The Annals of Mathematical Statistics
1972, Vol. 43, No. 2, 498-507

ON THE DUBINS AND SAVAGE CHARACTERIZATION OF OPTIMAL STRATEGIES[1]

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An elegant characterization of optimal strategies for gambling problems was given by Dubins and Savage in the finitely additive setting of their book *How to Gamble If You Must*. An exposition of their ideas is given here in a measurable, countably additive framework. With the additional measurability assumptions, it becomes possible to treat a larger class of payoff functions. Also, necessary and sufficient conditions are given for a strategy to be ε-optimal, a problem not considered by Dubins and Savage.

1. Definitions and preliminaries. This section establishes the framework for the sequel and reports certain technical measurability results needed there. Most of the notation and definitions below are borrowed or adapted from [3].

The term *Borel set* is used here to mean a Borel subset of a complete separable metric space. Let X be a Borel set. Denote by $\mathscr{B}(X)$ the Borel subsets of X and by $\mathscr{P}(X)$ the set of all countably additive probability measures on $\mathscr{B}(X)$. If $\mathscr{P}(X)$ is given the usual weak topology, then it has the structure of a Borel set and the Borel subsets of $\mathscr{P}(X)$ may be described as the smallest σ-field of subsets which makes $\gamma \to \gamma(A)$ a measurable function from $\mathscr{P}(X)$ to the Borel line for each A in $\mathscr{B}(X)$. (A thorough discussion of the weak topology is in Chapter II of [6] and the Borel structure of $\mathscr{P}(X)$ is explored in [2].)

Let F be a Borel set to be regarded as the set of fortunes of a gambler or possible states of a system. Set $\mathscr{B} = \mathscr{B}(F)$ and $\mathscr{P} = \mathscr{P}(F)$. An element γ of $\mathscr{P}$ will be called a *gamble*, although Dubins and Savage use that term to mean a finitely additive probability measure defined on all subsets of F. A *gambling house* Γ on F assigns to every f in F a non-void set $\Gamma(f)$ of gambles. It is assumed that the set $\{(f, \gamma): \gamma \in \Gamma(f)\}$ is a Borel subset of the product space $F \times \mathscr{P}$. The implications of such an assumption were first studied by Strauch in [7].

A *strategy* σ is a sequence $\sigma_0, \sigma_1, \cdots$, where σ_0 is a gamble and, for $n > 0$, σ_n is a measurable map from $F \times \cdots \times F$ (n-factors) into $\mathscr{P}$. Let H be the countably infinite product $F \times F \times \cdots$ with the product Borel structure. The symbol "h" will always denote a typical element or *history* $(f_1, f_2, \cdots)$ of H.

Received August 10, 1970; revised May 18, 1971.

[1] The research was supported by the National Science Foundation under NSF Grant GP-9556.

Key words. gambling, optimal strategy.

A history σ naturally induces a probability measure $\mu(\sigma)$ on $\mathscr{B}(H)$. That is, the $\mu(\sigma)$-marginal distribution of the first coordinate f_1 is σ_0 and, given the first n coordinates are $(f_1, \cdots, f_n)$, the conditional $\mu(\sigma)$-distribution of f_{n+1} is $\sigma_n(f_1, \cdots, f_n)$. When there is no danger of confusion, we shall use the same notation σ for the strategy σ and the corresponding measure $\mu(\sigma)$.

A strategy σ is *essentially available at f in Γ* if $\sigma_0 \in \Gamma(f)$, and, for $n > 0$, $\sigma_n(f_1, \cdots, f_n) \in \Gamma(f_n)$ σ-almost surely. Thus the gambler must choose gambles available in the house at his current fortune almost surely at each stage of play. For $f \in F$, let $\Gamma^\infty(f)$ be the set of all strategies essentially available at f in Γ and let $\Gamma^\infty = \{(f, \mu(\sigma)): \sigma \in \Gamma^\infty(f)\}$. Then the set Γ^∞ is a Borel subset of $F \times \mathscr{P}(H)$. For a proof see Theorem 2.1 of [10] and Theorem 5.1 of [12].

A *partial history p* is a finite sequence of elements of F. If σ is a strategy and $p = (f_1, \cdots, f_n)$ is a partial history, then the *conditional strategy* $\sigma[p]$ is defined by

$$(\sigma[p])_0 = \sigma_n(f_1, \cdots, f_n)$$

and, for $k > 0$,

$$(\sigma[p])_k(f_1', \cdots, f_k') = \sigma_{n+k}(f_1, \cdots, f_n, f_1', \cdots, f_k') .$$

If is easy to check that $\sigma[f_1, \cdots, f_n]$ is a version of the regular conditional σ-distribution of $(f_{n+1}, f_{n+2}, \cdots)$ given $(f_1, \cdots, f_n)$. Also, if $\sigma \in \Gamma^\infty(f)$, then $\sigma[f_1, \cdots, f_n] \in \Gamma^\infty(f_n)$ σ-almost surely.

Now let g be a measurable function from H to the extended reals. For $h \in H$, the value $g(h)$ is to be regarded as the payoff received by a gambler who experiences the history h and g is called the *payoff function*. To simplify the exposition, it is assumed that g is bounded above. (Many of our results would still hold if other conditions on g or on the house Γ were used to insure the existence of $\int g d\sigma$ for every available σ. The assumption here roughly corresponds to the common one of a nonnegative loss function.) A gambler who plays the strategy σ has expected winnings $\int g d\sigma$. Let

$$V(f) = \sup_{\sigma \in \Gamma^\infty(f)} \int g d\sigma .$$

The function V is called the *strategic utility* of the house Γ and $V(f)$ may be regarded as the most that can be achieved by a gambler with fortune f. The function V is bounded above and has been shown (Theorem 5.2, [12]) to be universally measurable. (Recall that a function is universally measurable if it is measurable with respect to the completion of every measure on the Borel sets.)

2. The Dubins and Savage payoff function. For a reader familiar with gambling theory, this section should serve as motivation for the sequel. It is not, however, a logical prerequisite for what follows.

Let u be a bounded function on F and let σ be a strategy. Dubins and Savage in [3] define the utility of σ to be

$$u(\sigma) = \limsup_{t\to\infty} \int u(f_t)\, d\sigma \,. \tag{1}$$

The lim sup is taken over the directed set of stop rules. There are no measurability requirements for u, σ, or the stop rules t. (For a definition of "strategy," "stop rule," and the integral in (1), consult [3].) If u is assumed to be $\mathscr{B}$-measurable and σ is a strategy in the sense of this note, then, by Theorem 3.2 of [12],

$$u(\sigma) = \int u^* d\sigma \,, \tag{2}$$

where

$$u^*(h) = \limsup_{n\to\infty} u(f_n) \,. \tag{3}$$

Thus the problem studied by Dubins and Savage is, under our measurability assumptions, of the type described in Section 1 where the payoff function g is u^*.

Now assume u to be $\mathscr{B}$-measurable and bounded above. Let σ be a strategy and define $u(\sigma)$ as in (1) except that the lim sup is to be taken over all measurable incomplete (i. e., not necessarily finite) stop rules t which are finite almost surely under σ. Then, by Theorem 2 of [11], formula (2) continues to hold, which is not the case if the stop rules are required to be everywhere finite as in [3].

EXAMPLE 1. Let $F = \{-1, 1\}$ and $\sigma_0 = \sigma_n(f_1, \cdots, f_n) = \frac{1}{2}\,\delta(1) + \frac{1}{2}\,\delta(-1)$ for all $(f_1, \cdots, f_n)$. Thus σ is the distribution of a fair coin toss process with path space H. Consider also F' = set of integers and $\sigma_0' = \sigma_0$, $\sigma_n'(f_1', \cdots, f_n') = \frac{1}{2}\,\delta(f_n' + 1) + \frac{1}{2}\,\delta(f_n' - 1)$ for all $(f_1', \cdots, f_n')$. The strategy σ' on H' is the distribution of the partial sums of a fair coin toss process. Let $u(f') = \min(f', 1)$ for all $f' \in F'$.

Suppose t' is an (everywhere finite) stop rule on H'. Let t be the stop rule on H given by

$$t(f_1, f_2, f_3, \cdots) = t'(f_1, f_1 + f_2, f_1 + f_2 + f_3, \cdots) \,.$$

Notice $f_1 + \cdots + f_t$ has the same distribution under σ as does $f_{t'}'$ under σ'. Now $u(f_1), u(f_1 + f_2), \cdots$ is an expectation decreasing semimartingale under σ and, since F is finite, t is bounded (Theorem 2.9.1 of [3]). Hence,

$$0 = \int u(f_1)\, d\sigma \geqq \int u(f_1 + \cdots + f_t)\, d\sigma = \int u(f_{t'}')\, d\sigma' \,,$$

for every stop rule t'. *A fortiori*, $\limsup_{t'\to\infty} \int u(f_{t'}')\, d\sigma' \leqq 0$. However, $u^* = 1$ σ'-almost surely, so that $\int u^* d\sigma' = 1$.

When it seems appropriate, results below will be specialized to the case when $g = u^*$ for some u and connections made with the original work in [3].

Notice that

$$u^*(f_1, f_2, \cdots) = u^*(f_2, \cdots)$$

for every $h = (f_1, f_2, \cdots)$.

3. Properties of V when the payoff function is shift-invariant. It is assumed for the remainder that the payoff function g, in addition to being bounded above and measurable, satisfies $g(f_1, f_2, \cdots) = g(f_2, \cdots)$ for every $h = (f_1, f_2, \cdots)$.

Intuitively, the gambler who has fortune f_1 after the first play should wish to play exactly as though he were entering the game with initial fortune f_1.

By the previous section, such shift-invariant payoff functions include, at least in a countably additive setting, those studied by Dubins and Savage. Moreover, many sequential optimization problems, which do not appear to have shift-invariant payoff functions, can be formulated so as to fit the model of this note.

EXAMPLE 2. Suppose r is a bounded function on F, $0 < \beta < 1$, and $g(h) = \sum_{n=1}^{\infty} \beta^n r(f_n)$. Such payoff functions have been studied by Blackwell [1] and others. Of course, g is not typically shift-invariant.

Let us follow Section 12.2 of [3] and set $f_n' = (f_1, \cdots, f_n)$ and $h' = (f_1', f_2', \cdots)$. Then, if $u(f_n') = \sum_{k=1}^{n} \beta^k u(f_k)$, we have $u^*(h') \equiv \limsup_{n\to\infty} u(f_n') = g(h)$, and u^* is shift-invariant on H'. Further examples and a more complete discussion are in [3].

A function Q on F is *excessive for* Γ if, for every $f \in F$ and $\gamma \in \Gamma(f)$, $\int Q d\gamma \leqq Q(f)$.

THEOREM 1. *The strategic utility V is excessive for Γ.*

Rather than prove Theorem 1, notice it is a special case of our next result, for whose statement we need another definition.

A *stopping variable* t is a measurable map defined on H, having values in the set $\{1, 2, \cdots, +\infty\}$, and such that, given $h = (f_1, f_2, \cdots)$ and $h' = (f_1', f_2', \cdots)$, if $t(h) = n$ and $f_i' = f_i$ for $1 \leqq i \leqq n$, then $t(h') = n$.

THEOREM 2. *If $f \in F$, $\sigma \in \Gamma^{\infty}(f)$, and t is a stopping variable such that $\sigma[t < +\infty] = 1$, then*

$$\int V(f_t)\, d\sigma \leqq V(f)\,.$$

(Here, $f_t(h) = f_{t(h)}$.)

PROOF. Let μ_t be the distribution of f_t under σ. Since V is measurable with respect to the completion of $\mathscr{B}$ under μ_t, there is a $\mathscr{B}$-measurable function Q such that $\mu_t\{f' : Q(f') = V(f')\} = 1$ and $Q(f') \leqq V(f')$ for all f' in F.

Let $\varepsilon > 0$ and define

$$A = \{(f', \mu(\sigma')): \int g d\sigma' \geqq Q(f') - \varepsilon, \sigma' \in \Gamma^{\infty}(f')\} .$$

Then A is a measurable subset of $F \times \mathscr{P}(H)$ and each f'-section of A is non-empty. Hence, by Theorem 6.3 of [4], there is a measurable map φ: $F \to \mathscr{P}(H)$ such that $\mu_t\{(f', \varphi(f')) \in A\} = 1$.

Now, for each $f' \in F$, chose a strategy $\bar{\sigma}(f')$ such that $\mu(\bar{\sigma}(f')) = \varphi(f')$ and choose the $\bar{\sigma}(f')$ so that $\mu_t\{f': \bar{\sigma}(f') \in \Gamma^{\infty}(f')\} = 1$.

Let σ' be the strategy which is the composition of σ with the family $\bar{\sigma}(\cdot)$ at time t. That is,

$$(\sigma')_0 = \sigma_0 ,$$

and, for every partial history $(f_1, \cdots, f_n)$,

$$\begin{aligned} (\sigma')_n(f_1, \cdots, f_n) &= \sigma_n(f_1, \cdots, f_n) && \text{if} \quad t > n , \\ &= (\bar{\sigma}(f_t))_0 && \text{if} \quad t = n , \\ &= (\bar{\sigma}(f_t))_{n-t}(f_{t+1}, \cdots, f_n) && \text{if} \quad t < n , \end{aligned}$$

where $t = t(f_1, \cdots, f_n, \cdots)$.

Then $\sigma' \in \Gamma^{\infty}(f)$ and

$$\int V(f_t)\, d\sigma = \int Q d\mu_t \leqq \int \{\int g d\bar{\sigma}(f_t) + \varepsilon\}\, d\mu_t = \int g d\sigma' + \varepsilon$$

(by Fubini (Theorem II. 14, [5]) and the shift-invariance of g) $\leqq V(f) + \varepsilon$. □

Theorem 2 is essentially an optional stopping theorem and implies that $V(f_1), V(f_2), \cdots$ is a (generalized) expectation decreasing semi-martingale with respect to any $\sigma \in \Gamma^{\infty}(f)$. Perhaps the reader should be reminded that such an optional stopping result does not hold for arbitrary (generalized) semi-martingales which are bounded from one side.

EXAMPLE 3. Let f_n, σ, and u be as in Example 1. Let $t(h) = \min\{n: f_1 + \cdots + f_n = 1\}$. Then $\sigma[t < \infty] = 1$ and $\int u(f_1 + \cdots + f_t)\, d\sigma = 1 > 0 = \int u(f_1)\, d\sigma$.

COROLLARY 1. *Let $f \in F$, $\sigma \in \Gamma^{\infty}(f)$, and t and s be stopping variables with $\sigma[t \leqq s < \infty] = 1$. Then*

$$\int V(f_t)\, d\sigma \geqq \int V(f_s)\, d\sigma .$$

PROOF. For every $h \in H$, let $p_t(h) = (f_1, \cdots, f_{t(h)})$. Then the conditional strategies $\sigma[p_t]$ are essentially available at f_t in Γ σ-almost surely. Let $s[p_t]$ be the conditional stopping variable defined by

$$s[p_t](f_1', f_2', \cdots) = s(f_1, \cdots, f_t, f_1', f_2', \cdots) - t .$$

Then, by Theorem 2,

$$\int V(f_{s[p_t]})\, d\sigma[p_t] \leqq V(f_t)$$

σ-almost surely. By Fubini's Theorem,

$$\int \{V(f_t) - V(f_s)\}\, d\sigma = \int \{V(f_t) - \int V(f_{s[p_t]})\, d\sigma[p_t]\}\, d\sigma\ .$$

The desired inequality follows. Interpret $\int V(f_{s[p_t]})\, d\sigma[p_t]$ as $V(f_t)$ when $s = t$. □

The next result is a version of the so-called functional equation of dynamic programming.

THEOREM 3. *For every $f \in F$, $V(f) = \sup_{\gamma \in \Gamma(f)} \int V\, d\gamma$.*

PROOF. Let $\varepsilon > 0$. Choose $\sigma \in \Gamma^{\infty}(f)$ such that $\int g\, d\sigma \geqq V(f) - \varepsilon$. Then $\sigma_0 \in \Gamma(f)$ and

$$\sup_{\gamma \in \Gamma(f)} \int V\, d\gamma \geqq \int V\, d\sigma_0 \geqq \int \{\int g\, d\sigma[f_1]\}\, d\sigma_0 = \int g\, d\sigma \geqq V(f) - \varepsilon\ .$$

This proves one of the needed inequalities. The other is immediate from Theorem 1. □

Now let σ be a strategy and $\{Q_n\}_{n \geqq 1}$ a sequence of universally measurable functions on F which are uniformly bounded above. Define

$$Q(\sigma) = \limsup_{t \to \infty} \int Q_t(f_t)\, d\sigma\ , \tag{4}$$

where the lim sup is over the directed set of stopping variables t such that $\sigma[t < \infty] = 1$. Also, define Q^* on H by

$$Q^*(h) = \limsup_{n \to \infty} Q_n(f_n)\ . \tag{5}$$

It will usually be the case below that all the Q_n are equal to some fixed function Q.

THEOREM 4. *If the functions Q_n are universally measurable and uniformly bounded above, then*

$$Q(\sigma) = \int Q^*\, d\sigma\ .$$

PROOF. If the Q_n are $\mathscr{B}$-measurable, the desired formula is a special case of Theorem 2 in [11].

If the Q_n are universally measurable, then there is a sequence R_n of $\mathscr{B}$-measurable functions such that

$$\sigma\{h\colon Q_n(f_n) = R_n(f_n) \quad \text{for} \quad n = 1, 2, \cdots\} = 1.$$

The R_n may be chosen to be uniformly bounded above. Hence,

$$\int Q^*\, d\sigma = \int R^*\, d\sigma = R(\sigma) = Q(\sigma)\ .\ \square$$

COROLLARY 2. *If V is the strategic utility of Γ and σ is any strategy, then*

$$V(\sigma) = \int V^*\, d\sigma\ .$$

Another formula for $V(\sigma)$ is given by

THEOREM 5. *If V is the strategic utility of Γ and $\sigma \in \Gamma^{\infty}(f)$ for some $f \in F$, then*

$$\int V(f_n)\, d\sigma \searrow V(\sigma) \quad \text{as} \quad n \to \infty .$$

PROOF. By Corollary 1, if $n \geqq m$, then $\int V(f_n)\, d\sigma \leqq \int V(f_m)\, d\sigma$. So the limit exists.

Again by Corollary 1, if t is a stopping variable, $\sigma[t < \infty] = 1$, and $t \geqq n$, then

$$\int V(f_t)\, d\sigma \leqq \int V(f_n)\, d\sigma .$$

Now take the lim sup as $t \to \infty$ and then the limit as $n \to \infty$ to get

$$V(\sigma) \leqq \lim_{n\to\infty} \int V(f_n)\, d\sigma .$$

But, by Fatou's Lemma and Corollary 2,

$$\lim_{n\to\infty} \int V(f_n)\, d\sigma \leqq \int V^*\, d\sigma = V(\sigma) . \quad \square$$

The next theorem exemplifies the sort of convergence result possible if the processes under consideration are not allowed to grow too large in a negative direction.

THEOREM 6. *Let $f \in F$ and $\sigma \in \Gamma^{\infty}(f)$.*

(a) *If $V(\sigma) > -\infty$, then $V(f_n) \to V^*(h)$ σ-almost surely as $n \to \infty$.*

(b) *If $\int g d\sigma > -\infty$, then $\int g d\sigma[f_1, \cdots, f_n] \to g(h)$ σ-almost surely as $n \to \infty$ and $\sigma[g > V^*] = 0$.*

PROOF. (a) Since $V(f_n)$ is an expectation decreasing process by Theorem 2 and $\inf_n \int V(f_n)\, d\sigma = V(\sigma) > -\infty$ by Theorem 5, $V(f_n)$ converges almost surely by a supermartingale convergence theorem (Theorem V. 17, [5]). The limit is, of course, V^* almost surely.

(b) By the assumption, g is integrable with respect to σ. Also, $\int g d\sigma[f_1, \cdots, f_n]$ is a version of the conditional σ-expectation of g given $(f_1, \cdots, f_n)$. Thus the convergence is by another martingale theorem (Theorem V. 18, [5]).

Finally,

$$V(f_n) \geqq \int g\, d\sigma[f_1, \cdots, f_n]$$

σ-almost surely since $\sigma[f_1, \cdots, f_n] \in \Gamma^{\infty}(f_n)$ σ-almost surely. Pass to the limit as $n \to \infty$ to prove the final assertion. $\square$

The final result of this section will be the starting point for the discussion of optimality here as its analogue was in [3].

THEOREM 7. *Let $f \in F$ and $\sigma \in \Gamma^{\infty}(f)$. Then $V(f) \geqq V(\sigma) \geqq \int g d\sigma$.*

PROOF. The first inequality is immediate from Theorem 2 and the definition of $V(\sigma)$ in (4). The second follows from Corollary 2 and Theorem 6(b). $\square$

4. Optimal strategies. Let $f \in F$ and $\sigma \in \Gamma^{\infty}(f)$ be fixed for the discussion in this section.

The strategy σ is said to be *optimal* (ε-*optimal*) *at* f *in* Γ iff $\int g d\sigma = V(f)$ ($\int g d\sigma \geqq V(f) - \varepsilon$).

The strategy σ is *thrifty* (ε-*thrifty*) iff $V(\sigma) = V(f)(V(\sigma) \geqq V(f) - \varepsilon)$.

The strategy σ is *equalizing* (ε-*equalizing*) iff $\int g d\sigma = V(\sigma)$ ($\int g d\sigma \geqq V(\sigma) - \varepsilon$).

An immediate consequence of Theorem 7 is

THEOREM 8. *A strategy is optimal* iff *it is thrifty and equalizing. A strategy is* ε-*optimal* iff *it is* ε_1-*thrifty and* ε_2-*equalizing for some* ε_1, ε_2 *such that* $\varepsilon_1 + \varepsilon_2 \leqq \varepsilon$.

Thus the notion of optimality will be characterized if we characterize thrifty strategies and equalizing strategies. The results which follow give formulae for the numbers $V(f) - V(\sigma)$ and $V(\sigma) - \int g d\sigma$, which measure the two ways in which σ may depart from optimality.

Let

$$\varepsilon_0 = V(f) - \int V \, d\sigma_0 ,$$

and, for $n > 0$ and $h \in H$, let

$$\varepsilon_n(h) = \varepsilon_n(f_1, \cdots, f_n) = V(f_n) - \int V \, d\sigma_n(f_1, \cdots, f_n) .$$

Notice that, by Theorem 1, $\varepsilon_n \geqq 0$ σ-almost surely for all n.

THEOREM 9. *The strategy* σ *is* ε-*thrifty* iff

$$\int (\textstyle\sum_{n=0}^{\infty} \varepsilon_n) \, d\sigma \leqq \varepsilon .$$

In fact,

$$V(f) - V(\sigma) = \int (\textstyle\sum_{n=0}^{\infty} \varepsilon_n) \, d\sigma .$$

PROOF. It is easy to see by induction on n that

$$\int V(f_n) \, d\sigma = V(f) - \int (\textstyle\sum_{k=0}^{n-1} \varepsilon_k) \, d\sigma .$$

Let $n \to \infty$ and the result follows from the monotone convergence theorem and Theorem 5. □

The result corresponding to Theorem 9 for a finitely additive theory of gambling is in [9].

THEOREM 10. *The strategy* σ *is thrifty* iff, *for all* n, $\varepsilon_n = 0$ σ-*almost surely.*

PROOF. Immediate from Theorem 9. □

Thus a gambler is thrifty iff the $V(f_n)$ process is a (generalized) martingale with respect to his strategy. In intuitive terms, he must play in such a way that his prospective optimal winnings (i.e., his current values for V) almost never decrease in expectation.

Stated still another way, a strategy is thrifty if the gambler almost always chooses gambles γ so that $\int V \, d\gamma$ attains the supremum of the functional

equation in Theorem 3. A common error is to assume that strategies so constructed are necessarily optimal, which brings us to equalizing strategies.

THEOREM 11. *If* $V(\sigma) = -\infty$, *then* σ *is equalizing. If* $V(\sigma) > -\infty$ *and* $\varepsilon > 0$, *then* σ *is* ε*-equalizing* iff $\int (V^* - g)\, d\sigma \leqq \varepsilon$ *and, in fact,*

$$V(\sigma) - \int g\, d\sigma = \int (V^* - g)\, d\sigma .$$

PROOF. The first statement is obvious and the second follows from Corollary 2. □

A sufficient condition for σ to be ε-equalizing is given next. The notation "i.o." is used below as an abbreviation for the phrase "for infinitely many n."

THEOREM 12. *If* $\varepsilon > 0$ *and* $\sigma\{h\colon V(f_n) \leqq g(h) + \varepsilon \text{ i.o.}\} = 1$, *then* σ *is* ε*-equalizing.*

PROOF. We can assume $V(\sigma) > -\infty$. Then, by the hypothesis and Theorem 6(a), $\sigma[V^* \leqq g + \varepsilon] = 1$. Now use Theorem 11. □

The corresponding necessary condition is in

THEOREM 13. *If* $\varepsilon > 0$, σ *is* ε^2*-equalizing and* $V(\sigma) > -\infty$, *then*

$$\sigma\{h\colon V(f_n) \leqq g(h) + \varepsilon \text{ i.o.}\} \geqq 1 - \varepsilon .$$

PROOF. By assumption, $\int g\, d\sigma \geqq V(\sigma) - \varepsilon^2 > -\infty$. So, by Theorem 6(b), $V^* - g \geqq 0$ σ-almost surely. Also, by Theorem 11, $\int (V^* - g)\, d\sigma \leqq \varepsilon^2$. Hence, $\sigma\{h\colon V^*(h) - g(h) < \varepsilon\} \geqq 1 - \varepsilon$. The conclusion now follows from the definition of V^* as $\limsup_{n\to\infty} V(f_n)$. □

The previous two theorems imply

THEOREM 14. *Suppose* $V(\sigma) > -\infty$. *Then* σ *is equalizing* iff, *for all* $\varepsilon > 0$,

$$\sigma\{h\colon V(f_n) \leqq g(h) + \varepsilon \text{ i.o.}\} = 1 .$$

Let us now assume that u is a $\mathscr{B}$-measurable function from F to the reals which is bounded above, and consider the associated Dubins and Savage payoff function u^*. It is of some interest to reinterpret our results for such a payoff function and thus make explicit contact with the original work in [3].

THEOREM 15. *Let* $g = u^*$ *and suppose* $V(\sigma) > -\infty$. *Then the following are equivalent:*

(a) σ *is equalizing*;
(b) $u(\sigma) = V(\sigma)$;
(c) *for all* $\varepsilon > 0$, $\sigma\{h\colon u^*(h) \geqq V(f_n) - \varepsilon \text{ i.o.}\} = 1$;
(d) *for all* $\varepsilon > 0$, $\sigma\{h\colon u(f_n) \geqq V(f_n) - \varepsilon \text{ i.o.}\} = 1$;
(e) *for all* $\varepsilon > 0$, $v_\varepsilon(\sigma) = 1$, *where* v_ε *is the indicator function of the set* $\{f\colon u(f) \geqq V(f) - \varepsilon\}$.

PROOF. Since $u(\sigma) = \int u^* \, d\sigma$, (a) and (b) are equivalent. Also, (a) and (c) are equivalent by the previous theorem; and (d) and (e) are easily seen to be equivalent if Q is set equal to v_ε in Theorem 4. It will be enough to show (c) and (d) are equivalent.

Let $\{x_n\}$ and $\{y_n\}$ be sequences of real numbers and let $x^* = \limsup_{n\to\infty} x_n$. The following implications can be checked for any $\varepsilon > 0$:

(i) if $x_n \geqq y_n - \varepsilon$ i.o., then $x^* \geqq y_n - 2\varepsilon$ i.o.;

(ii) if $x^* \geqq y_n - \varepsilon$ i.o. and y_n converges to a finite limit, then $x_n \geqq y_n - 2\varepsilon$ i.o.

By (i), we see that (d) $\Rightarrow$ (c). By Theorem 6(a), $V(f_n)$ converges σ-almost surely to V^*, so that the opposite implication follows from (ii). □

Condition (b) was used in [3] to define equalizing and was shown there to be equivalent to (e). Theorems 13 and 14 can also be reinterpreted for $g = u^*$.

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