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A GENERALIZATION OF THE WEAK VERSION OF THE GLIVENKO-CANTELLI THEOREM

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0. Introduction. In this note, a theorem, generalizing the weak version of the Glivenko-Cantelli Theorem, is presented. It is furthermore pointed out that the result is of interest in the area of Goodness of Fit Tests of the Seshadri, Csörgö, Stephens type, [1].

1. The Theorem. *Let $X_1, X_2, \cdots, X_n, \cdots$ be a sequence of positive random variables satisfying the Strong Law of Large Numbers. Let $S_n = X_1 + \cdots + X_n$ and*

$$T_{j,n} = S_j/S_n\,, \qquad 1 \leqq j < n < \infty\,.$$

Let $\mu > 0$ be the a.e. *limit of S_n/n.*

Then: for every $\varepsilon > 0$,

$$P[\lim_{N\to\infty} \sup_{n>N} (\max_{1\leqq j<n} |T_{j,n} - (j/n)|) > \varepsilon] = 0\,. \tag{1.1}$$

PROOF. Let ε, η be arbitrary positive numbers with $\varepsilon < 4$ and $\eta < 1$. By Egoroff's Theorem, [2], page 339, there exists a set A, of measure at least $(1-\eta)$, such that $S_n/n \to \mu$, uniformly on A. We now restrict our attention to the set A.

Set $\varepsilon' = \varepsilon\mu/8$. Then there exists a positive integer $M = M(\varepsilon, \eta)$, such that whenever $n > j \geqq M$, $\mu - \varepsilon' < S_j/j < \mu + \varepsilon'$ and $\mu - \varepsilon' < S_n/n < \mu + \varepsilon'$, on A. The above easily gives:

$$|T_{j,n} - (j/n)| < \varepsilon/2 \qquad \text{on } A, \text{ for} \quad n > j \geqq M\,.$$

Next, choose $N_0 = N_0(\varepsilon, M)$, such that

$$M/N_0 < \varepsilon/2\,.$$

Let $k < M$, $n \geqq N_1 \geqq N_0$.

$$\begin{aligned}|T_{k,n} - (k/n)| &\leqq \max\,[T_{M,N_0}, (M/N_0)] \\ &\leqq |T_{M,N_0} - (M/N_0)| + \varepsilon/2 \\ &< \varepsilon \qquad\qquad \text{on } A, \text{ for all} \quad k < M\,.\end{aligned}$$

Hence:

$$P[\sup_{n\geqq N_0} (\max_{1\leqq j<n} |T_{j,n} - (j/n)|) > \varepsilon] < \eta\,.$$

Since N_0 is a function of ε, η, the arbitrary constants, the proof is complete.

2. Specialization. Let the X_i be independent, identically distributed positive random variables with mean, μ. Under these conditions, a necessary and sufficient condition that the random variables $T_{j,n}$ behave as $(n-1)$ order statistics from the uniform pdf on $(0, 1)$, is that the X_i have pdf $f(x, \lambda) = \lambda \exp(-\lambda x)$. This result is proved in [1].

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Hence (1.1), when weakened to:

$$\lim_{n\to\infty} P[\max_{1\le j\le n} |T_{j,n} - (j/n)| > \varepsilon] = 0 \tag{2.1}$$

with the X_i independent identically distributed exponential random variables, becomes the weak version of Glivenko-Cantelli Theorem (i.e., the uniform weak consistency of the empirical cdf.)

Seshadri, Csörgö, and Stephens, [1], use the quantity

$$\max_{1\le j<n} [|T_{j,n} - (j/n)|] \quad \text{to test a random sample ,}$$

$X_1, \cdots, X_n$ for exponentiality.

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[2] TITCHMARSH, E. C. (1939). *The Theory of Functions*, 2nd ed. Oxford Univ. Press.