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UPPER BOUNDS FOR THE ASYMPTOTIC MAXIMA OF CONTINUOUS GAUSSIAN PROCESSES

By M. B. Marcus

Northwestern University

Upper bounds are obtained for $|X(t)|/Q(t)$ as $t \to \infty$, where $X(t)$ is a continuous Gaussian process with $EX^2(t) \leqq Q^2(t)$, $Q(t)$ non-decreasing. Our results are extensions of some work of Pickands (1967), Nisio (1967) and Orey (1971) to larger classes of Gaussian processes, i.e. fewer restrictions are imposed on the covariance functions. The results follow from Fernique's lemma (1964) and a recent lemma on the maximum of Gaussian sequences due to Landau, Shepp, Fernique and the author (see Marcus, Shepp (1971) for further references to this lemma).

0. Introduction. Let $X(t)$ be a continuous Gaussian process, $EX(t) = 0$ $EX(t)^2 \leqq Q^2(t)$, $Q(t)$ non-decreasing. A considerable amount of attention has been given to studying the behavior of $X(t)/Q(t)$ as $t \to \infty$. We refer the reader to the work of Berman (1971), Orey (1971), Pickands (1967, 1969) and Watanabe (1970). A common characteristic of these papers is the class of Gaussian processes that they study. Roughly speaking, when dealing with stationary processes the requirement is imposed that $E(X(t) - X(s))^2 \leqq 0(|t - s|^\alpha)$, and when studying processes for which $Q(t) \uparrow \infty$ as $t \to \infty$ that $Q(ts)/Q(t) = 0(s^\alpha)$. As more stringent conditions are imposed on $Q(t)$ (and on the covariance of the process in the stationary case) more detailed results are obtained; results that are much more precise than those that we obtain. However, our concern is different. What we want to do is to get an idea of the asymptotic maxima for as wide a class of continuous Gaussian processes as possible. Therefore our direction is to weaken conditions on the covariance of the processes. Many standard results are extended to larger classes of processes and we exhibit processes with upper bounds between $(\log \log t)^{\frac{1}{2}}$ and $(\log t)^{\frac{1}{2}}$.

Our results are the following:

In Section 1 we consider stationary processes $X(t)$, $E(X(t+h) - X(t))^2 = \sigma^2(h)$, $EX^2(t) = 1$. Theorem 1.1 states that

$$\text{(0.1)} \qquad \limsup_{t\to\infty} (|X(t)| - (2 \log t)^{\frac{1}{2}}) \leqq 0 \quad \text{a.s.}$$

when $\sigma^2(h) = O(1/|\log|h||^\alpha)$, for $h \to 0$, $\alpha > 1$. This extends a result of Pickands (1967) in which the condition is $\sigma^2(h) = O(h^\alpha)$, $\alpha > 0$. In Theorem 1.4 we show that with no conditions other than stationarity and continuity

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$$\limsup_{t\to\infty} \frac{|X(t)|}{(2\log t)^{\frac{1}{2}}} \leqq 1 \quad \text{a.s.}$$

This extends Nishio's result (1967) in which it is assumed that $\sigma(h)$ satisfies Fernique's integral condition (see (1.1) below).

In Section 2 we consider processes $X(t)$ with stationary increments. We first show that it is possible to construct such processes with a large variety of functions $Q(t)$, ranging from $Q(t)$ bounded to $Q(t) \sim$ (Const.) t. Corollary 2.4 states that

$$\limsup_{t\to\infty} \frac{|X(t)|}{[2(Q^2(t) + Q^2(1))\log t]^{\frac{1}{2}}} \leqq 1 \quad \text{a.s.},$$

no conditions other than stationary increments and continuity are imposed. In Theorem 2.6 conditions are obtained on $Q(t)$ so that

$$\limsup_{t\to\infty} \left(\frac{|X(t)|}{Q(t)} - (2\log\log t)^{\frac{1}{2}}\right) \leqq 0 \quad \text{a.s.} \tag{0.2}$$

The conditions are similar to Orey's (1971) for t large but we show that the behavior of $Q(t)$ for t near zero is immaterial. We also obtain examples of processes with $Q(t) \neq 0(t^\alpha)$, $\alpha > 0$ for which (0.2) continues to hold. In Theorem 2.8 we sacrifice some precision to get a general picture of how the asymptotic maxima behaves,

$$P\left[\frac{|X(t)|}{Q(t)} \geqq (3+\varepsilon)(2\log\log t)^{\frac{1}{2}} + 13\left(\frac{1}{Q(t)}\int_0^t \frac{Q(u)}{u}\,du\right)^{\frac{1}{2}} \text{ i.o.}\right] = 0\,,$$

the assumptions are that the process is continuous and has stationary increments. It is possible to find values of $Q(t)$ for which the last term is $(\log t)^{(1-\gamma)/2}$ $0 \leqq \gamma < 1$.

Actually stationarity conditions are not required for many of the results given in Sections 1 and 2. In Section 3 we discuss how these results can be applied to continuous Gaussian processes in general. Also, examples are given that suggest that it is impossible to obtain general results about the asymptotic behavior of continuous Gaussian processes without additional conditions being imposed.

One object in initiating this study was to see how much information Fernique's lemma (1964), (see also Marcus (1970)), would give about the asymptotic maxima of continuous Gaussian processes. It turns out that the lemma is quite sharp, as we show in Corollaries 1.3 and 2.7 in which the results in (0.1) and (0.2) are sharpened. Nevertheless, Watanabe's results (1970) are sharper still; the reason for this in terms of the limits of applicability of Fernique's lemma is interesting; it is discussed following Corollary 1.3.

We do not answer the question of whether our bounds are best possible, some results along this line can be found in Berman (1971), Orey (1971), Pickands III (1967, 1969) and Watanabe (1970).

Finally, we are concerned only with separable, real valued, continuous, zero mean Gaussian processes which we will generally refer to simply as continuous Gaussian processes.

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1. Stationary processes.

THEOREM 1.1. *Let* $X(t)$ *be a stationary Gaussian process* $EX(t)^2 = 1$, $E(X(t) - X(s))^2 = \sigma^2(|t-s|)$ *where* $\sigma^2(h) = O(1/|\log|h||^\alpha)$, $\alpha > 1$; *then*

$$\limsup_{t\to\infty} (|X_t| - (2\log t)^{\frac{1}{2}}) \leqq 0 \quad \text{a.s.}$$

PROOF. For $|t-s|$ sufficiently small $\sigma(|t-s|) \leqq k/|\log|t-s||^\alpha$, for some $\alpha > 1$ hence $\int^\infty \sigma(e^{-x^2})\,dx < \infty$. Therefore we can use Fernique's lemma stated below in a form similar to the one that appears in Marcus (1970).

LEMMA 1.2. *Let* $X(t)$ *be a Gaussian process on* $[0, 1]$. *Suppose* $E(X(t)-X(s))^2 \leqq \psi^2(|t-s|)$, $\psi(t) \uparrow \infty$ *as* $t \to \infty$ *and*

$$\int^\infty \psi(e^{-x^2})\,dx < \infty\,. \tag{1.1}$$

Let $c(p) = n^{2^p}$ *for* n *a fixed integer*, $n > 3$; *then*

$$P\{\|X\|_\infty \geqq a\Gamma + \textstyle\sum_{p=1}^\infty (a_1 + \log c(p+1))^{\frac{1}{2}}\psi(c(p)^{-1})\}$$
$$\leqq n^2 \int_a^\infty e^{-x^2/2}\,dx + \textstyle\sum_{p=1}^\infty c(p)^2 \int_{(a_1+2\log c(p+1))^{1/2}}^\infty e^{-x^2/2}\,dx$$

where $EX(t)^2 \leqq \Gamma^2$ *and* $\|X\|_\infty = \sup_{t\in[0,1]} |X(t)|$. *In our proof* $\Gamma = 1$.

Define $Y_k(t) = X(k+t)$; the lemma will be applied to $Y_k(t)$. Substitute

$$a = (2\log k)^{\frac{1}{2}} + \frac{2\log n + (1/2+\varepsilon)\log\log k}{(2\log k)^{\frac{1}{2}}}\,, \tag{1.2}$$

$$a_1 = (1+\varepsilon)(2\log k)\,, \tag{1.3}$$

and note that the result holds for all $\sigma(h) \leqq k/(\log 1/h)^\alpha$ if we substitute this value for $\psi(h)$. Therefore,

$$P\Big[\sup_{t\in[k,k+1]} |X(t)| \geqq a + \text{Const.} \textstyle\sum_{p=1}^\infty (a_1 + \log c(p+1))^{\frac{1}{2}} \frac{1}{(\log n)^\alpha 2^{p\alpha}}\Big] \tag{1.4}$$
$$\leqq \frac{1}{k(\log k)^{1+\varepsilon}} + \frac{1}{k^{1+\varepsilon}} \textstyle\sum_{p=1}^\infty \frac{1}{2^{\frac{1}{2}(p+1)}}\,.$$

The right side of (1.4) is a term of a convergent series independent of n; we can obtain our result by means of the Borel-Cantelli lemma if we can find an n as a function of k so that

$$\frac{2 \log n + (\frac{1}{2} + \varepsilon) \log \log k}{(2 \log k)^{\frac{1}{2}}} \to 0 \quad \text{as} \quad k \uparrow \infty\,, \tag{1.5}$$

$$\sum_{p=1}^{\infty} (a_1 + \log c(p+1))^{\frac{1}{2}} \frac{1}{(\log n)^{\alpha} 2^{p\alpha}} \to 0 \quad \text{as} \quad k \uparrow \infty\,. \tag{1.6}$$

(Throughout this paper we shall write n as a function of k without bothering to assure that n is an integer. Since we are concerned with asymptotic results it does not matter to us whether $n = f(k)$ or $n = [f(k)]$.) Refer to (1.6); the term $(a_1 + \log c(p+1))^{\frac{1}{2}} \leqq a_1^{\frac{1}{2}} + (\log c(p+1))^{\frac{1}{2}}$. The expression

$$\sum_{p=1}^{\infty} (\log c(p+1))^{\frac{1}{2}} \frac{1}{(\log n)^{\alpha} 2^{p\alpha}} = \sum_{p=1}^{\infty} \frac{2^{\frac{1}{2}}}{(\log n)^{\alpha - \frac{1}{2}}} \frac{1}{2^{p(\alpha - \frac{1}{2})}}\,.$$

This term approaches zero as long as $n \to \infty$. The critical term is

$$\frac{a_1^{\frac{1}{2}}}{(\log n)^{\alpha}} \sum_{p=1}^{\infty} \frac{1}{2^{p\alpha}} \leqq \frac{3(\log k)^{\frac{1}{2}}}{(\log n)^{\alpha}}\,, \tag{1.7}$$

Clearly we can satisfy (1.5) and (1.7) for any $\alpha > 1$ if we set $\log n = (\log k)^{\frac{1}{2} - \varepsilon}$ for ε sufficiently small.

Finally the lemma follows by observing that the asymptotic limits are the same when $\log t$, $t \in [k, k+1]$ is substituted for $\log k$.

This method of proof leads to sharper results than those given in Theorem 1 if additional conditions are imposed on the covariance. In particular we get

COROLLARY 1.3. *Let* $\sigma^2(h) = O(h^{\alpha})$, $0 < \alpha \leqq 2$. *Then*

$$P\left[|X(t)| - (2 \log t)^{\frac{1}{2}} \geqq \frac{[2/\alpha + (1/2 + \varepsilon)] \log \log t}{(2 \log t)^{\frac{1}{2}}} \text{ i.o.}\right] = 0\,.$$

PROOF. In this case

$$\begin{aligned} &\sum_{p=1}^{\infty} (a_1 + \log c(p+1))^{\frac{1}{2}} \psi(c(p)^{-1}) \\ &\qquad \leqq \sum_{p=1}^{\infty} (a_1 + \log c(p+1))^{\frac{1}{2}} \frac{1}{(n^{\alpha/2}) 2^{p}} \\ &\qquad \leqq C_1 \frac{(\log k)^{\frac{1}{2}}}{n^{\alpha}} + C_2 \frac{(\log n)^{\frac{1}{2}}}{n^{\alpha}} \end{aligned} \tag{1.8}$$

for some constants C_1 and C_2. Take $n = (\log k)^{1/\alpha + \varepsilon_1}$, then

$$a = (2 \log k)^{\frac{1}{2}} + \frac{[2/\alpha + (1/2 + 3\varepsilon_1)] \log \log k}{(2 \log k)^{\frac{1}{2}}}\,, \tag{1.8a}$$

and the second term in (1.8a) dominates (1.8). This completes the proof.

Corollary 1.3 is not as sharp as Watanabe's result (1970) wherein $2/\alpha$ is replaced by $1/\alpha$, but the method that we use can be pushed no further. The reason for this is rather interesting. In Fernique's lemma we dominate the probability of the union of sets by the sums of the probabilities of the sets (i.e. we make statements like $P[\max_j (X_j > a)] \leqq \sum_j P(X_j > a)$ for random variables X_j). These estimates do not exploit the covariance structure of the random variables. In Watanabe's proof, which is similar to the derivation of Fernique's lemma up to a point, he achieves the result of Corollary 1.3 by first using the same crude estimates, but the final improvement necessitates a sharpening of this estimate in which he utilizes specific restrictions on the covariances of processes that he is studying. Perhaps the most interesting aspect of Fernique's lemma is that it can give such sharp results using such a crude method of estimating the maximum. This, of course, depends strongly on the fact that the random variables are Gaussian.

The condition $\sigma^2(h) = O(1/|\log|h||)^a$, $a > 1$ in Theorem 1.1 does not include all processes for which (1.1) is satisfied; the following result is true for all continuous, stationary Gaussian processes. (Condition (1.1) is a sufficient but not necessary condition for continuity; see Marcus, Shepp (1970)).

THEOREM 1.4. *Let $X(t)$ be a continuous stationary Gaussian process, $EX^2(t) = 1$, then*

$$\limsup_{t\to\infty} \frac{|X(t)|}{(2\log t)^{\frac{1}{2}}} \leqq 1 \quad \text{a.s.}$$

PROOF. This result is a trivial consequence of the following lemma due to Fernique, Landau, Shepp and the author; see Marcus, Shepp (1971) for further reference.

LEMMA 1.5. *Let $\{X_n\}$ be a sequence of bounded Gaussian random variables, i.e. $P(\sup_n |X_n| < \infty) = 1$.*

Let $\sup_n EX_n^2 = a^2$ then

$$\lim_{t\to\infty} \frac{1}{t^2} \log P(\sup_n |X_n| > t) = -(2a^2)^{-1}.$$

By separability, Lemma 1.5 implies

$$P[\sup_{t\in[0,1]} |X(t)| > \tau] \leqq \exp\left(-\frac{\tau^2}{2(1+\varepsilon)}\right) \tag{1.9}$$

for τ sufficiently large. However, by stationarity, (1.9) holds in every interval. Therefore

$$P[\sup_{t\in[k,k+1]} \frac{|X(t)|}{(2\log k)^{\frac{1}{2}}} \geqq 1 + 2\varepsilon] < \frac{1}{k^{1+\varepsilon}}.$$

The theorem follows from the Borel-Cantelli lemma.

Finally, we remark that (1.4) can be used to obtain results that lie between Theorems 1.1 and 1.4 in sharpness in those cases when (1.1) is satisfied but the hypothesis of Theorem (1.1) is not.

2. Processes with stationary increments. Let $X(t)$ be a continuous Gaussian process with stationary increments, $X(0) = 0$. For such a process $E(X(t) - X(s))^2 = EX(t-s)^2$ Let Q be a non-decreasing function such that $EX(t-s)^2 \leqq Q^2(|t-s|)$. We will investigate upper bounds for the asymptotic growth of $|X(t)|/Q(t)$. Before proceeding we show that it is possible to construct Gaussian processes with stationary increments with a variety of corresponding Q functions. We call two functions f and g comparable if $C_1 \leqq f(x)/g(x) \leqq C_2$, $0 < C_1 \leqq C_2 < \infty$ for x sufficiently large.

LEMMA 2.1. *Let $H(t)$ be a monotonically decreasing regularly varying function, $H(t) \leqq 1$, $t^2H(t) \geqq 1$ for $t > 1$. We can find continuous Gaussian processes with stationary increments for which $Q^2(t)$ is comparable to $t^2H(t)$.*

PROOF. Let $Y(t)$ be a continuous stationary Gaussian process. Let $r(\tau) = \int_0^\infty \cos u\tau \, dF(u)$ be its covariance function. Define $X(t) = \int_0^t Y(u)\,du$; this is a continuous Gaussian process with stationary increments.

$$EX(t)^2 = \int_0^\infty \frac{1-\cos ut}{u^2}\, dF(u) \,. \tag{2.1}$$

Breaking up the integral in (2.1) into two parts over the intervals $[0, 1/t]$ and $[1/t, \infty)$ we get

$$\tfrac{1}{4}\, t^2F(1/t) \leqq EX(t)^2 \leqq \tfrac{1}{2}\, t^2F(1/t) + 2\int_{1/t}^\infty \frac{dF(u)}{u^2} \,.$$

If the measure $F(u)$ is regularly varying near zero

$$2\int_{1/t}^\infty \frac{dF(u)}{u^2} \sim \text{Const. } t^2F(1/t) \,.$$

We choose $H(t) = F(1/t)$.

Note that it is only the values of the spectrum of $Y(t)$ near the origin that determines the rate of increase of $EX(t)^2$ for t large when $EX(t)^2$ is unbounded. We next show that the asymptotic growth of $|X(t)|/Q(t)$ is not influenced by the values of $Q(t)$ for t small.

LEMMA 2.2. *Let $X(t)$ be a continuous Gaussian process with stationary increments. Then*

$$P\left[\lim_{N\to\infty} \sup_{t\in[k,k+1],\, k=N,N+1,\cdots} \frac{|X(t)-X(k)|}{(2\log k)^{\frac{1}{2}}} \leqq Q(1)\right] = 1 \,.$$

PROOF. The proof is identical to the proof of Theorem 1.4 since the increments of the process are stationary. It should be obvious that the same result holds for t in any sequence of intervals of fixed length.

For future reference we mention the next lemma whose proof follows immediately by the Borel-Cantelli lemma.

LEMMA 2.3. *Let $X(t)$ be a continuous Gaussian process with stationary increments. Then*

$$P\left[\limsup_{n\to\infty} \frac{|X(n)|}{(2Q^2(n)\log n)^{\frac{1}{2}}} \leqq 1\right] = 1 ,$$

where the limit is taken along the integers.

Lemmas 2.2 and 2.3 yield the following asymptotic upper bound for $|X(t)|/Q(t)$.

COROLLARY 2.4. *Let $X(t)$ be a continuous Gaussian process with stationary increments, $EX^2(t) \leqq Q^2(t)$, $Q(t)$ non-decreasing. Then*

$$\limsup_{t\to\infty} \frac{|X(t)|}{[2(Q^2(t) + Q^2(1))\log t]^{\frac{1}{2}}} \leqq 1 \quad \text{a.s.}$$

The following lemma is our most general result on the growth rate; it will be followed by some theorems and corollaries expressing specific results in more comprehensible forms.

LEMMA 2.5. *Let $X(t)$ be a continuous Gaussian process with stationary increments $EX(t)^2 \leqq Q^2(t)$, $Q(t)$ non-decreasing. Then*

$$(2.2) \qquad P\Bigg[\sup_{t\in[2^k,2^{k+1}]}\left|\frac{X(t)}{Q(t)}\right| \geqq a + 4\sum_{p=1}^{p(k)} (a_1 + \log c(p+1))^{\frac{1}{2}} \frac{Q(2^k c(p)^{-1})}{Q(2^k)} + C\frac{Q(1)k^{\frac{1}{2}}}{Q(2^k)} \text{ i.o.}\Bigg] = 0 ,$$

where

$$(2.3) \qquad a = (2\log k)^{\frac{1}{2}} + \frac{2\log n + (\frac{1}{2}+\varepsilon)\log\log k}{(2\log k)^{\frac{1}{2}}} \quad \textit{or}$$

$$(2.4) \qquad \begin{aligned} a &= (1+\varepsilon)(2\log k)^{\frac{1}{2}} + (2\log n)^{\frac{1}{2}} \\ a_1 &= (1+\varepsilon)(2\log k) , \end{aligned}$$

and $c(p) = n^{2^p}$ for n an integer, $n > 3$ and $p(k)$ the largest value of p for which $2^k c(p)^{-1} > 1$.

PROOF. We define the least monotone majorant of $EX^2(t)$

$$R^2(t) = \sup_{u\leqq t} EX^2(u) .$$

Let $||X(t)|| = (EX^2(t))^{\frac{1}{2}}$. Then $||X(t+h) - X(0)|| \leqq ||X(t+h) - X(t)|| + ||X(t) - X(0)||$. Therefore, since the process has stationary increments

$$||X(t+h) - X(0)|| \leqq R(h) + R(t) , \qquad h \geqq 0 .$$

Since R is the least monotone majorant for $||X(t+h) - X(0)||$ we have

$$(2.5) \qquad R(t+h) \leqq R(h) + R(t) ,$$

$$(2.5a)\quad E\left(\frac{X(t)}{R(t)}-\frac{X(s)}{R(s)}\right)^2=\frac{EX^2(t)}{R^2(t)}+\frac{EX^2(s)}{R^2(s)}-\frac{(EX^2(t)+E^2(s)-EX^2(t-s))}{R(t)R(s)}$$
$$=\frac{EX^2(t-s)}{R(t)R(s)}+\left(\frac{EX^2(s)}{R(s)}-\frac{EX^2(t)}{R(t)}\right)\left(\frac{1}{R(s)}-\frac{1}{R(t)}\right).$$

Without loss of generality assume $s \leqq t$. By the triangle inequality $||X(t) - X(0)|| \geqq |||X(s) - X(0)|| - ||X(t) - X(s)|||$, therefore

$$\frac{EX^2(s)}{R(s)}-\frac{EX^2(t)}{R(t)}\leqq\frac{EX^2(s)}{R(s)}-\frac{EX^2(s)}{R(t)}-\frac{EX^2(t-s)}{R(t)}+\frac{(EX^2(s))^{\frac{1}{2}}(EX^2(t-s))^{\frac{1}{2}}}{R(t)}.$$

By (2.5) $R(t)-R(s)\leqq R(t-s)$, so (2.5a)$\leqq R^2(t-s)/R(t)R(s)+3(R^2(t-s)/R^2(t))$. It follows that for $s,\,t\in[2^k,\,2^{k+1}]$

$$(2.5b)\qquad E\left(\frac{X(t)}{R(t)}-\frac{X(s)}{R(s)}\right)^2\leqq 4\,\frac{R^2(t-s)}{R^2(2^k)}.$$

The following inequality is a version of Fernique's lemma in which the interval $[2^k, 2^{k+1}]$ is partitioned into increments of length $2^k/c(p)$, $p = 1, 2, \cdots, p(k), p(k) + 1$. The partitioning stops the first time the length of the subpartition is less than 1 and then Lemma 1.5 is used. We also use (2.5b)

$$(2.5c)\qquad P\Bigg[\sup_{t\in[2^k,2^{k+1}]}\left|\frac{X(t)}{R(t)}\right|>a+2\sum_{p=1}^{p(k)}(a_1+\log c(p+1))^{\frac{1}{2}}\frac{R(2^kc(p)^{-1})}{R2(^k)}$$
$$+2\,\frac{(\log c(p(k)+1)+a_1)^{\frac{1}{2}}R(1)}{R(2^k)}+(1+2\varepsilon)\,2\,\frac{(2\log 2^{k+1})^{\frac{1}{2}}R(1)}{R(2^k)}\Bigg]$$
$$<n^2\int_a^\infty e^{-x^2/2}\,dx+\sum_{p=1}^\infty c(p)^2\int_{(a_1+2\log c(p+1))^{1/2}}^\infty e^{-x^2/2}\,dx+\frac{N}{2^{k\varepsilon}},$$

where N is a constant.

The event being measured in (2.5c) is

$$(2.5d)\qquad \Bigg\{|X(t)|>aR(t)+\sum_{p=1}^{p(k)}B(p)\frac{R(2^kc(p)^{-1})}{R(2_k)}R(t)+C\frac{R(t)}{R(2_k)}+D\frac{R(t)}{R(2^k)};$$
$$\text{for some}\quad t\in[2^k,2^{k+1}]\Bigg\}.$$

The abbreviations $B(p)$, C, D are self evident.

Note. $R(t)/R(2^k) \leqq R(2^{k+1})/R(2^k) \leqq 2$ by (2.5). Therefore, the probability of (2.5d) is greater than the probability of

$$(2.5e)\qquad \{|X(t)|>aR(t)+2\sum B(p)Q(2^kc(p)^{-1})+2C+2D;$$
$$\text{for some}\quad t\in[2^k,2^{k+1}]\}$$

since $R(t) \leqq Q(t)$. Divide both sides of this inequality by $Q(t)$ and replace $R(t)/Q(t)$ by 1 and $1/Q(t)$ on the right of the inequality by $1/Q(2^k)$. The probability of this new event is less than the probability of (2.5d). The right side of (2.5c) is a term of a convergent sequence. The Borel-Cantelli lemma completes the proof.

THEOREM 2.6. *Let $X(t)$ be a Gaussian process with stationary increments, $EX(t)^2 \leqq Q^2(t)$, $Q(t)$ non-decreasing. Then*

$$\limsup_{t\to\infty}\left(\frac{|X(t)|}{Q(t)} - (2\log\log t)^{\frac{1}{2}}\right) \leqq 0 \quad \text{a.s.} \tag{2.6}$$

if either of the following conditions is satisfied:

(i) $\dfrac{Q(2^k c(p)^{-1})}{Q(2^k)} \leqq (c(p)^{-1})^{\alpha/2}, \qquad \alpha > 0 .$

(ii) $Q(t) = \exp\left(\dfrac{\log t}{(\log\log t)^\alpha}\right)$ for $t > T, \quad \alpha < \frac{1}{2}$.

PROOF. To prove the result under condition (i), substitute the value of a given by (2.3) into Lemma 2.5. Replace the quotient $Q(2^k c(p)^{-1})/Q(2^k)$ by $c(p)^{-\alpha/2}$ in (2.2). The first two terms to the right of the inequality sign in (2.2) are exactly the same as Corollary 1.3 except now $\log k \sim \log\log 2^k$ can be interpolated to give $\log\log t$. Condition (i) implies that $k^{\frac{1}{2}}/Q(2^k) \to 0$; thus (i) implies (2.6).

To show that (ii) also implies (2.6) refer again to Lemma 2.5 with the value of a given by (2.3). Clearly, for each k we must find a value of n so that

$$\frac{\log n}{(\log k)^{\frac{1}{2}}} \to 0 ,$$

$$(\log k)^{\frac{1}{2}} \sum_{p=1}^{p(k)} \frac{Q(2^k/c(p))}{Q(2^k)} \to 0 , \tag{2.7}$$

$$(\log n)^{\frac{1}{2}} \sum_{p=1}^{p(k)} 2^{p/2} \frac{Q(2^k/c(p))}{Q(2^k)} \to 0 .$$

Let ε be a number for which $\alpha + \varepsilon < \frac{1}{2}$. Then choosing $\log n = (\log k)^{\alpha+\varepsilon/2}$ all three conditions in (2.7) are satisfied for the values of $Q(t)$ in ii).

Condition (i) is Orey's (1971) condition of $Q(t)$ for $t > 1$; actually we could obtain the same result for t greater than any constant. Our contribution here is that we require no condition on $Q(t)$ for t near zero, since Lemma 2.2 enabled us to disregard $Q(t)$ for t small. (To be more precise we require that $k^{\frac{1}{2}}/Q(2^k) \to 0$, but this is implied by condition (i)).

The purpose of condition (ii) is simply to show that there are other processes for which the iterated logarithm law as given in (2.6) holds. Actually, whenever the first two conditions of (2.7) are satisfied for some value of Q (2.6) will be true.

In our proof of Theorem 2.6 when condition (i) applied we actually obtained the stronger result, which we mention as a corollary.

COROLLARY 2.7. *Let $X(t)$ be a continuous Gaussian process with stationary increments. Suppose that* $Q(2^k c(p)^{-1})/Q(2^k) \leqq (c(p)^{-1})^{\alpha/2}$ $0 < \alpha \leqq 2$. *Then*

$$P\left[\frac{|X(t)|}{Q(t)} - (2\log\log t)^{\frac{1}{2}} \geqq \frac{(2/\alpha + 1/2 + \varepsilon)\log\log\log t}{2(\log\log t)^{\frac{1}{2}}} \text{ i.o.}\right] = 0 .$$

Just as in the case of Corollary 1.3 this result is not as sharp as Watanabe's in which $2/\alpha$ is replaced by $1/\alpha$. The reason is the same as the one given in Section 1. The same discussion applies.

In the next Theorem, at the expense of maximum accuracy in those cases when (2.6) holds or when Corollary (2.4) might be more informative, we give an upper bound for the asymptotic maximum of $|X(t)|/Q(t)$ for a wide class of processes characterized by the functions $Q(t)$.

THEOREM 2.8. *Let $X(t)$ be a continuous Gaussian process with stationary increments $EX(t)^2 \leqq Q(t)^2$, $Q(t)$ non-decreasing, then*

$$P\left[\frac{|X(t)|}{Q(t)} \geqq (3+\varepsilon)(2\log\log t)^{\frac{1}{2}} + 13\left(\frac{1}{Q(t)}\int_1^t \frac{Q(u)}{u}\,du\right)^{\frac{1}{2}} \text{i.o.}\right] = 0 .$$

PROOF. Refer to (2.5e) and substitute the value of a given by (2.4). We choose

$$\log n = \max\left(\int_1^{2k} \frac{Q(u)}{u}\,du, \log 4\right).$$

We have

$$\begin{aligned}(2.8)\qquad \textstyle\sum_{p=1}^{p(k)} (a_1 + \log c(p+1))^{\frac{1}{2}} Q(2^k/c(p)) &\leqq (1+\varepsilon)(2\log k)^{\frac{1}{2}} Q(2^k/n^2) \\ &+ (1+\varepsilon)(2\log k)^{\frac{1}{2}} \int_1^{p(k)} Q(2^k/c(p))\,dp + (2\log n)^{\frac{1}{2}} Q(2^k/n^2) \\ &+ (2\log n)^{\frac{1}{2}} \int_1^{p(k)} 2^{p/2} Q(2^k/c(p))\,dp \leqq (2+\varepsilon)(2\log k)^{\frac{1}{2}} Q(2^k) \\ &+ 2\left(\int_1^t \frac{Q(u)}{u}\,du\right)^{\frac{1}{2}}.\end{aligned}$$

Therefore the probability that

$$\begin{aligned}(2.9)\qquad \Big\{|X(t)| > aR(t) &+ (2+\varepsilon)(\log k)^{\frac{1}{2}} Q(2^k) + 8\left(\int_1^t \frac{Q(u)}{u}\,du\right)^{\frac{1}{2}} \\ &+ 2(k+2)^{\frac{1}{2}} Q(1) + \tfrac{3}{2}\, 2(k+1)^{\frac{1}{2}} Q(1); \\ &\text{for some} \quad t \in [2^k, 2^{k+1}]\Big\},\end{aligned}$$

is a term of a convergent sequence. The last two terms can be absorbed by the integral. Dividing by $Q(t)$ and applying the Borel-Centelli lemma the result follows.

We have shown that the upper bound for the asymptotic maxima of processes with stationary increments lies between Const. $(2\log\log t)^{\frac{1}{2}}$ and

Const. $(\log t)^{\frac{1}{2}}$. It is also possible for intermediate values to occur since for $Q(t) = \exp[(\log t)^{\gamma}]$, $0 < \gamma < 1$

$$\left(\frac{1}{Q(t)} \int_1^t \frac{Q(u)}{u}\,du\right)^{\frac{1}{2}} \sim \frac{1}{\gamma^{\frac{1}{2}}} (\log t)^{(1-\gamma)/2} .$$

3. Gaussian processes with no explicit stationarity requirements. Continuous Gaussian processes can be constructed that will exhibit any kind of growth rate, as the following examples indicate.

(1) Let $b(t)$ be Brownian motion, $g(t)$ a continuous increasing function. Define $Y(t) = b(g(t))$, $Q^2(t) = EY(t)^2$. Then $|Y(t)|/Q(t)$ has $(1+\varepsilon)(2\log\log g(t))^{\frac{1}{2}}$ as an upper limit for its asymptotic growth rate, and this limit is essentially the best that can be obtained.

(2) In Mascus (1968) there are many examples of stationary Gaussian processes and associated functions $f(h)$, $f(h) \uparrow \infty$ as $h \downarrow 0$, for which

$$C_0 \leqq \limsup_{h\to 0} \frac{|X(h) - X(0)|}{\sigma(h)f(h)} \leqq C_1 , \tag{3.1}$$

where $0 < C_0$, $C_1 < \infty$ and $\sigma^2(h) = E(X(t+h) - X(t))^2$. Define

$$Z(t) = Z(1/h) = \frac{|X(h) - X(0)|}{\sigma(h)}, \qquad h < 1 .$$

Then $EZ(t)^2 = 1$ and an upper bound for the asymptotic maximum of $|Z(t)|$ is $C_1 f(1/t)$, and we see by (3.1) that this cannot be appreciably improved.

(3) Let $X(t)$ be any continuous stationary process with covariance function $\gamma(\tau)$ with the property that $\lim_{\tau\to\infty} \gamma(\tau) = 0$. Define $Y(t) = Q(t)X(t)$ for any $Q(t) \uparrow \infty$ as $t \to \infty$. Then by Pickands (1967) and Theorem 1.4

$$P\left[\limsup_{t\to\infty} \frac{|Y(t)|}{(2Q^2(t)\log t)^{\frac{1}{2}}} = 1\right] = 1$$

for any $Q(t)$.

These examples show that we cannot make statements on the maxima of Gaussian processes without imposing certain conditions on the processes. The only general statement that we can make is that Lemma 2.3 gives an upper bound for $\limsup_{n\to\infty} |X(n)|/Q(n)$ if the limit is taken along the integers. But example (1) gives processes which grow faster than $(2\log n)^{\frac{1}{2}}$, therefore for these processes, their local behavior is the main factor influencing their growth rate.

Although it seems impossible to make general statements, most of the results of Sections 1 and 2 hold with only minor modifications if the stationarity conditions are dropped. To be more specific, in Theorem 1.1 and Corollary 1.3 stationarity is not used at all. The result holds for any continuous Gaussian process $X(t)$, $EX(t)^2 \leqq V(t)^2$, as long as

$$E\left|\frac{X(t)}{V(t)} - \frac{X(s)}{V(s)}\right|^2 \leqq \psi^2(|t-s|) \quad \text{for} \quad t, s \in [n, n+1], \tag{3.2}$$

$n = 0, 1, 2, \cdots$ and $\psi(|t-s|) = O(1/|\log|t-s||)^\alpha$, $\alpha > 1$.

In proving Theorems 2.6 and 2.8 (and Corollary 2.7) the property of stationary increments was used to enable us to use Lemma 2.2 and to provide an increasing function R so that

$$\left[E\left(\frac{X(t)}{R(t)} - \frac{X(s)}{R(s)}\right)^2\right]^{\frac{1}{2}} \leqq \frac{2R(|t-s|)}{\min(R(t), R(s))}, \tag{3.3}$$

$2^k \leqq t, s \leqq 2^{k+1}$, $|t-s| > 1$, $k = k_0, k_0 + 1, \cdots$. These results apply to any continuous Gaussian process as long as a function R can be found satisfying (3.3) and an equivalent to Lemma 2.2 can be obtained. The latter could be accomplished, for example, if one could find an increasing function ψ, satisfying (1.1) such that

$$\left[E\left(\frac{X(t)}{R(t)} - \frac{X(s)}{R(s)}\right)^2\right]^{\frac{1}{2}} < \psi(|t-s|), \qquad |t-s| \leqq 1.$$

The conditions required by Watanabe (1970) are similar to (3.2) and (3.3).

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