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UNIFORM INTEGRABILITY OF SQUARE INTEGRABLE MARTINGALES

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Let $(M_t, \mathscr{F}_t)_{t \geqq 0}$ be a continuous square integrable martingale and let A_t be the natural increasing process in the Doob decomposition of M_t^2. Extending a result of Burgess Davis we show that there exist constants C_1 and C_2 such that

$$C_1 E[A_t^{\frac{1}{2}}] \leqq E[\sup_{s \leqq t} |M_s|] \leqq C_2 E[A_t^{\frac{1}{2}}]$$

for all $t > 0$. Now if $A_\infty = \lim_{t\to\infty} A_t$, we find moment conditions on A_∞ which relate to uniform integrability of M_t. In particular, $E[A_\infty^{\frac{1}{2}}] < \infty$ implies M_t is uniformly integrable which implies $E[A_\infty^{1/\delta}] < \infty$ for all $\delta > 4$.

In [1] B. Davis shows that if $f = (f_1, f_2, \cdots)$ is a martingale then there exists constants C_1 and C_2 independent of f such that

$$C_1 E\{(\sum_{k=1}^\infty (f_{k+1} - f_k)^2)^{\frac{1}{2}}\} \leqq E[\sup_n |f_n|] \leqq C_2 E\{(\sum_{k=1}^\infty (f_{k+1} - f_k)^2)^{\frac{1}{2}}\} .$$

In this paper we will extend this to continuous time square integrable martingales and consider a consequence of the extension.

LEMMA. *Let $(M_t)_{t \geqq 0}$ be a continuous square integrable martingale. Let $A_t = \langle M \rangle_t$ be the natural increasing process in the Doob decomposition of M_t^2. Then there exists constants C_1 and C_2 such that for every $t > 0$*

$$C_1 E[A_t^{\frac{1}{2}}] \leqq E[\sup_{s \leqq t} |M_s|] \leqq C_2 E[A_t^{\frac{1}{2}}] .$$

PROOF. If $\{t_0, t_1, \cdots, t_n\}$ is a partition of $[0, t]$ then by Davis' theorem we have

$$C_1 E\{(\sum_{k=1}^n (M_{t_k} - M_{t_{k-1}})^2)^{\frac{1}{2}}\} \leqq E[\sup_k |M_{t_k}|] \leqq C_2 E\{(\sum_{k=1}^n (M_{t_k} - M_{t_{k-1}})^2)^{\frac{1}{2}}\} .$$

It is known [3] that for a continuous square integrable martingale we have $\sum_{k=1}^n (M_{t_k} - M_{t_{k-1}})^2 \to A_t$ in L_1 as $n \to \infty$ and the $\max_k |t_k - t_{k-1}| \to 0$. Hence it is easy to show

$$E\{(\sum_{k=1}^n (M_{t_k} - M_{t_{k-1}})^2)^{\frac{1}{2}}\} \to E[A_t^{\frac{1}{2}}]$$

as $n \to \infty$ and $\max_k |t_k - t_{k-1}| \to 0$. Now $\sup_k |M_{t_k}|$ is monotone nondecreasing as the partition $\{t_k\}$ becomes finer so by the monotone convergence theorem,

$$\lim_{n\to\infty} E[\sup_{k \leqq n} |M_{t_k}|] = E[\sup_{s \leqq t} |M_s|] .$$

Therefore, the lemma follows by taking limits of above inequality.

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THEOREM. *Let $(M_t, \mathscr{F}_t)_{t\geqq 0}$ be a continuous square integrable martingale and let A_t be as in the lemma. Then*

(a) $\lim_{t\to\infty} E[(A_t)^{\frac{1}{2}}] < \infty$ *implies M_t is uniformly integrable.*

(b) *if M_t is uniformly integrable, then* $\lim_{t\to\infty} E[(A_t)^{1/\delta}] < \infty$ *for all $\delta > 4$.*

PROOF. In order to show part (a) one simply applies the lemma and gets $E[\sup_t |M_t|] < \infty$ which in turn implies uniform integrability.

For part (b) we assume $\lim_{t\to\infty} E[(A_t)^{1/\delta}] = \infty$ for some $\delta > 4$. Let $\alpha = \delta - 4 > 0$ and let $A_\infty = \lim_{t\to\infty} A_t$. By assumption we have $P[(A_\infty)^{1/\delta} > \lambda] \geqq 4\lambda^{-1-(\alpha/8)}$ for infinitely many arbitrarily large λ. Let $\{\lambda_k\}$ be a sequence of such λ's with $\lim_{k\to\infty} \lambda_k = \infty$. Now choose $\{t_k\}$ such that $P[A_{t_k} \geqq \lambda_k^{\delta}] \geqq 2\lambda_k^{-1-(\alpha/8)}$ and $t_k \geqq t_{k-1}$. We will now show that $E\{|M_t|\}$ is not bounded which contradicts uniform integrability. Recall that M_t can be expressed as a Brownian motion with a time change, $M(t, \omega) = X[A(t, \omega), \omega]$ [2]. For notational convenience we will write $X[A(t, \omega), \omega] = X[A(t)]$. Now

$$E\{|M_{t_k}|\} = E\{|X[A(t_k)]|\} \geqq E\{|X[A(t_k) \wedge \lambda_k^{\delta}]|\} \geqq E\{|X(\lambda_k^{\delta})| \cdot 1_{[A(t_k)\geqq \lambda_k^{\delta}]}\}\,.$$

Define $\beta_k = \lambda_k^{\delta/2-(1+(\alpha/8))}$. Now $X(\lambda_k^{\delta}) \sim N(0, \lambda_k^{\delta})$ so

$$P[|X(\lambda_k^{\delta})| \leqq \beta_k] = 2^{\frac{1}{2}}\pi^{-\frac{1}{2}}\lambda_k^{-\delta/2} \int_0^{\beta_k} \exp\{-x^2/2\lambda_k^{\delta}\}\,dx \leqq 2^{\frac{1}{2}}\pi^{-\frac{1}{2}}\lambda_k^{-1-(\alpha/8)} \leqq P[A(t_k) \geqq \lambda_k^{\delta}]\,.$$

Using this fact it follows that

$$E\{|X(\lambda_k^{\delta})| \cdot 1_{[A(t_k)\geqq\lambda_k^{\delta}]}\} \geqq E\{|X(\lambda_k^{\delta})| \cdot 1_{[|X(\lambda_k^{\delta})|\leqq\beta_k]}\}$$

since on the right-hand side, the integration is over a set of smaller probability and $|X(\lambda_k^{\delta})|$ is required to assume all its small values. However,

$$\begin{aligned} E\{|X(\lambda_k^{\delta})| \cdot 1_{[|X(\lambda_k^{\delta})|\leqq\beta_k]}\} &= 2^{\frac{1}{2}}\pi^{-\frac{1}{2}}\lambda_k^{-\delta/2} \int_0^{\beta_k} x\exp\{-x^2/2\lambda_k^{\delta}\}\,dx \\ &= 2^{\frac{1}{2}}\pi^{-\frac{1}{2}}\lambda_k^{\delta/2}[1 - \exp\{-2^{-1}\lambda_k^{-2-(\alpha/4)}\}] \\ &= 2^{\frac{1}{2}}\pi^{-\frac{1}{2}}\lambda_k^{2+(\alpha/2)}[1 - \exp\{-2^{-1}\lambda_k^{-2-(\alpha/4)}\}] \geqq \tfrac{1}{8}(\lambda_k^{\alpha/4}) \end{aligned}$$

for λ_k large. Hence $E\{|M_{t_k}|\}$ can be made arbitrarily large by taking k large so M_t is not uniformly integrable.

REMARK. The moments $\frac{1}{2}$ and $\frac{1}{4}$ have in no way been shown to be best so the question of whether or not a single moment of A_∞ characterizes uniform integrability remains open.

REFERENCES

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