationary* GI/M/1 *queue is given by*

$$R(t) = \frac{1}{1-t} - \frac{(1-\xi)t}{(2-\xi)\xi'(1)(1-t)^2} + \frac{(1-\xi)^3 \xi(t)}{\xi^2(2-\xi)(1-t)(1-\xi(t))} + \frac{(1-\xi)^2 t\xi(t)}{\xi(2-\xi)\xi'(1)(1-t)^2(1-\xi(t))} \tag{1}$$

where for $0 \leqq t \leqq 1$ $v(t) = \mu(1-\xi(t))$ *is the unique positive solution of* $\mu - \theta - \mu t\alpha(\theta) = 0$. *The correlation coefficients are given by* $(n = 1, 2, \cdots)$

$$r_n = 1 - \frac{(1-\xi)n}{(2-\xi)\xi'(1)} + \frac{(1-\xi)^3}{\xi^2(2-\xi)} \sum_{k=1}^n B_k + \frac{(1-\xi)^2}{\xi(2-\xi)\xi'(1)} \sum_{k=1}^n (n-k)B_k \tag{2}$$

where

$$B_n = \Pr(W_m \neq 0 (m = 1, 2, \cdots, n), W_0 = 0) = \frac{1}{n!}\left(\frac{d^{n-1}}{dt^{n-1}} \frac{[\alpha(\mu(1-t))]^n}{(1-t)^2}\right)\Bigg|_{t=0}. \tag{3}$$

PROOF. The proof is accomplished by using techniques similar to those of Daley (1968d). Since $\{W_n\}$ is a Markov chain we have

$$E(W_0 W_n) = \int_0^\infty E(W_n \mid W_1 = x) d_x E(W_0; W_1 \leqq x) \qquad (n = 1, 2, \cdots).$$

We evaluate the integrating function by first finding

(4) $$E(e^{-\theta W_0}; W_1 \leqq x) = \int_{W_1 \leqq x} e^{-\theta W_0}\, d\,\Pr$$
$$= \int_0^\infty \int_0^{x+y} [1-\xi+\xi v(1-e^{-(\theta+v)(x+y-z)})/(\theta+v)]\mu e^{-\mu z}\, dz\, dA(y)$$
$$= [(\theta(1-\xi)+v)(1-\alpha(\mu)e^{-\mu x}) - \mu v\xi(\alpha(\theta+v)e^{-(\theta+v)x} - \alpha(\mu)e^{-\mu x})$$
$$\div(\mu - v - \theta)]/(\theta+v).$$

On observing that $\alpha'(v) = -(1-\xi/\xi'(1))/\mu$, $\Pr(W_1 \leqq x \mid W_0 = 0) = 1-e^{-\mu x}\alpha(\mu)$ and $E(W_1; W_1 \leqq x) = \xi[(1-e^{-vx})/v - xe^{-vx}]$, differentiation of (4) yields

$$E(W_0; W_1 \leqq x) = E(W_1; W_1 \leqq x) + ((1-\xi)/\mu\xi)\Pr(W_1 \leqq x \mid W_0 = 0)$$
$$-(\Pr(W_1 \leqq x))/\mu\xi'(1) - [((1-\xi)/\xi - 1/\xi'(1))/\mu]H(x)$$

where $H(x)$ is the unit step function with its jump at $x = 0$. Thus, by stationarity, we obtain

$$E(W_0 W_n) - E(W_0 W_{n-1}) = ((1-\xi)/\mu\xi)(E(W_n - W_{n-1} \mid W_0 = 0))$$
$$+(E(W_{n-1} \mid W_0 = 0) - E(W))/\mu\xi'(1) \qquad (n = 1, 2, \cdots).$$

Equation (1) now follows by taking generating functions and noting that $\sigma_w^2 = (2\xi - \xi^2)/v^2$ and that the generating function of $\{E(W_n \mid W_0 = 0)\}$ may be found explicitly (e.g. from Takacs (1962) page 121 (28)).

Lagrange's theorem on the reversion of power series shows that $\xi(t)/(1-\xi(t)) = \sum_{n=1}^\infty B_n t^n$ where B_n is given by the right-hand member of (3), and the first equality follows from Pakes (1971) (last equation in Section 5). Equation (2) now follows from (1), and the proof is complete.

We shall see below that for a stationary GI/M/1 queue $\{\tau_n\}$ is a convex sequence and it is known (Daley (1968a)) that for a stationary M/G/1 system $\{r_n\}$ is convex. However it is not clear from (2) whether $\{r_n\}$ is convex for the stationary GI/M/1 queue. Similarly, it is unclear whether, or not, $\{\tau_n\}$, for the stationary M/G/1 system, is convex.

4. Estimation and heavy traffic approximations. In order to estimate the mean of a stationary stochastic process such as is the waiting time process $\{W_n\}$, given a sample of observations $(W_1, \cdots, W_N)$, we form the unbiased estimator $m = (\sum_{n=1}^N W_n)/N$ whose variance is given exactly by

$$\operatorname{Var}(m) = \sigma_w^2\{N + 2\textstyle\sum_{n=1}^{N-1}(N-n)r_n\}/N^2.$$

If Σr_n converges, the variance of the sample mean is, when N is large, asymptotically equal to

(5) $$\sigma_w^2\{1 + 2\textstyle\sum_{n=1}^\infty r_n\}/N.$$

Thus for the purposes of estimating w_1 in this way, we are interested in the convergence of Σr_n, and it follows from the nonnegativity of r_n and Abel's theorem that $\sum_{n=0}^\infty r_n < \infty$ iff $\lim_{t\uparrow 1} R(t) < \infty$, and the two limits are equal.

LEMMA 1. *For the stationary* GI/M/1 *queue,* $\sum_{n=0}^{\infty} r_n < \infty$ *and is given by*

$$(6)\qquad \sum_{n=0}^{\infty} r_n = [1/\xi + \xi'(1)/\xi(1-\xi) - \xi'(1)(1-\xi)/\xi^2 + \xi''(1)/2\xi\xi'(1)]/(2-\xi)$$
$$= 1/(1-\xi)(1+\mu\alpha') + \mu^2\alpha''/2(2-\xi)(1+\mu\alpha')^2$$

where α' and α'' *are the first and second derivatives of* $\alpha(\theta)$ *respectively, evaluated at* $\mu(1-\xi)$.

PROOF. That the series does converge follows from Daley's result mentioned at the end of Section 2, and a two-fold application of L'Hospital's rule to (1) gives the first equality in (6). Now $\xi(t)$ satisfies $\xi(t) = t\alpha[\mu(1-\xi(t))]$ and this equation can be used to eliminate $\xi'(1)$ and $\xi''(1)$ in (6) to give the second equality.

Let $\rho = 1/\mu\lambda_1 < 1$ be the traffic intensity. Under actual operating conditions interest often attaches to the case in which ρ approaches unity from below, the well-known condition of heavy traffic. We shall now obtain the heavy traffic forms of expressions (6) and (2).

LEMMA 2. *For the stationary* GI/M/1 *queueing system with fixed interarrival time distribution function and* $\lambda_2 < \infty$,

$$(7)\qquad \sum_{n=0}^{\infty} r_n = [\lambda_2/\lambda_1{}^2 + o(1)]/(1-\rho)^2 \qquad (\rho\uparrow 1).$$

REMARKS. (i) Strictly speaking, the result is to be proved for a family of stationary GI/M/1 systems with common $A(\cdot)$ and with μ such that $0 < 1-\rho \ll 1$.

(ii) The lemma shows that to achieve reasonable sampling accuracy in estimating the mean waiting time, the sample size, N, should satisfy $N \approx K(1-\rho)^{-2}$ for some constant K, when ρ is near one.

(iii) The lemma is contained in Corollary 2.1 of Blomqvist (1969) but his result (which is for a stationary GI/G/1 system) requires $\lambda_4 < \infty$. If we make this assumption here then we can show that $\sum_{n=1}^{\infty} r_n = a_1/(1-\rho)^2 + a_2/(1-\rho) + O(1)$ $(\rho\uparrow 1)$ where $a_1 = \lambda_2/\lambda_1{}^2$ and a_2 is a function of λ_1, λ_2 and λ_3. Daley (1968a) has obtained a similar expansion for a family of stationary M/G/1 systems having common service time distribution functions with $\mu_4 < \infty$. The leading term of his expansion is $(\mu_2/\mu_1{}^2)/(1-\rho)^2$.

PROOF. Noting that $\xi \to 1$ as $\rho\uparrow 1$ and substituting the expressions

$$(8)\qquad 1-\xi = 2(1-\rho)/\rho\mu^2\alpha''(\theta_1) \qquad (0 < \theta_1 < \mu(1-\xi))$$
$$1+\mu\alpha' = (1-\rho)[2\alpha''(\theta_2)/\alpha''(\theta_1) - 1]/\rho \qquad (0 < \theta_2 < \mu(1-\xi))$$

into (6) yields (7). Expressions (8) are derived using Taylor's theorem; see Daley (1968b) for the details.

THEOREM 2. *For a family of* GI/M/1 *queues with fixed interarrival time distribution functions such that* $\lambda_3 < \infty$, *and with service distribution function parameter* μ *such* *that* $0 < 1-\rho < \delta < 1$, *the serial correlation coefficients*, r_n, *of a stationary sequence of waiting times satisfy*

(9) $$|1-\Omega n(1-\rho)^2 - r_n| < K(1-\rho)^3$$

for any finite set of integers $n = 1, 2, \cdots, N$, *where* $\Omega = 2\lambda_1{}^2/\lambda_2$ *and* K *is a constant depending on* N.

REMARKS. Blomqvist (1969) has obtained a theorem for the stationary GI/G/1 queue which shows that, under our conditions, $1-\Omega n(1-\rho)^2 - r_n = o[(1-\rho)^2]$ $(\rho\uparrow 1)$ for each fixed positive integer n. In our case we only need $\lambda_2 < \infty$ for this result to hold.

Daley (1968a) has obtained a result of the form (9) for the stationary M/G/1 system with $\mu_3 < \infty$ and $\Omega = 2\mu_1{}^2/\mu_2$.

PROOF. The first part of (8) can be cast into the form

$$1-\xi = \Omega(1-\rho) + 2\lambda_1{}^2(1-\rho)(\lambda_2 - \alpha''(\theta_1))/\lambda_2\alpha''(\theta_1) + O[(1-\rho)^2]$$

where $0 < \theta_1 < \mu(1-\xi)$. Thus we have

$$0 \leqq \lambda_2 - \alpha''(\theta_1) \leqq \int_0^\infty x^2(1-e^{-\mu(1-\xi)x})\,dA(x) \leqq \mu(1-\xi)\int_0^\infty x^3\,dA(x).$$

The right-hand side is $O(1-\rho)$ by virtue of (8) and $\lambda_3 < \infty$. Hence

(10) $$1-\xi = \Omega(1-\rho) + O[(1-\rho)^2]$$

and a similar argument shows that

(11) $$1/\xi'(1) = (1+\mu\alpha')/\xi = 1-\rho + O[(1-\rho)^2].$$

Since $\alpha(\mu(1-x))$ is a regular function of x in $|x| < \varepsilon < 1$ and a continuous function of ρ in a closed interval containing $\{0 \leqq 1-\rho < \delta < 1\}$, the theorem follows on substituting (10) and (11) into (2) and noting that the summations therein form a finite set of finite sums of finite order derivatives of $\alpha(\mu(1-x))/(1-x)^2$ and are thus uniformly bounded in the set $\{x = 0; 0 \leqq 1-\rho \leqq \delta < 1\}$.

5. Sojourn times. From the definition of the sojourn time, V_n, of C_n in a GI/G/1 queue, it is easily seen that $V_{n+1} = (V_n - T_n)^+ + S_{n+1}$ and our assumptions imply that $\{V_n; n = 0, 1, \cdots\}$ is a Markov chain defined on the nonnegative real line. The transition probability of this Markov chain is given by

$$\begin{aligned}\Pr\{V_{n+1} \leqq x \mid V_n = y\} &= \int_0^x \Pr\{(y-T_n)^+ \leqq x-z\}\,dB(z) \\ &= B(x) - \int_0^x A(z-x+y-0)\,dB(z) \quad (n = 0, 1, \cdots; x, y \geqq 0).\end{aligned}$$

For each fixed $x \geqq 0$, this function is a non-increasing function of y, thus the Markov chain $\{V_n\}$ is stochastically monotone and a result of Daley (1968c) allows us to assert

THEOREM 3. *In the queue* GI/G/1 *with* $\mu_3 < \infty$ *and* $E(U_n) < 0$, *the serial correlation coefficients*, $\{\tau_n\}$, *of a stationary sequence of sojourn times, decrease monotonically to zero.*

Since S_n is independent of S_0, W_0 and W_n we have

$$\operatorname{Cov}(V_0, V_n) = \operatorname{Cov}(W_0, W_n) + \operatorname{Cov}(S_0, W_n) \qquad (n = 1, 2, \cdots). \tag{12}$$

For a stationary GI/G/1 queue define the functions $C_n(x) = E(S_0; W_n \leqq x) = \int_{W_n \leqq x} S_0 \, d\Pr(n = 1, 2, \cdots; x \geqq 0)$ so that

$$E(S_0 W_n) = \int_0^\infty x \, dC_n(x) = \int_0^\infty (\mu_1 - C_n(x)) \, dx \tag{13}$$ and

$$C_{n+1}(x) = \int_{0-}^\infty U(x-y) \, dC_n(y) = \int_{-\infty}^x C_n(x-y) \, dU(y).$$

Letting $\chi\{A\}$ be the indicator function of the set A, we have

$$\begin{aligned} C_1(x) &= E(S_0 \chi\{(W_0 + S_0 - T_0)^+ \leqq x\}) \\ &= E(E(S_0 \chi\{(W_0 + S_0 - T_0)^+ \leqq x\} \mid W_0, T_0)) \\ &\leqq E(E(S_0) E(\chi\{(W_0 + S_0 - T_0)^+ \leqq x\} \mid W_0, T_0)) = \mu_1 G(x) \end{aligned}$$

since $\chi\{(W_0 + S_0 - T_0)^+ \leqq x\}$ is a non-increasing function of S_0 and Gurland's (1967) lemma has been applied; see also Daley (1968d). This bound and stationarity together with (13) imply that $C_n(x) \leqq \mu_1 G(x)$ ($x \geqq 0$; $n = 1, 2, \cdots$) which with the observation that $C_n(x) \uparrow \mu_1 (x \to \infty)$, implies that

$$\operatorname{Cov}(S_0, W_n) = \int_0^\infty (\mu_1 G(x) - C_n(x)) \, dx \geqq 0,$$

and thus $\operatorname{Cov}(V_0, V_n) \geqq \operatorname{Cov}(W_0, W_n)$. This, together with Daley's theorem mentioned at the end of Section 2, proves the first half of

THEOREM 4. *If* $\{V_n\}$ *is a stationary sequence of sojourn times of the* GI/G/1 *queue with* μ_3, $\lambda_1 < \infty$ *and* $E(U_n) < 0$, *then the necessary and sufficient condition that* $\sum_{n=0}^\infty \operatorname{Cov}(V_0, V_n) < \infty$ *is that* $\mu_4 < \infty$.

PROOF OF SUFFICIENCY. In view of Daley's theorem it suffices to prove that $\sum_{n=1}^\infty \operatorname{Cov}(S_0, W_n) < \infty$ if $\mu_3 < \infty$. Define the functions

$$R_n(y) = a\rho_n(y) = \int_0^y (\mu_1 G(x) - C_n(x)) \, dx \qquad (y \geqq 0; n = 1, 2, \cdots)$$

where $a = \sigma_w (\operatorname{Var} S_0)^{\frac{1}{2}} > 0$. The integrand is nonnegative and so for each $n = 1, 2, \cdots$, $\rho_n(\cdot)$ is a continuous, non-decreasing and nonnegative function with $\rho_n(0) = 0$ and $\lim_{y \to \infty} \rho_n(y) = (\operatorname{Cov}(S_0, W_n))/a \leqq 1$. Thus each $\rho_n(\cdot)$ is the distribution function of a (possibly defective) random variable. By using substantially the same argument as in the proof of Theorem 2 of Daley (1968a) we can show that

$$\sum_{n=2}^\infty \operatorname{Cov}(S_0, W_n) = \int_0^\infty H(x) \, dR_1(x)$$

where $H(x)$ is the unique nonnegative renewal function satisfying

$$H(x) = V(x) + \int_{-\infty}^{x} H(x-y)\,dV(y) \quad (x \geqq 0),\ V(x) = 1 - U(-x-0)$$

and which also satisfies the inequality

$$cx/2 + K_1 \leqq H(x) \leqq 2cx + K_2 \qquad (x \geqq 0)$$

where K_1 and K_2 are finite constants and $1/c = E(-U_n) > 0$. Thus we see that $\sum_{n=1}^{\infty} \mathrm{Cov}(S_0, W_n)$ converges or diverges with $\int_0^\infty x\,dR_1(x)$.

Now by definition

$$\begin{aligned} C_1(x) = E(S_0;(W_0+S_0)^+ \leqq x) &= \iiint_{(u+y-v)^+ \leqq x;u,v,y \geqq 0} y\,dB(y)\,dG(u)\,dA(v) \\ &= \int_0^\infty \int_0^\infty yG(x+v-y)\,dA(v)\,dB(y) \\ &\qquad\qquad \text{(using Fubini's theorem)} \\ &\geqq \int_0^\infty yG(x-y)\,dB(y). \end{aligned}$$

This gives

$$\begin{aligned} \int_0^\infty x\,dR_1(x) &\leqq \int_0^\infty x[\mu_1 G(x) - \int_0^\infty yG(x-y)\,dB(y)]\,dx \\ &= \int_0^\infty x \int_0^\infty y(G(x)-G(x-y))\,dB(y)\,dx. \end{aligned}$$

Use of Fubini's theorem shows this to be

$$\begin{aligned} \int_0^\infty y \int_0^\infty x(G(x)-G(x-y))\,dx\,dB(y) &= \int_0^\infty (w_1 y^2 + y^3/2)\,dB(y) \\ &= w_1\mu_2 + \mu_3/2 \end{aligned}$$

and this completes the proof.

By using the representation given above for $\sum_{n=2}^{\infty} \mathrm{Cov}\,(S_0, W_n)$ it can be shown that under certain conditions (see Blomqvist (1969) Theorem 2) $\sum_{n=0}^{\infty} \mathrm{Cov}(V_0, V_n) \sim \sum_{n=0}^{\infty} \mathrm{Cov}(W_0, W_n)$ as $E(U_n) \uparrow 0$. A similar argument to that which leads to Theorem 6 in Blomqvist (1969) enables us to state that

$$\sigma_v^2 - \mathrm{Cov}(V_0, V_n) + nw_1 E(U_n) = o(1) \qquad (E(U_m) \uparrow 0)$$

for each fixed n, provided certain extra conditions are satisfied and for which we refer to the paper cited above.

Consider now the stationary GI/M/1 queue. In view of (12) and Theorem 1, it is only necessary to find $\mathrm{Cov}\,(S_0, W_n)$ in order to calculate $\mathrm{Cov}\,(V_0, V_n)$. Since $\{W_n\}$ is a Markov chain,

$$E(S_0 W_n) = \int_{0-}^{\infty} E(W_n \mid W_1 = x)\,d_x E(S_0; W_1 \leqq x) \qquad (n = 1, 2, \cdots).$$

Using the notation of Section 2, we have

$$\begin{aligned} E(e^{-\theta S_0}; W_1 \leqq x) &= \int_0^\infty \int_0^{x+y} e^{-(\theta+\mu)z}(1-\xi e^{-\nu(x+y-z)})\,dz\,dA(y) \\ &= \frac{\mu}{\theta+\mu} - \frac{\mu\xi^2}{\theta+\mu\xi} e^{-\nu x} - \frac{\theta\nu\alpha(\theta+\mu)}{(\theta+\mu)(\theta+\mu\xi)} e^{-(\theta+\mu)x}. \end{aligned}$$

Differentiation yields

$$E(S_0; W_1 \leqq x) = [1-\xi e^{-\nu x}-(1-\xi)(1-\alpha(\mu)e^{-\mu x})]/\mu\xi$$
$$= [\Pr(W_1 \leqq x)-(1-\xi)\Pr(W_1 \leqq x \mid W_0 = 0)]/\mu\xi$$

and so

$$E(S_0 W_n) = [w_1-(1-\xi)E(W_n \mid W_0 = 0)]/\mu\xi.$$

(Since $\{E(W_n \mid W_0 = 0)\}$ is monotone non-decreasing we see that $\{\text{Cov}(S_0, W_n); n = 1, 2, \cdots\}$ is a non-increasing sequence. This property holds for any stationary GI/G/1 queue (with $\mu_3 < \infty$) as can be seen by applying Gurland's lemma to $S_0((W_n+U_n)^+ - W_n)$). This expression yields

$$\sum_{n=1}^{\infty} \text{Cov}(S_0, W_n)t^n = [t-(1-\xi)\xi(t)/\xi(1-\xi(t))]/\mu^2(1-t)$$

and combining this with the generating function of $\{\text{Cov}(W_0, W_n)\}$ through (12) and recalling that $\sigma_v{}^2 = 1/(\mu(1-\xi))^2$ leads to an expression for $\sum_{n=0}^{\infty} \tau_n t^n$ which is identical to the generating function of the serial correlation coefficients of a stationary sequence of queue lengths embedded at arrival epochs (for example see Daley (1968b) or Pakes (1971)).

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