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DIFFUSION APPROXIMATIONS OF BRANCHING PROCESSES[1]

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For $n = 1, 2, \cdots$ let $z_t^{(n)}$, $t \geqq 0$, be an age-dependent branching process starting from n ancestors. Suppose it has the reproduction generating function f_n, $f_n'(1) = 1+\alpha/n+o(n^{-1})$, $f_n''(1) = 2\beta_n \to 2\beta$, $f_n'''(1-) \leqq$ some constant, and the life-length distribution L with $L(0) = 0$ and $\lambda = \int_0^\infty tL(dt) < \infty$. Then, it is shown that the finite dimensional distributions of $n^{-1}z_{nt}^{(n)}$ converge, as $n \to \infty$, to the corresponding laws of the diffusion $t \to x_t$ with drift $(\alpha/\lambda)x$ and infinitesimal variance $(2\beta/\lambda)x$.

1. Introduction and summary. Let x_t, $t \geqq 0$, be a one-dimensional diffusion with drift αx and infinitesimal variance $2\beta x$, $\alpha \in R$, $\beta > 0$, $x \geqq 0$, describing the growth of a large population with independent individuals. Feller (1951) sketched how this process might appear as the limit of a sequence of Galton-Watson processes, where the nth population has n ancestors and is measured in units of n individuals; the nth time unit equals n time units of the first process; the number of offspring per individual in the nth process has expectation $1+\alpha/n+o(n^{-1})$, a finite variance $2\beta_n$ converging to 2β, and a third moment bounded in n. The rigorous formulation and proof of this fact are due to Jiřina (1969). More general problems of Galton-Watson processes with transformed times and states have been considered by Lamperti in a sequence of papers. But the Feller–Jiřina scheme is attractive in yielding a natural and explicit limit process.

We shall generalize it to age-dependent branching processes. Suppose that f_n, $n \in N$, is a sequence of generating functions of probability measures on the nonnegative integers, N, satisfying $m_n = f_n(1) = 1+\alpha/n+o(n^{-1})$, $2\beta_n = f_n''(1) \to 2\beta$, $f_n'''(1-) \leqq c < \infty$, $\alpha \in R$, $\beta > 0$, $c > 0$. Let L be a probability distribution on the nonnegative reals, R_+, with $L(0) = 0$ and $\lambda = \int_0^\infty tL(dt) < \infty$. All reproduction generating functions denoted by f_n or f are assumed nonlinear. Denote by $z_t(n)$, $t \in R_+$, $n \in N$ an age-dependent branching process with off-spring generating function f_n and life-length distribution L, started from n ancestors at time zero. We shall prove that, for any t, $x_n(t) = n^{-1}z_{nt}(n)$ converges, as $n \to \infty$, in distribution to the value x_t at time t of a diffusion with drift $\alpha x/\lambda$ and infinitesimal variance $2\beta x/\lambda$. It will be clear from the proof that the condition on $\{f_n'''(1-)\}$ may be relaxed.

The approach is the following: If $F_n(s, t)$ is the generating function of a branching process defined by f_n and L but with one ancestor—an (f_n, L) process in Sevastyanov's terminology—then $x_n(t)$ has the generating function $F_n^{\,n}(s^{1/n}, nt)$, $s \in [0, 1]$. But

$$\lim_{n\to\infty} F_n^{\,n}(s^{1/n}, nt) = \exp - a(s, t)$$

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if (and only if)

$$\lim_{n\to\infty} n[1-F_n(s^{1/n}, nt)] = a(s,t).$$

And this is exactly what we shall show with

$$a(s,t) = \frac{\sigma e^{\alpha t/\lambda}}{1+\sigma(\beta/\alpha)(e^{\alpha t/\lambda}-1)} \qquad \text{if} \quad \alpha \neq 0,$$

$$\frac{\sigma}{1+\sigma\beta t/\lambda} \qquad \text{if} \quad \alpha = 0.$$

Here $\sigma = -\log s$, $s \in (0, 1]$, and $\exp - a(s,t)$ is the generating function of the diffusion with the stated drift and variance [see Jiřina (1969)]. The convergence in distribution then follows from the continuity theorem for Laplace transforms.

The method consists in a study of the basic integral equation of (f, L) processes:

$$F(s,t) = s[1-L(t)] + \int_0^t f \circ F(s, t-y) L(dy) \tag{1}$$

by means of a Taylor expansion. Once the convergence of $x_n(t)$ is established, a recursive argument applied to the corresponding equation for

$$F^{(k)}(s_1, \cdots, s_k; t_1, \cdots, t_k) = E[s_1^{z_{t_1}} \cdots s_k^{z_{t_k}}],\ s_i \in [0, 1],\ t_i \in R_+,\ 1 \leqq i \leqq k,\ z_t$$

an (f, L) process, namely

$$\begin{aligned} F^{(k)}(s_1, \cdots, s_k, t_1, \cdots, t_k) &= s_1 \cdots s_k[1-L(t_k)] \\ &+ s_1 \cdots s_{k-1} \int_{t_{k-1}}^{t_k} f \circ F(s_k, t_k - y) L(dy) \\ &+ s_1 = s_{k-2} \int_{t_{k-2}}^{t_{k-1}} f \circ F^{(2)}(s_{k-1}, s_k; t_{k-1} - y, t_k - y) L(dy) + \cdots + \\ &+ \int_0^{t_1} f \circ F^{(k)}(s_1, \cdots, s_k; t_1 - y, \cdots, t_k - y) L(dy), \end{aligned}$$

would yield the convergence of all finite-dimensional distributions of $x_n(t)$. This, however, involves lengthy calculations and is omitted.

It is easy to give the sample space of (suitably normalized) branching processes the Skorohod J_1-topology: if the process is not supercritical, define its Malthusian parameter, μ, to equal zero and consider for any branching process z_t the process $w_t = e^{-\mu t} z_t$. This is a right continuous process with left limits at any point and $\lim_{t\to\infty} w_t$ exists almost surely under simple conditions (Jagers (1968)). Hence, $w_{\tan \pi t/2}$, $0 \leqq t \leqq 1$, is a random element of $D[0, 1]$. But we have not been able to find any neat tightness conditions in terms of f_n and L.

2. Some simple properties of branching processes.

PROPOSITION 2.1. *Let q be the extinction probability of an (f, L) process with generating function F. Then, for $0 \leqq s \leqq q$, $s \leqq F(s,t) \leqq q$, and for $q \leqq s \leqq 1$, $q \leqq F(s,t) \leqq s$.*

PROOF. Suppose that $0 \leqq s \leqq q$ and take $\varepsilon > 0$. Set $t_0 = \inf\{t; F(s,t) \leqq s-\varepsilon\}$. We wish to prove that $t_0 = \infty$, i.e. $F(s,t) > s-\varepsilon$ for all t. Since $L(t) = 0$ implies that $F(s, t) = s$, and $F(s,\cdot)$ is right continuous, then $L(t_0) > 0$. But if $t_0 < \infty$,

$$\begin{aligned} s-\varepsilon \geqq F(s,t_0) &= s[1-L(t_0)]+\textstyle\int_0^{t_0} f\circ F(s,t_0-y)L(dy) \\ &> s[1-L(t_0)]+f(s-\varepsilon)L(t_0) \\ &> s[1-L(t_0)]+(s-\varepsilon)L(t_0) \geqq s-\varepsilon. \end{aligned}$$

This contradiction for all $\varepsilon > 0$ shows that there is no t such that $F(s, t) < s$. On the other hand, if $t_1 = \inf\{t; F(s,t) \geqq q\}$ and $s < q$, then $L(t_1) > 0$ and

$$q \leqq F(s,t_1) < s[1-L(t_1)]+qL(t_1) \leqq q$$

showing that $0 \leqq F(s, t) < q$ if $0 \leqq s < q$. Since the basic integral equation has only one solution between zero and one, it is evident that $F(q, t) = q$ identically. And for $s \geqq q$, $F(s, t) \geqq F(q, t) = q$, whereas an argument like the one given yields $F(s, t) \leqq s$.

PROPOSITION 2.2. *For any (f, L) process, $F(s,\cdot)$ is nondecreasing if $0 \leqq s \leqq q$ and nonincreasing if $s \geqq q$.*

PROOF. Fix $s \leqq q$ and put $M(u) = \sup_{0\leqq t\leqq u} F(s, t)$.

$$\begin{aligned} F(s,t) &= s+\textstyle\int_0^t [f\circ F(s,t-y)-s]L(dy) \\ &\leqq s+\textstyle\int_0^t [f\circ M(u-y)-s]L(dy) \leqq s+\int_0^u [f\circ M(u-y)-s]L(dy) \end{aligned}$$

for $0 \leqq t \leqq u$, since $f\circ M(u-y) \geqq f\circ F(s, t-y) \geqq f(s) \geqq s$. Hence

$$M(u) \leqq s[1-L(u)]+\textstyle\int_0^u f\circ M(u-y)L(dy).$$

Define for $n \in N$ $\varphi_n: R_+ \to [0, 1]$ by

$$\varphi_0 = 1$$

$$\varphi_{n+1}(t) = s[1-L(t)]+\textstyle\int_0^t f\circ\varphi_n(t-y)L(dy).$$

By induction $M \leqq \varphi_n$. But $\varphi_n \downarrow F(s,\cdot)$ [2, p. 132]. Thus $M = F(s,\cdot)$.

For $s \geqq q$ the same reasoning applied to $I(u) = \inf_{0\leqq t\leqq u} F(s, t)$ and a sequence ψ_n with $\psi_0 = 0$ yields the proposition.

PROPOSITION 2.3. *If, for $\alpha \geqq 0$, q_n is the smallest nonnegative root of $f_n(x) = x$, then*

$$q_n = 1-\alpha/\beta n+o(n^{-1}), \qquad \text{as } n\to\infty.$$

The proof is left for the reader.

3. The convergence of generating functions. We start from the basic integral equation for (f_n, L) processes,

$$F_n(s,t) = s[1-L(t)]+\textstyle\int_0^t f_n\circ F_n(s,t-y)L(dy).$$

Fix $s \in (0, 1)$ and set $g_n(t) = h[1-F_n(s^{1/n}, nt)]$, $t \in R_+$, $\sigma_n = n(1-s^{1/n})$. Expanding f_n around 1 gives

$$g_n(t) = \sigma_n[1-L(nt)] + m_n \int_0^t g_n(t-y)L(n\,dy)$$
$$- \beta_n/n \int_0^t g_n^2(t-y)L(n\,dy) + n\int_0^t r_n \circ g_n(t-y)L(n\,dy),$$

where $|r_n(x)| \leqq c(x/n)^3$, $x \geqq 0$. Take Laplace-Stieltjes transforms (denoted by circumflexes) of this:

$$\hat{g}_n(z) = \sigma_n[1-\hat{L}(z/n)] + m_n\hat{g}_n(z)\hat{L}(z/n) - \beta_n/n\widehat{g_n^2}(z)\hat{L}(z/n) + n(\widehat{r_n \circ g_n})(z)\hat{L}(z/n),$$

$z > 0$. Then,

$$\beta_n\hat{L}(z/n)\widehat{g_n^2}(z) + n[1-m_n\hat{L}(z/n)]g_n(z) - \sigma_n n[1-\hat{L}(z/n)] - n^2(\widehat{r_n \circ g_n})(z)\hat{L}(z/n) = 0.$$

Evidently, $\beta_n\hat{L}(z/n) \to \beta$, $n[1-m_n\hat{L}(z/n)] \to \lambda z - \alpha$ and $\sigma_n n[1-\hat{L}(z/n)] \to \lambda z\sigma = -\lambda z \log s$. Furthermore,

$$n^2(\widehat{r_n \circ g_n})(z) = zn^2 \int_0^\infty r_n \circ g_n(t)\, e^{-zt}\, dt$$
$$\leqq cn^{-1} \int_0^\infty g_n^3(t) z\, e^{-zt}\, dt \leqq K/n$$

for some K, since $g_n(t) = n[1-F_n(s^{1/n}, nt)] \leqq n(1-s^{1/n}) + n(1-q_n)$, which is bounded by 2.3. Hence, as $n \to \infty$, the equation (loosely speaking) approaches the equation in the following proposition:

PROPOSITION 3.1. *If* $\alpha \neq 0$, *the equation*

$$\beta\widehat{x^2}(z) + (\lambda z - \alpha)\hat{x}(z) - \sigma\lambda z = 0$$

has the solution

$$a(t) = \frac{\sigma\, e^{\alpha t/\lambda}}{1+\sigma(\beta/\alpha)(e^{\alpha t/\lambda}-1)}.$$

For $\alpha = 0$

$$\frac{\sigma}{1+\sigma\beta t/\lambda}$$

is a solution

PROOF. Assume that $\alpha \neq 0, L(t) = 1-e^{-\gamma t}, \gamma = 1/\lambda, f_n(x) = 1+(1+\alpha/n)(x-1)+\beta(x-1)^2$. The equation for g_n has a sense also if f_n is not a probability generating function and it reduces to a Riccati differential equation

$$g_n' = \alpha\gamma g_n - \beta\gamma g_n^2,$$
$$g_n(0) = \sigma_n.$$

The solution is

$$g_n(t) = \frac{\alpha\sigma_n\, e^{\alpha\gamma t}}{\alpha + \beta\sigma_n(e^{\alpha\gamma t}-1)}$$

which tends to a as $n \to \infty$. Therefore

$$\beta \frac{\gamma n}{z+\gamma n}\widehat{g_n^2}(z)+n\left[1-(1+\alpha/n)\frac{\gamma n}{z+\gamma n}\right]\widehat{g_n}(z)-\sigma_n \frac{zn}{z+\gamma n}=0.$$

And letting $n \to \infty$ completes the proof. The same argument applies to the case $\alpha = 0$.

PROPOSITION 3.2. *There is no other function than those given in Proposition* 3.1 *which satisfy the equation there with initial value* σ.

PROOF. Assume that A is also a solution for $\alpha \neq 0$. Then,

$$\hat{A}(z)-\hat{a}(z)=\frac{\beta}{\lambda z-\alpha}[\widehat{a^2}(z)-\widehat{A^2}(z)], \qquad z > \alpha/\lambda.$$

Since $\beta/(\lambda z-\alpha)$ is the transform of $\beta/\alpha \exp \alpha t/\lambda$,

$$A(t)-a(t)=\int_0^t [a^2(y)-A^2(y)]\, e^{-\alpha y/\lambda}\, dy\, e^{\alpha t/\lambda}\, \beta/\lambda.$$

A must be differentiable,

$$A'(t)-a'(t)=\alpha/\lambda[A(t)-a(t)]+\beta/\lambda[a^2(t)-A^2(t)]$$

and

$$A(t)-a(t)=K\exp \lambda^{-1}[\alpha t-\beta \textstyle\int_0^t [A(y)+a(y)]\, dy].$$

Since $A(0) = \sigma = a(0)$, the constant $K = 0$.

Assume now that $\alpha > 0$. If $\exp(-\alpha/2\beta) \leqq s \leqq 1$, then (by Proposition 2.3) $s^{1/n} \geqq q_n$ for n larger than some $n(\alpha, \beta)$ and (by Proposition 2.2) $g_n(t) = n[1-F_n(s^{1/n}, nt)]$ increases from $g_n(0) = n(1-s^{1/n})$ to $g_n(\infty) = n(1-q_n)$ with t. Moreover, the sequence $\{g_n\}$ is bounded by some constant and from any subsequence of the natural numbers we may by Helly's selection theorem choose a new subsequence on which $\{g_n\}$ is weakly convergent. Since the limit must solve the equation in 3.1, $g_n \to a$.

If $\alpha \leqq 0$, we choose s small instead (as we might indeed have done above) and repeat the argument for g_n, now nonincreasing. This completes the proof of the convergence.

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