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NOTE ON A "KOLMOGOROV-TYPE" INEQUALITY

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An inequality, described as similar to Kolmogorov's but without the assumption of independence, was proved in [1] and accompanied by several corollaries (cf. also the highly favorable review #6370, Math. Reviews **39** (1970)). The purpose of this note is to give a much shorter proof of the inequality and to point out that the corollaries are all trivial even without using the inequality. The reader may also easily check that Loève's [2] page 241–242 propositions, implying the corollaries, are *not* trivial.

To make the note shorter we shall limit ourselves to the case of Theorem 1, [1] when $0 < r \leqq 1$ and thus $(a+b)^r \leqq a^r + b^r$ for all nonnegative numbers a, b. (For Theorem 2, when $r > 1$, one would use the Minkowski inequality.) Now let X_i be random variables, $0 < r \leqq 1$, $\varepsilon > 0$ and $\{c_i\}$ a non-increasing sequence of positive numbers. Then the result of Theorem 1 can be formally strengthened, by replacing X_i by $|X_i|$, to

$$(*) \quad P\{\max_{m \leqq k \leqq n} c_k \sum_1^k |X_i| \geqq \varepsilon\} \leqq \varepsilon^{-r}[c_m{}^r \sum_1^m E|X_i|^r + \sum_{m+1}^n c_i{}^r E|X_i|^r] .$$

But it is easy to verify that the random variable appearing at the left-hand side is less or equal to $c_m \sum_{i=1}^m |X_i| + \sum_{i=m+1}^n c_i |X_i|$ which has the rth moment less or equal ε^r times the right-hand side in (*). The Markov inequality yields (*).

As for the corollaries take Corollary 2 as a typical example: If $E|X_i| = v < +\infty$ and $q > 0$ then $\sum_1^\infty |X_i|/i^{q+1}$ is integrable and thus is finite a.e. The Kronecker lemma gives then the desired result $n^{-q-1} \sum_1^n X_i \to 0$ a.e. The example following Corollary 3 follows immediately from the Borel-Cantelli lemma.

REFERENCES

[1] KOUNIAS, EUSTRATIOS G. and WENG, TENG-SHAN (1969). An inequality and almost sure convergence. *Ann. Math. Statist.* **40** 1091–1093.

[2] LOÈVE, M. (1955). *Probability Theory*. Van Nostrand, Princeton.

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