The Annals of Mathematical Statistics
1972, Vol. 43, No. 2, 480-489

THE RANDOMIZATION MODEL FOR INCOMPLETE BLOCK DESIGNS

BY J. ROBINSON

University of Connecticut and University of Sydney

1. Introduction. The randomization scheme for incomplete block designs consists of assigning treatments to a plan of blocks and plots and allocating the block positions and the positions of plots within blocks randomly to the experimental units. It is the purpose of this paper to consider the asymptotic properties of estimates of treatment contrasts and of certain permutation tests under the randomization models appropriate to these designs. Similar results have been obtained for the randomized block design by Wald and Wolfowitz (1944) and for the completely randomized design by Silvey (1954).

In Section 2, two theorems are derived, giving the limiting distribution of linear forms in the universe of restricted permutations obtained by the randomization procedure for incomplete block designs. These theorems are applied in Section 3, to show that the permutation distributions of certain test statistics have the same limiting form as their limiting distributions under the usual normal theory models. In Section 4, combined estimation of treatment contrasts, using both intra-block and inter-block information, is considered and the limiting distribution of a combined test statistic is obtained.

2. Two combinatorial limit theorems. Let $c_{nij}, a_n(i, j)$ $(i = 1, \cdots, n, j = 1, \cdots, k)$ be $2kn$ real numbers defined for every positive integer n. Let $(I_{n1}, \cdots, I_{nn})$ be the random variable taking each permutation of $(1, \cdots, n)$ with equal probability and let $(J_{i1}, \cdots, J_{ik})$, $(i = 1, \cdots, n)$ be n independent random variables each taking each permutation of $(1, \cdots, k)$ with equal probability independently of $(I_{n1}, \cdots, I_{nn})$. We consider the asymptotic distribution of certain random variables of the form

$$L_n = \sum_{i=1}^{n} \sum_{j=1}^{k} c_{nij} a_n(I_{ni}, J_{ij}) , \tag{1}$$

as n tends to infinity.

Write $\mathbf{I}_n = (I_{n1}, \cdots, I_{nn})$. We can show immediately that

$$E(L_n \mid \mathbf{I}_n) = k \sum_{i=1}^{n} c_{ni\cdot} a_n(I_{ni}, \cdot) \tag{2}$$

and then that

$$E(L_n) = nkc_{n\cdot\cdot} a_n(\cdot, \cdot) \tag{3}$$

Received May 26, 1970; revised August 18, 1971.

and that

$$V(L_n \mid \mathbf{I}_n) = \frac{1}{k-1} \sum_{i=1}^{n} [\sum_{j=1}^{k} (c_{nij} - c_{ni\cdot})^2] \times [\sum_{j=1}^{k} \{a_n(I_{ni}, j) - a_n(I_{ni}, \cdot)\}^2] . \tag{4}$$

Then

$$E[V(L_n \mid \mathbf{I}_n)] = \frac{1}{n(k-1)} \sum_{i=1}^{n} \sum_{j=1}^{k} (c_{nij} - c_{ni\cdot})^2 \times \sum_{i=1}^{n} \sum_{j=1}^{k} [a_n(i, j) - a_n(i, \cdot)]^2 \tag{5}$$

and

$$V[E(L_n \mid \mathbf{I}_n)] = \frac{k^2}{n-1} \sum_{i=1}^{n} (c_{ni\cdot} - c_{n\cdot\cdot})^2 \sum_{i=1}^{n} [a_n(i, \cdot) - a_n(\cdot, \cdot)]^2 , \tag{6}$$

and hence

$$\begin{aligned} V(L_n) &= E[V(L_n \mid \mathbf{I}_n)] + V[E(L_n \mid \mathbf{I}_n)] \\ &= \frac{1}{n(k-1)} \sum_{i=1}^{n} \sum_{j=1}^{k} (c_{nij} - c_{ni\cdot})^2 \sum_{i=1}^{n} \sum_{j=1}^{k} [a_n(i, j) - a_n(i, \cdot)]^2 \\ &\quad + \frac{k^2}{n-1} \sum_{i=1}^{n} (c_{ni\cdot} - c_{n\cdot\cdot})^2 \sum_{i=1}^{n} [a_n(i, \cdot) - a_n(\cdot, \cdot)]^2 . \end{aligned} \tag{7}$$

We use the notation that the arithmetic mean over a subscript is denoted by replacing the subscript with a dot.

Suppose the $a_n(i, j)$ satisfy

$$\frac{\sum_{i=1}^{n} \{\sum_{j=1}^{k} [a_n(i, j) - a_n(i, \cdot)]^2\}^2}{\{\sum_{i=1}^{n} \sum_{j=1}^{k} [a_n(i, j) - a_n(i, \cdot)]^2\}^2} = o(1) \tag{8}$$

or all $a_n(i, j) - a_n(i, \cdot)$ are zero; and

$$\frac{\sum_{i=1}^{n} [a_n(i, \cdot) - a_n(\cdot, \cdot)]^s}{\{\sum_{i=1}^{n} [a_n(i, \cdot) - a_n(\cdot, \cdot)]^2\}^{s/2}} = o(1) , \qquad s = 3, 4, \cdots, \tag{9}$$

or all $a_n(i, \cdot) - a_n(\cdot, \cdot)$ are zero.

Further, suppose the c_{nij} satisfy

$$\frac{\frac{1}{n(k-1)} \sum_{i=1}^{n} [\sum_{j=1}^{k} (c_{nij} - c_{ni\cdot})^2]^2}{\left[\frac{1}{n(k-1)} \sum_{i=1}^{n} \sum_{j=1}^{k} (c_{nij} - c_{ni\cdot})^2\right]^2} = O(1) , \tag{10}$$

or all $c_{nij} - c_{ni\cdot}$ are zero; and

$$\frac{\frac{1}{n} \sum_{i=1}^{n} (c_{ni\cdot} - c_{n\cdot\cdot})^s}{\left[\frac{1}{n} \sum_{i=1}^{n} (c_{ni\cdot} - c_{n\cdot\cdot})^2\right]^{s/2}} = O(1) , \qquad s = 3, 4, \cdots, \tag{11}$$

or all $c_{ni\cdot} - c_{n\cdot\cdot}$ are zero.

We may replace the conditions (9) and (11) by a condition like that proposed by Hoeffding (1951),

$$(12) \qquad \lim_{n\to\infty} n^{(r/2)-1} \frac{\sum_{i=1}^n (c_{ni\cdot} - c_{n\cdot\cdot})^r \sum_{i=1}^n [a_n(i, \cdot) - a_n(i, \cdot)]^r}{\{\sum_{i=1}^n (c_{ni\cdot} - c_{n\cdot\cdot})^2 \sum_{i=1}^n [a_n(i, \cdot) - a_n(\cdot, \cdot)]^2\}^{r/2}} = 0\,, \qquad r = 3, 4, \cdots$$

or all $(c_{ni\cdot} - c_{n\cdot\cdot})$ or all $a_n(i, \cdot) - a_n(\cdot, \cdot)$ are zero. The conditions (8) and (10) may be replaced by the condition

$$(13) \qquad \lim_{n\to\infty} n \frac{\sum_{i=1}^n d_{ni}^2 \sum_{i=1}^n f_{ni}^2}{\{\sum_{i=1}^n d_{ni} \sum_{i=1}^n f_{ni}\}^2} = 0$$

or all d_{ni} or all f_{ni} are zero; where

$$d_{ni} = \sum_{j=1}^k (c_{nij} - c_{ni\cdot})^2$$

and

$$f_{ni} = \sum_{j=1}^k [a_n(i, j) - a_n(i, \cdot)]^2\,.$$

THEOREM 1. *If the sets of numbers $a_n(i, j)$ and c_{nij} satisfy the conditions* (12) *and* (13) *then*

$$(14) \qquad L_n{}^0 = \frac{L_n - E(L_n)}{(V(L_n))^{\frac{1}{2}}}\,,$$

where L_n is defined by (1), *has a limiting normal distribution with mean zero and variance* 1, *unless L_n is a constant.*

PROOF. First, assume that neither $E[V(L_n \mid \mathbf{I}_n)]$ nor $V[E(L_n \mid \mathbf{I}_n)]$ is zero.

Let

$$L_n{}^* = \frac{L_n - E(L_n \mid \mathbf{I}_n)}{[V(L_n \mid \mathbf{I}_n)]^{\frac{1}{2}}} = \frac{\sum_{i=1}^n Y_{ni}}{[\sum_{i=1}^n V(Y_{ni} \mid \mathbf{I}_n)]^{\frac{1}{2}}}\,,$$

where

$$Y_{ni} = \sum_{j=1}^k (c_{nij} - c_{ni\cdot})[a_n(I_{ni}, J_{ij}) - a_n(I_{ni}, \cdot)]\,.$$

Given $\mathbf{I}_n$, the Y_{ni} are independently distributed, so from the Liapounov version of the central limit theorem, if

$$(15) \qquad R(\mathbf{I}_n) = \frac{\sum_{i=1}^n E(Y_{ni}^4 \mid \mathbf{I}_n)}{[\sum_{i=1}^n E(Y_{ni}^2 \mid \mathbf{I}_n)]^2} \to 0$$

then

$$(16) \qquad |P[L_n{}^* < x \mid \mathbf{I}_n] - \Phi(x)| \to 0 \quad \text{for any} \quad x,$$

where $\Phi(x)$ is the distribution function of a standardized normal variate.

Now by the Cauchy inequality $Y_{ni}^2 \leqq d_{ni} f_{nI_{ni}}$, so

$$\frac{\sum_{i=1}^n E(Y_{ni}^4 \mid \mathbf{I}_n)}{[\sum_{i=1}^n E(Y_{ni}^2 \mid \mathbf{I}_n)]^2} \leqq \left\{\frac{E[V(L_n \mid \mathbf{I}_n)]}{V(L_n \mid \mathbf{I}_n)}\right\}^2 \frac{\sum_{i=1}^n d_{ni}^2 f_{nI_{ni}}^2}{\left[\frac{1}{n(k-1)} \sum_{i=1}^n d_{ni} \sum_{i=1}^n f_{ni}\right]^2} .$$

The second factor on the right tends to zero in probability since it is nonnegative and its expected value tends to zero from (13). $V(L_n \mid \mathbf{I}_n)/E[V(L_n \mid \mathbf{I}_n)]$ tends to 1 in probability, since this ratio has expectation 1 and condition (13) implies that its variance tends to 0. So $R(\mathbf{I}_n) \to 0$ in probability.

That is, given $\varepsilon > 0$, $\eta > 0$, there exists an $n(\varepsilon, \eta)$ such that for $n > n(\varepsilon, \eta)$

$$P\{|R(\mathbf{I}_n)| < \varepsilon\} > 1 - \eta .$$

So from the result (16), given $\delta > 0$ we can find an $n(\delta)$ such that for $n > \max(n(\delta), n(\varepsilon, \eta))$

$$|P[L_n^* < x \mid \mathbf{I}_n] - \Phi(x)| < \delta$$

with probability greater than $1 - \eta$. That is $P[L_n^* < x \mid \mathbf{I}_n] \to \Phi(x)$ in probability.

Hoeffding (1951) has shown that condition (12) is sufficient for the asymptotic normality of

$$L_n^{**} = \frac{E(L_n \mid \mathbf{I}_n) - E(L_n)}{\{V[E(L_n \mid \mathbf{I}_n)]\}^{\frac{1}{2}}} .$$

Now

$$L_n^0 = \left\{\frac{E[V(L_n \mid \mathbf{I}_n)]}{V(L_n)}\right\}^{\frac{1}{2}} \left\{\frac{V(L_n \mid \mathbf{I}_n)}{E[V(L_n \mid \mathbf{I}_n)]}\right\}^{\frac{1}{2}} L_n^* + \left\{\frac{V[E(L_n \mid \mathbf{I}_n)]}{V(L_n)}\right\}^{\frac{1}{2}} L_n^{**} . \tag{17}$$

The factor $\{V(L_n \mid \mathbf{I}_n)/E[V(L_n \mid \mathbf{I}_n)]\}^{\frac{1}{2}}$ tends to one in probability and the sum of squares of the two constant factors is one. This, together with the asymptotic normality of the conditional distribution of L_n^* and of the distribution of L_n^{**}, implies that L_n^0 has a standard normal limit distribution.

If one of $E[V(L_n \mid \mathbf{I}_n)]$ and $V[E(L_n \mid \mathbf{I}_n)]$ is zero then, from (17), it follows that L_n^0 has a standard normal limit distribution, since one term would tend to 0 in probability. If both are zero then L_n is a constant.

REMARK. Condition (12), which is sufficient for the asymptotic normality of L_n^{**}, could be replaced by a necessary and sufficient condition of Lindeberg type as proposed by Hájek (1961).

Consider the s linear forms

$$L_{nm} = \sum_{i=1}^n \sum_{j=1}^k c_{nij}^{(m)} a_n(I_{ni}, J_{ij}) , \qquad m = 1, \cdots, s , \tag{18}$$

each of which satisfy the conditions of Theorem 1 and which are also orthogonal, in the sense that

$$(19)\qquad \sum_{i=1}^{n}\sum_{j=1}^{k}(c_{nij}^{(m)}-c_{ni\cdot}^{(m)})(c_{nij}^{(m')}-c_{ni\cdot}^{(m')})=0\,,\qquad m\neq m'\,,$$

and

$$(20)\qquad \sum_{i=1}^{n}(c_{ni\cdot}^{(m)}-c_{n\cdot\cdot}^{(m)})(c_{ni\cdot}^{(m')}-c_{n\cdot\cdot}^{(m')})=0\,,\qquad m\neq m'\,.$$

Then we may prove a theorem concerning their joint distribution.

THEOREM 2. *If each of the s linear forms given in* (18) *satisfies the conditions of Theorem* 1 *and if they satisfy the conditions* (19) *and* (20), *then the joint limiting distribution of*

$$(21)\qquad L_{nm}^{0}=\frac{L_{nm}-E(L_{nm})}{[V(L_{nm})]^{\frac{1}{2}}}\,,\qquad m=1,\cdots,s\,,$$

is multivariate normal with means zero and variance covariance matrix **I**, *unless some* L_{nm} *are constant.*

PROOF. Consider the random variable $\sum_{m=1}^{s}\alpha_m L_{nm}^0$. To show that this random variable has an asymptotically normal distribution, we need to show that $\sum_{m=1}^{s}\alpha_m c_{nij}^{(m)}$ and $a_n(i,j)$ satisfy the conditions (12) and (13). Now

$$\begin{aligned}\sum_{i=1}^{n}\sum_{j=1}^{k}[\textstyle\sum_{m=1}^{s}\alpha_m(c_{nij}^{(m)}-c_{ni\cdot}^{(m)})]^2 &= \sum_{i=1}^{n}\sum_{j=1}^{k}\sum_{m=1}^{s}\sum_{m'=1}^{s}\alpha_m\alpha_{m'}(c_{nij}^{(m)}-c_{ni\cdot}^{(m)})(c_{nij}^{(m')}-c_{ni\cdot}^{(m')})\\ &= \sum_{m=1}^{s}\alpha_m^{\,2}\sum_{i=1}^{n}\sum_{j=1}^{k}(c_{nij}^{(m)}-c_{ni\cdot}^{(m)})^2\,.\end{aligned}$$

Also

$$\begin{aligned}\sum_{i=1}^{n}\{\textstyle\sum_{j=1}^{k}[\sum_{m=1}^{s}\alpha_m(c_{nij}^{(m)}-c_{ni}^{(m)})]^2\}^2 &\leqq \sum_{i=1}^{n}[\textstyle\sum_{j=1}^{k}(\sum_{m=1}^{k}\alpha_m^{\,2})\{\sum_{m=1}^{s}(c_{nij}^{(m)}-c_{ni\cdot}^{(m)})^2\}]^2\\ &\leqq s^2(\textstyle\sum_{m=1}^{s}\alpha_m^{\,2})^2\sum_{m=1}^{s}\sum_{i=1}^{n}[\sum_{j=1}^{k}(c_{nij}^{(m)}-c_{ni\cdot}^{(m)})^2]^2\,.\end{aligned}$$

We may consider each α_m to be nonzero. Then unless

$$\sum_{i=1}^{n}\sum_{j=1}^{k}(c_{nij}^{(m)}-c_{ni\cdot}^{(m)})^2=0\quad\text{or}\quad\sum_{i=1}^{n}\sum_{j=1}^{k}[a_n(i,j)-a_n(i,\cdot)]^2=0\,,$$

$$\begin{aligned}&n\frac{\sum_{i=1}^{n}\left[\sum_{j=1}^{k}\left\{\sum_{m=1}^{s}\alpha_m(c_{nij}^{(m)}-c_{ni\cdot}^{(m)})\right\}^2\right]^2\sum_{i=1}^{n}\left\{\sum_{j=1}^{k}[a_n(i,j)-a_n(i,\cdot)]^2\right\}^2}{\left\{\sum_{i=1}^{n}\sum_{j=1}^{k}\left[\sum_{m=1}^{s}\alpha_m(c_{nij}^{(m)}-c_{ni\cdot}^{(m)})\right]^2\sum_{i=1}^{n}\sum_{j=1}^{k}[a_n(i,j)-a_n(i,\cdot)]^2\right\}^2}\\ &\leqq ns^2\left(\sum_{m=1}^{s}\alpha_m^{\,2}\right)^2\sum_{m=1}^{s}\frac{\sum_{i=1}^{n}\left[\sum_{j=1}^{k}(c_{nij}^{(m)}-c_{ni\cdot}^{(m)})^2\right]^2\sum_{i=1}^{n}\left\{\sum_{j=1}^{k}[a_n(i,j)-a_n(i,\cdot)]^2\right\}^2}{\left\{\sum_{m=1}^{s}\alpha_m^{\,2}\sum_{i=1}^{n}\sum_{j=1}^{k}(c_{nij}^{(m)}-c_{ni\cdot}^{(m)})^2\sum_{i=1}^{n}\sum_{j=1}^{k}[a_n(i,j)-a_n(i,\cdot)]^2\right\}^2}\,.\end{aligned}$$

Since each $\alpha_m^{\,2}>0$, each term of this summation tends to zero. Thus $\sum_{m=1}^{s}\alpha_m c_{nij}^{(m)}$ and $a_n(i,j)$ satisfy the condition (13). Similarly, it may be shown that they satisfy condition (12).

Hence, every random variable of the form $\sum_{m=1}^{s} \alpha_m L_{nm}^0$ is asymptotically normally distributed with mean zero and variance $\alpha_1^2 + \cdots + \alpha_s^2$ and the theorem follows by exactly the argument used in Fraser ((1957), page 242), unless some of the exceptional conditions hold when some L_{nm} are constant.

3. Permutation tests for incomplete block designs. Let $\mathbf{\Phi}$ be the incidence matrix of the treatments, where

$$\mathbf{\Phi}' = (\mathbf{\Phi}_1', \cdots, \mathbf{\Phi}_b')$$

and the (i, j)th element of $\mathbf{\Phi}_m'$ $(m = 1, \cdots, b)$ is 1 if treatment i is applied to the jth plot of block m. Then

$$\mathbf{\Phi}'\mathbf{\Phi} = \mathbf{R} = \text{diag}\,(r_1, \cdots, r_v)$$

and

$$\mathbf{N} = \mathbf{\Phi}'(\mathbf{I}_b \times \mathbf{1}_k)\,,$$

where $\mathbf{1}_k$ is a k-vector with each element 1. $\mathbf{N}$ is the incidence matrix of the design. Let

$$\mathbf{L} = ((bk)^{-\frac{1}{2}}\mathbf{R}^{\frac{1}{2}}\mathbf{1}_v, \mathbf{l}_1, \cdots, \mathbf{l}_{v-1})$$

be an orthogonal matrix such that

$$\mathbf{L}'\mathbf{R}^{-\frac{1}{2}}\mathbf{N}\mathbf{N}'\mathbf{R}^{-\frac{1}{2}}\mathbf{L} = D = \text{diag}\,(\rho_0, \rho_1, \cdots, \rho_{v-1})\,.$$

Suppose

$$\rho_0 = \cdots = \rho_{\alpha-1} = k \qquad \text{and} \qquad \rho_{v-\gamma} = \cdots = \rho_{v-1} = 0\,.$$

We wish to consider a test of a null hypothesis that all treatments have the same effect. Let

$$\mathbf{y}' = (y_{11}, \cdots, y_{1k}, y_{21}, \cdots, y_{bk})$$

be a vector of observations and let $\mathbf{Y}$ be a random vector taking, with equal probability, each of the possible values

$$(y_{i_1 j_{11}}, \cdots, y_{i_1 j_{1k}}, y_{i_2 j_{21}}, \cdots, y_{i_b j_{bk}})\,,$$

where $(i_1, \cdots, i_b)$ is a permutation of $(1, \cdots, b)$ and $(j_{11}, \cdots, j_{ik}), \cdots, (j_{b1}, \cdots, j_{bk})$ are permutations of $(1, \cdots, k)$. Let $\mathbf{V} = \mathbf{\Phi}'\mathbf{Y}$ be a vector of treatment totals, let $\mathbf{B} = (\mathbf{I}_b \times \mathbf{1}_k)\mathbf{Y}$ be a vector of block totals, let $\mathbf{T} = \mathbf{N}\mathbf{B}$ and let $\mathbf{Q} = \mathbf{V} - \mathbf{T}/k$. We will assume that the elements of $\mathbf{y}$ satisfy conditions (8) and (9).

Consider the linear forms

$$\mathbf{l}_i'\mathbf{R}^{-\frac{1}{2}}\mathbf{Q} = \mathbf{l}_i'\mathbf{R}^{-\frac{1}{2}}\left(\mathbf{\Phi}' - \frac{1}{k}\mathbf{N} \times \mathbf{1}_k'\right)\mathbf{Y}\,, \qquad i = \alpha, \cdots, v-1\,.$$

We may notice that

$$\mathbf{l}_i'\mathbf{R}^{-\frac{1}{2}}\left(\mathbf{\Phi}' - \frac{1}{k}\mathbf{N} \times \mathbf{1}_k'\right)\left(\mathbf{\Phi} - \frac{1}{k}\mathbf{N}' \times \mathbf{1}_k\right)\mathbf{R}^{-\frac{1}{2}}\mathbf{l}_j = \mathbf{l}_i'\mathbf{R}^{-\frac{1}{2}}\left(\mathbf{R} - \frac{1}{k}\mathbf{N}\mathbf{N}'\right)\mathbf{R}^{-\frac{1}{2}}\mathbf{l}_j ,$$
$$= 0 , \qquad i \neq j ,$$
$$= 1 - \rho_i/k , \qquad i = j .$$

Further, if we choose the $r_1, \cdots, r_v$ to satisfy the condition

$$\lim_{b\to\infty} r_i/b > 0 , \qquad \text{for} \quad i = 1, \cdots, v ,$$

then the elements of the vectors

$$\mathbf{l}_i'\mathbf{R}^{-\frac{1}{2}}\left(\mathbf{\Phi}' - \frac{1}{k}\mathbf{N} \times \mathbf{1}_k'\right) , \qquad i = \alpha, \cdots, v - 1 ,$$

satisfy the conditions (10) and (11). So, if we write

$$S^2 = \frac{1}{b(k-1)} \sum_{i=1}^{b} \sum_{j=1}^{k} (y_{ij} - y_{i\cdot})^2 ,$$

then the $v - \alpha$ linear forms

$$\frac{\mathbf{l}_i'\mathbf{R}^{-\frac{1}{2}}\mathbf{Q}}{S(1 - \rho_i/k)^{\frac{1}{2}}} , \qquad i = \alpha, \cdots, v - 1 ,$$

satisfy the conditions of Theorem 2 and so they are asymptotically distributed as independent normal variates with zero means and unit variances.

Hence

$$\sum_{i=\alpha}^{v-1} \frac{(\mathbf{l}_i'\mathbf{R}^{-\frac{1}{2}}\mathbf{Q})^2}{S^2(1 - \rho_i/k)}$$

has limiting chi-square distribution with $v - \alpha$ degrees of freedom. This statistic is readily seen to be $b(k-1)$ times the ratio of the treatment sum of squares adjusted for blocks and the sum of this sum of squares and the error sum of squares in the usual analysis of variance.

Also, consider the linear forms

$$\mathbf{l}_i'\mathbf{R}^{-\frac{1}{2}}\mathbf{T} = \mathbf{l}_i'\mathbf{R}^{-\frac{1}{2}}(\mathbf{N} \times \mathbf{1}_k')\mathbf{Y} , \qquad i = 1, \cdots, v - \gamma .$$

Again, we may notice that

$$\mathbf{l}_i'\mathbf{R}^{-\frac{1}{2}}\mathbf{N}\mathbf{N}'\mathbf{R}^{-\frac{1}{2}}\mathbf{l}_j = 0 , \qquad i \neq j ,$$
$$= \rho_i , \qquad i = j ,$$

and that the elements of the vectors $\mathbf{l}_i'\mathbf{R}^{-\frac{1}{2}}(\mathbf{N} \times \mathbf{1}_k')$ satisfy the conditions (10) and (11). So, if we write

$$U^2 = \frac{k}{b-1} \sum_{i=1}^{b} (y_{i\cdot} - y_{\cdot\cdot})^2 ,$$

then the $v - \gamma$ linear forms

$$\frac{\mathbf{l}_i'\mathbf{R}^{-\frac{1}{2}}\mathbf{T}}{U(k\rho_i)^{\frac{1}{2}}} , \qquad i = 1, \cdots, v - \gamma ,$$

satisfy the conditions of Theorem 2 and so they are asymptotically distributed as independent normal variates with zero means and unit variances. Hence

$$\sum_{i=1}^{v-\gamma} \frac{(\mathbf{l}_i'\mathbf{R}^{-\frac{1}{2}}\mathbf{T})^2}{U^2 k \rho_i}$$

has limiting chi-square distribution with $v - \gamma$ degrees of freedom. This statistic is $b - 1$ times the ratio of the treatment component of the block sum of squares ignoring treatments and the total block sum of squares ignoring treatments.

4. Estimation from the randomization model assuming additivity. Let

$$\mathbf{z}' = (z_{11}, \cdots, z_{1k}, \cdots, z_{bk})$$

be a vector of plot errors and let $\mathbf{Z}$ be a random vector taking, with equal probability, all the possible values

$$(z_{i_1 j_{11}}, \cdots, z_{i_1 j_{1k}}, \cdots, z_{i_b j_{bk}}) ,$$

where $(i_1, \cdots, i_b)$ is a permutation of $(1, \cdots, b)$ and $(j_{11}, \cdots, j_{1k}), \cdots, (j_{b1}, \cdots, j_{bk})$ are permutations of $(1, \cdots, k)$. If $\mathbf{Y}$ is the vector of possible observations and if we assume that the treatment effects are additive, then an appropriate model for the design is

$$\mathbf{Y} = \mathbf{\Phi t} + \mathbf{Z} .$$

Consider the contrasts $\mathbf{l}_i'\mathbf{R}^{-\frac{1}{2}}\mathbf{Q}$, $i = \alpha, \cdots, v - 1$. These have expectations $\mathbf{l}_i'\mathbf{R}^{\frac{1}{2}}\mathbf{t}(1 - \rho_i/k)$ and variances $\sigma^2(1 - \rho_i/k)$, $i = \alpha, \cdots, v - 1$, where

$$\sigma^2 = \frac{1}{b(k-1)} \sum_{i=1}^{b} \sum_{j=1}^{k} (z_{ij} - z_{i\cdot})^2 .$$

Also, the contrasts $\mathbf{l}_i'\mathbf{R}^{-\frac{1}{2}}\mathbf{T}$, have expectations $\rho_i \mathbf{l}_i'\mathbf{R}^{\frac{1}{2}}\mathbf{t}$ and variances $k\sigma_1^2\rho_i$, $i = 1, \cdots, v - \gamma$, where

$$\sigma_1^2 = \frac{k}{b-1} \sum_{i=1}^{b} (z_{i\cdot} - z_{\cdot\cdot})^2 .$$

Thus a combined estimate of $\mathbf{l}_i'\mathbf{R}^{\frac{1}{2}}\mathbf{t}$ is

$$\frac{w\mathbf{l}_i'\mathbf{R}^{-\frac{1}{2}}\mathbf{Q} + (w'/k)\mathbf{l}_i'\mathbf{R}^{-\frac{1}{2}}\mathbf{T}}{(1 - \rho_i/k)w + w'\rho_i/k} , \qquad i = 1, \cdots, v - 1 ,$$

where

$$w = 1/\sigma \qquad \text{and} \qquad w' = 1/\sigma_1^2 ,$$

and the variance of this estimate is

$$\frac{1}{(1 - \rho_i/k)w + w'\rho_i/k} , \qquad i = 1, \cdots, v - 1 .$$

The error sum of squares

$$\sum_{i=1}^{b} \sum_{j=1}^{k} (y_{ij} - y_{i\cdot})^2 - \sum_{m=\alpha}^{v-1} (\mathbf{l}_m'\mathbf{R}^{-\frac{1}{2}}\mathbf{Q})^2/(1 - \rho_m/k)$$

has expectation

$$\frac{b(k-1)-(v-\alpha)}{b(k-1)} \sum_{i=1}^{b} \sum_{j=1}^{k} (z_{ij} - z_{i\cdot})^2 ,$$

so the error mean square has expectation σ^2. The block sum of squares adjusted for treatments

$$k \sum_{i=1}^{b} y_{i\cdot}^2 + \sum_{m=\alpha}^{v-1} (\mathbf{l}_m'\mathbf{R}^{-\frac{1}{2}}\mathbf{Q})^2/(1 - \rho_m/k) - \mathbf{V}'\mathbf{R}^{-1}\mathbf{V}$$

has expectation

$$\frac{bk-v}{b-1} \sum_{i=1}^{b} (z_{i\cdot} - z_{\cdot\cdot})^2 + \frac{v-k\alpha}{kb(k-1)} \sum_{i=1}^{b} \sum_{j=1}^{k} (z_{ij} - z_{i\cdot})^2 ,$$

so the mean square of blocks adjusted for treatments has expectation

$$\frac{bk-v}{k(b-\alpha)} \sigma_1^2 + \frac{v-k\alpha}{k(b-\alpha)} \sigma^2 .$$

Thus an estimate of w is given by

$$\hat{w} = 1/E$$

and an estimate of w' is given by

$$\hat{w}' = \frac{bk-v}{k(b-\alpha)B - (v-k\alpha)E} ,$$

where B and E are the block mean square adjusted for treatments and the error mean square, respectively. It may be noted that these are the same formulae as are obtained in the usual way from the infinite model.

The error mean square may be shown to be equal to

$$(bk - b - v + \alpha)^{-1} \Big\{ \sum_{i=1}^{b} \sum_{j=1}^{k} (z_{ij} - z_{i\cdot})^2 - \sum_{m=\alpha}^{v-1} \Big[\mathbf{l}_m'\mathbf{R}^{-\frac{1}{2}}\Big(\mathbf{\Phi}' - \frac{1}{k}\mathbf{N} \times \mathbf{1}_k'\Big)\mathbf{Z}\Big]^2 \Big/ (1 - \rho_m/k) \Big\} .$$

The second term divided by σ^2 is asymptotically distributed as a chi-square variate with $v - \alpha$ degrees of freedom so as b tends to infinity the error mean square divided by σ^2 tends to 1 in probability. Similarly we may show that the block mean square adjusted for treatments divided by

$$\frac{bk-v}{k(b-\alpha)} \sigma_1^2 + \frac{v-k\alpha}{k(b-\alpha)} \sigma^2$$

tends to 1 in probability. Hence the combined estimates using estimated weights have the same asymptotic distribution as the combined estimates when the weights are known.

If we write

$$u_m = \frac{\hat{w}\mathbf{l}_m{}'\mathbf{R}^{-\frac{1}{2}}\mathbf{Q} + (\hat{w}'/k)\mathbf{l}_m{}'\mathbf{R}^{-\frac{1}{2}}\mathbf{T}}{(1 - \rho_m/k)\hat{w} + \hat{w}'\rho_m/k}$$

then the asymptotic distribution of

$$\sum_{m=1}^{v-1} u_m{}^2[(1 - \rho_m/k)\hat{w} + \hat{w}'\rho_m/k]$$

is that of a chi-square variate with $v - 1$ degrees of freedom, under the null hypothesis. This is the combined test of significance proposed by Rao (1947).

I wish to thank a referee who pointed out errors in an earlier proof of Theorem 1 and suggested the shorter and more general proof presented here.

REFERENCES

FRASER, D. A. S. (1957). *Nonparametric Methods in Statistics.* Wiley, New York.

HÁJEK, J. (1961). Some extensions of the Wald-Wolfowitz-Noether theorem. *Ann. Math. Statist.* **32** 506–523.

HOEFFDING, W. (1951). A combinatorial limit theorem. *Ann. Math. Statist.* **22** 558–566.

RAO, C. R. (1947). General methods of analysis for incomplete block designs. *J. Amer. Statist. Assoc.* **42** 541–561.

SILVEY, S. D. (1954). The asymptotic distribution of statistics arising in certain non-parametric tests. *Proc. Glasgow Math. Assoc.* **2** 47–51.

WALD, A. and WOLFOWITZ, J. (1944). Statistical tests based on permutations of the observations. *Ann. Math. Statist.* **15** 358–372.