

STATIONARY MEASURES FOR A CLASS OF STORAGE MODELS DRIVEN BY A MARKOV CHAIN

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A storage model in which the growth rate is proportional to the level of the dam plus a factor, both proportionality constant and factor depending on a finite Markov chain, is shown to be a Hunt process. Hitting distributions are shown to satisfy certain integral equations, communicating and recurrence classes are studied, and stationary measures are shown to exist when the dam is finite and the level has at least one recurrent linear growth phase, and in some other cases as well.

1. Introduction. Recently Brockwell [5] considered a continuous time storage model in which the level Y_t , when not at a boundary, satisfies the differential equation $(d/dt)Y_t = Z_t$ where Z_t is a finite state Markov chain. Many results were obtained and it seemed not unlikely that much of the theory could be carried over to the case in which

$$(1.1) \quad \frac{d}{dt} Y_t = a(Z_t)Y_t + b(Z_t)$$

off the boundary, where a and b are real-valued functions on the finite state space of the Markov chain Z . Indeed, most of the calculations are essentially identical.

On the other hand, it is quite impossible to use Brockwell's techniques to prove the existence and uniqueness of invariant measures for the process (Y_t, Z_t) . But the results given by Azéma, Kaplan-Duflo and Revuz [2] provide ready answers to these questions. In fact, this reference should be of help in treating a much more general class of storage models, including those considered by Weldon [8] where $(d/dt)Y_t = Z_t$ is a finite state semi-Markov process. We will consider these questions elsewhere.

For a more detailed account of the history of these and other related models we refer the reader to Brockwell [5].

In this paper we show that the process (Y_t, Z_t) is a Hunt process, where Z_t is a finite state Markov chain and Y_t satisfies (1.1). We then examine the potential theoretic exceptional sets and derive integral equations satisfied by hitting distributions in order to study recurrence properties. Reference [2] provides invariant measures when there exist recurrent classes, conditions for which are given in Section 5.

2. The process. The exterior forces governing the change in the storage level Y_t are assumed related to the (right-continuous) Markov chain Z_t , with finite

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state space $M = \{1, \dots, m\}$ and transition matrix $\exp(tQ)$, $Q = (q_{ij})$, $i, j \in M$, $q_i = -q_{ii}$, by the functions $a, b: M \rightarrow R$. The level Y_t is supposed to satisfy (1.1) while Y_t is in $J = (\alpha, \beta)$, $-\infty \leq \alpha < \beta \leq +\infty$, and Y_t remains at the boundary of J until the driving conditions change suitably to force the level back into J .

More precisely, writing $a_i = a(i)$, $b_i = b(i)$, being given $y \in R$ define

$$(2.1) \quad h(t, y, i) := e^{ta_i y} + \int_0^t e^{(t-s)a_i} b_i ds$$

as the unique solution to the initial value problem

$$\frac{d}{dt} h(t) = a_i h(t) + b_i, \quad h(0) = y,$$

introduce the deterministic exit time

$$(2.2) \quad \tau_{yi} := \inf \{t \geq 0 \mid h(t, y, i) \in J^{-c}\}$$

($-$ = closure in the usual topology of R , c = complement operator, infimum of the empty set is taken as ∞), and finally define the (known, deterministic, nonrandom) function

$$(2.3) \quad k(t, y, i) := h(t \wedge \tau_{yi}, y, i)$$

(\wedge = minimum). The process Y_t is obtained by piecing together the k -functions as follows: for a given path $Z_t(\omega)$ with jumps at times $t_1 < t_2 < \dots$ through the states z_0 to z_1 to $z_2 \dots$ in M , if $Y_0 = y \in J^-$ is given, define inductively

$$(2.4) \quad Y_t(\omega) := k(t - t_n, Y_{t_n}(\omega), z_n)$$

for $t_n \leq t \leq t_{n+1}$, $n = 0, 1, \dots$, $t_0 = 0$.

Notice that if $Y_s(\omega) = \alpha$, then $Y_t(\omega) = \alpha$ until the first t_n after s for which $\tau_{\alpha z_n} > 0$.

It follows from the fact that Y_{t+s} is uniquely determined by Y_t and the driving coefficients $a(Z_r)$, $b(Z_r)$, $t \leq r < t + s$, that the pair (Y_t, Z_t) is a right-continuous Markov process with state space $E = J^- \times M$. But much more is true.

To see that $X_t = (Y_t, Z_t)$ is a very nice Markov process define the function classes on $E = J^- \times M$ (with discrete topology on M , and product topology on E): B = bounded measurable, C = bounded continuous, C_0 = continuous vanishing at infinity.

Define the semigroup of operators P_t on B , $t \geq 0$, by

$$(2.5) \quad P_t f(x) = E^x f(X_t) = E(f(Y_t, Z_t) \mid (Y_0, Z_0) = x).$$

(2.6) THEOREM. *If H is any of the above function classes, $P_t H \subset H$.*

PROOF. If $\nu_t(\omega)$ equals the number of jumps of $Z_s(\omega)$ in $[0, t)$, then

$$P_t f(y, i) = \sum_{n=0}^{\infty} f_n(y, i)$$

where $f_n(y, i) = E^{\nu_i}(f(X_t); \nu_t = n)$.

But $|\sum_{n=k}^{\infty} f_n(y, i)| \leq \|f\|_{\infty} P^{\nu_i}(\nu_t \geq k)$, hence the series representation of $P_t f$

converges uniformly for $f \in B$, and it suffices to show that $f_n \in H$ if f is. Now, for example,

$$f_0(y, i) = P^{v_i}(v_t = 0)f(k(t, y, i), i) \tag{and}$$

$$f_1(y, i) = P^{v_i}(v_t = 1) \sum_{j \neq i} \int_0^t f(k(t-s, k(s, y, i), j))q_{ij}e^{-q_{ii}s} ds,$$

and the remaining f_n have a similar form. Because k is linear in y , $f_n \in H$ if f is. \square

(2.7) COROLLARY. $P_t f(y, i) \rightarrow f(y, i)$ as $t \rightarrow 0$, uniformly for $f \in C_0$ and pointwise for $f \in C$.

PROOF. Since ν depends only on the Markov chain Z_t , $P^{v_i}(v_t \neq 0) \rightarrow 0$ uniformly in y . The result now follows from the above series representation of $P_t f$ and the fact that $k(t, y, i) \rightarrow y$ as $t \rightarrow 0$. \square

Combining these results with the basic existence theorem for Hunt processes given in Blumenthal and Gettoor ([4] page 46—whose notation and terminology we follow consistently), we have

(2.8) COROLLARY. *The process $X_t = (Y_t, Z_t)$ is equivalent to a Hunt process and a (non-strong) Feller process.*

REMARK. If $X_t^\wedge = (Y_t^\wedge, Z_t^\wedge)$ is the Hunt version of our process with state space $E = J^- \times M$, then Z^\wedge is a right continuous version of the Markov chain Z . If we construct our process $X = (Y, Z^\wedge)$ using Z^\wedge then Y is right continuous by construction and hence the paths of Y and Y^\wedge are equal a.s., i.e. for each initial point (y, i) there is a measurable set $\Lambda = \Lambda_{y,i}$ such that $P^{v_i}\Lambda = 1$ and for $\omega \in \Lambda$, $Y_t(\omega) = Y_t^\wedge(\omega)$ for $t \geq 0$. This will be used without comment in the following sections.

From now on we write $X = (Y, Z)$ for the above Hunt version $X^\wedge = (Y^\wedge, Z^\wedge)$.

The results of Azéma, Kaplan-Duflo and Revus [1], [2], [3] imply the existence of invariant measures for X on recurrent classes if X satisfies hypothesis (L) (see [2] page 158, [7] page 160, [4] page 196 and Theorem (3.6) below).

In order to determine if X has this property we must look at potential theoretic exceptional sets and the fine topology for X . This we do in the following section.

3. Exceptional sets and fine topology for X . Looking at the functions h in (2.1) we see that

$$a_i h_i + b_i = 0 \text{ implies } h(t, h_i, i) \equiv h_i \text{ for all } t \in R.$$

Such points $h_i = (h_i, i) \in E$ are called *holding points* for k . Notice that each line (segment) $E_i := J^- \times \{i\}$ has at most one holding point unless $a_i = 0 = b_i$, in which case E_i is termed a *holding line*.

Clearly, if $a_i < 0$, $h_i \in E$ is an *attracting holding point* (for k) in the sense that $k(t, y, i) \rightarrow h_i$ all $y \in E_i$ as $t \rightarrow \infty$, while if $-\infty < h_i < \alpha$ and $a_i < 0$ then $k(t, y, i) \rightarrow \alpha$ for all $y \in E_i$ as $t \rightarrow \infty$ and $\alpha_i = (\alpha, i)$ is termed an *attracting*

boundary point while $\beta_i = (\beta, i)$ is a *repelling boundary point*. In case $h_i = \alpha_i$, α_i is given the obvious rubric. If $\alpha_i > 0$, attracting and repelling are everywhere interchanged in the terminology above. If $\alpha_i = 0 < b_i$, think of β_i as an attracting boundary point (in E if $\beta < \infty$).

Observe that an attracting holding point h_i is reached by $k(t, y, i)$ in a finite time only when $y = h_i$.

To discuss exceptional sets and fine topology of X we partition the boundary operator ∂ in the product E -topology into four components: given $A, B \subset E$, denote the E -complement by $A^c = E \setminus A$, $B \setminus C = B \cap C^c$, define

$$\begin{aligned}
 H &= \{\text{holding points and attracting boundary points for } k \text{ in } E\}, \\
 \partial_{ex} A &= \{(y, i) \in \partial A \setminus H \mid \exists \varepsilon = \varepsilon(y, i) > 0 \ni k(t, y, i) \in A^c, 0 < t < \varepsilon\} \\
 (3.1) \quad \partial_{en} A &= \{(y, i) \in \partial A \setminus H \mid \exists \varepsilon = \varepsilon(y, i) > 0 \ni k(k, y, i) \in A, 0 < t < \varepsilon\}, \\
 \partial_{alt} A &= \{(y, i) \in \partial A \setminus H \mid \exists t_n, s_n \downarrow 0 \ni k(t_n, y, i) \in A, k(s_n, y, i) \in A^c\} \quad \text{and} \\
 \partial_h A &= \partial A \cap H.
 \end{aligned}$$

Notice that $\partial_{en} A = \partial_{en} A^\circ = \partial_{ex} A^c$ ($^\circ =$ interior operator in product topology of E).

The points in H are *holding points* for X ([4] page 91), while those in H^c are *instantaneous*. The boundary operators are termed *exit*, *entrance*, *alternating* and *holding*, respectively, for obvious reasons.

This decomposition makes it easy to characterize *finely open* sets ([4] page 85), i.e. sets from which the process does not immediately exit.

Introducing the *entrance* and *hitting* times for $A \subset E$:

$$\begin{aligned}
 (3.2) \quad D_A(\omega) &= \inf \{t \geq 0 \mid X_t(\omega) \in A\} \\
 T_A(\omega) &= \inf \{t > 0 \mid X_t(\omega) \in A\}
 \end{aligned}$$

and observing that X_t follows the known path $(k(t, y, i), i)$ until the first jump, we perceive that given $A \subset E$ and $x \in A$, $x \in A^\circ \cup \partial_{en} A$ implies $P^x(T_{A^c} > 0) = 1$, $x \in \partial_h A$ implies $P^x(T_{E \setminus \{x\}} > 0) = 1$, $x \in \partial_{ex} A$ implies $P^x(T_{A^c} = 0) = 1$, and $x \in \partial_{alt} A$ implies $P^x(T_{B_x} = 0) = 1$, where for $x = (y, i)$, $B_x = \{(k(s_n, y, i), i) \in A^c \mid s_n \downarrow 0\}$. This gives

$$(3.3) \quad \text{THEOREM. } A \text{ set } A \subset E \text{ is finely open iff } A \subset A^\circ \cup \partial_{en} A^\circ \cup \partial_h A.$$

Thus, for example, an interval $[c, d) \subset E_i$ is finely open and finely closed if $k(t, c, i)$ is a strictly increasing function of t while (c, d) is just finely open. If there are no holding lines, every finely open set of the form $A^\circ \cup \partial_{en} A^\circ \cup \partial_h A$ is a countable union of certain half open intervals and points. On the other hand, every subset of a holding line is finely open.

For a (nearly) Borel set A , the set A^r of *regular points for A* are those points starting from which A is hit immediately ([4] page 61), i.e. $A^r = \{x \mid P^x(T_A = 0) = 1\}$.

We exhibit A^r below as well as the finely closed sets (i.e. complements of finely open sets). Part (a) below follows from remarks preceding Theorem 3.3;

the statement in part (c) is true in general ([4] pages 87, 199) but it and the representation are immediate in the present set-up.

(3.4) THEOREM. (a) $A^r = A^\circ \cup \partial_{en} A^\circ \cup \partial_{alt} A \cup (A \cap \partial_h A)$. (b) $A \setminus A^r = \partial_{ex} A \cap A$. (c) A is finely closed iff $A^r \subset A$ and the fine closure of A is $A \cup A^r = A \cup \partial_{en} A^\circ \cup \partial_{alt} A$.

We now characterize the potential theoretic exceptional sets associated with X . (See [4] page 79 for precise definitions.)

Let λ denote Lebesgue measure on E .

(3.5) THEOREM. Let A be a subset of E . (a) A is polar iff $A = \emptyset$. (b) A is thin iff A is countable and $A \cap H = \emptyset$ and there exists no sequence $\{t_n\}$ strictly decreasing to zero such that $k(t_n, y, i) \in A$ all n and some $(y, i) \in E$. (c) A is semipolar iff A is countable and $A \cap H = \emptyset$. (d) A universally measurable set A ([4] page 2) is null (has potential zero) iff $\lambda(A) = 0$ and $A \cap H = \emptyset$.

PROOF. (a) Clear. (b) By definition A is thin iff $A^r = \emptyset$ iff (by Theorem 3.4) $A^\circ = \emptyset$, $A \cap \partial_h A = A \cap A^- \cap H = A \cap H = \emptyset$ and $\partial_{alt} A = \emptyset$. The first of these requirements follows if A is countable and the third if there is no such sequence of t 's. Conversely, let A be thin. Then $A \cap H = \emptyset$ and no such t sequence can exist, else $(y, i) = \lim_n (k(t_n, y, i), i) \in A^r$. It remains to show A countable when thin. Write each $E_i \setminus H$ as a countable union of compact intervals I_{ij} with end points in A and $I_{ij} \cap H = \emptyset$. If any $I_{ij} = [a_1, a_2]$ is such that $A_{ij} = I_{ij} \cap A$ is uncountable we get a contradiction: assuming $k(t, a_1, i)$ is increasing in t , there is a smallest $t_0 > 0$ such that $T_0 = \{t \leq t_0 \mid k(t, a_1, i) \in A_{ij}\}$ is uncountable, else there is a t -sequence. But then $T_0 \setminus \{t_0\} = \bigcup_{n=1}^\infty \{t \leq t_0 - n^{-1} \mid k(t, a_1, i) \in A_{ij}\}$ is countable. Replace a_1 by a_2 if $k(t, a_1, i)$ is decreasing. This proves (b). (c) A semipolar set is defined as a countable union of thin sets, nonholding points are thin, so (c) follows trivially from (b). (d) Now suppose A is null. If $x = (y, i)$ is a holding point for X , then starting from x the average time in x is at least $1/q_i > 0$; so $A \cap H = \emptyset$. Writing $A = \bigcup_{i \in M} \bigcup_{j=1}^\infty A_{ij}$ as in (b), with $A_{ij} = [a_{ij1}, a_{ij2}] \cap A \cap E_i$ and $k(t_{ij}, a_{ij1}, i) = a_{ij2}$, $t_{ij} > 0$, starting from a_{ij1} the average time in A_{ij} is at least $\exp[-q_i t_{ij}] \int_0^{t_{ij}} I_{A_{ij}}(k(s, a_{ij1}, i)) ds$, implying $\lambda(A_{ij}) = 0$ if A is null. The obvious modification gives the result when $t_{ij} < 0$. Conversely, if $\lambda(A) = 0$ and $A \cap H = \emptyset$, on each visit to E_i X can spend only zero time in A_{ij} (it has a positive velocity across $[a_{ij1}, a_{ij2}]$). But each path visits E_i only countably often and thus A is null. \square

(3.6) COROLLARY. The following are equivalent: (a) X satisfies hypothesis (L), (b) there exists a finite (reference) measure ρ on the universally measurable sets such that $\rho(A) = 0$ exactly when A is null, (c) X has no holding lines.

PROOF. The equivalence of the latter two statements is obvious from (3.5), and that of the first two is well known (the proof of this is nearly that of ([4] page 197, Proposition (1.2)). \square

The existence of an invariant measure for X with no holding lines is now insured by [2] if X has any recurrent states.

In order to determine which states, if any, are recurrent we derive integral equations satisfied by the hitting distributions of X in the next section. Included are integral equations for other functions of interest. In the section following we take up the question of invariant measures.

4. Calculations. In this section we obtain integral equations satisfied by various functions of interest to anyone applying our models to storage problems. We characterize the solutions corresponding to these probabilistic functions and give iterative procedures for their calculation. This is important, for the integral equations will not always have unique solutions.

The technique used is the powerful "method of first jump." Since all the proofs are basically the same we will content ourselves with proving only one of the following theorems. Nor will we state in great detail any but the theorem on hitting distributions.

Recall that the finite Markov chain Z on $M = \{1, \dots, m\}$ has transition matrix $\exp(tQ)$. Write $q_i = -q_{ii}$, write $\pi_{ij} = q_{ij}/q_i$ and $\pi_{ii} = 0$ if $q_i \neq 0$, and $\pi_{ij} = 0$, $\pi_{ii} = 1$ if $q_i = 0$, $i, j \in M$, $i \neq j$. From $\Pi = (\pi_{ij})$, $i, j \in M$.

We first look at the infinitesimal generator \mathcal{A} of the semigroup P_t :

(4.1) THEOREM. For $f \in B$,

$$(4.2) \quad P_t f(y, i) = e^{-tq_i} f(k(t, y, i), i) + \sum_j \pi_{ij} \int_0^t P_s f(k(t-s, y, i), j) q_i \exp[-q_i(t-s)] ds,$$

and thus for $f \in C^1$

$$(4.3) \quad \mathcal{A}f(y, i) = \sum_j q_{ij} f(y, j) + (a_i y + b_i) f'(y, i),$$

where ' is the appropriate one-sided derivative on the repelling boundary point for k .

For U any measurable subset of E , write $U_i = U \cap E_i$ and $D = D_{E \setminus U}$, where "—" denotes product topology closure and D_λ is defined in (3.2). Denote the deterministic exit time by

$$(4.4) \quad t_{y_i} = \inf \{t \geq 0 \mid k(t, y, i) \in E \setminus U\}.$$

(4.5) THEOREM. For $h \in B$, set $f(x) = E^x h(X_D)$. (a) f is a solution of the integral equation

$$(4.6) \quad \varphi(y, i) = \exp[-q_i t_{y_i}] h(k(t_{y_i}, y, i), i) + \sum_j \pi_{ij} \int_0^{t_{y_i}} \varphi(k(s, y, i), j) q_i e^{-q_i s} ds.$$

(b) For $h \in B$ and nonnegative, f is the smallest nonnegative solution of (4.5).

(c) f is the unique bounded solution to (4.6) in case Z is irreducible and there is some i for which $\sup \{t_{y_i} \mid y \in U_i\} = K < \infty$.

(d) The functions $f_\nu(y, i) = E^{y_i}(h(X_D); \nu \leq n)$, $\nu =$ number of jumps of Z before time D , increase to f and can be calculated inductively by noting that

$$(4.7) \quad \begin{aligned} f_0(y, i) &= \exp[-q_i t_{yi}]h(k(t_{yi}, y, i), i) \\ f_{n+1}(y, i) &= f_0(y, i) + \sum_j \pi_{ij} \int_0^{t_{yi}} f_n(k(s, y, i), j)q_i e^{-q_i s} ds. \end{aligned}$$

(e) For U open, f satisfies the differential equation

$$(4.8) \quad \mathcal{A}f(y, i) = (a_i y + b_i)f'(y, i) + \sum_j q_{ij}f(y, j) = 0, \quad (y, i) \in U \setminus H \cup \partial_{en} U$$

with boundary conditions

$$\begin{aligned} \sum_j q_{ij}f(y, i) &= 0, \quad (y, i) \in (U \cap H) \cup \partial_h U \\ f(y, i) &= h(y, i), \quad (y, i) \in \partial_{ex} U \cup \partial_{alt} U \cup E \setminus U^-. \end{aligned}$$

Again ' is one-sided differentiation in $\partial_{en} U$.

PROOF. (a) Use the strong Markov property, noting that prior to the first jump of Z , X is deterministic. The first term in (4.6) corresponds to no Z -jump before X enters $E \setminus U^-$, the second to a jump to j by Z at time s before X enters $E \setminus U^-$. (b) and (d) (4.7) follows from the strong Markov property, and if h is nonnegative and φ is a nonnegative solution of (4.6) then by induction $f_n \leq \varphi$, implying $f = \lim f_n \leq \varphi$. (c) If φ_1, φ_2 are bounded and satisfy (4.6), then $\psi = \varphi_1 - \varphi_2$ satisfies

$$\psi(y, l) = \sum_j \pi_{lj} \int_0^{t_{yl}} \psi(k(s, y, l), j)q_l e^{-q_l s} ds.$$

Set $\psi_l = \sup \{|\psi(y, l)| | y \in E_l\}$, $l \in M$, so that

$$(*) \quad \psi_l \leq \sum_j \pi_{lj} \psi_j \quad \text{and} \quad \psi_i \leq \sum_j \pi_{ij} \psi_j \int_0^K q_i e^{-q_i s} ds.$$

In obvious matrix notation, $0 \leq \Psi \leq \Pi\Psi$. Irreducibility of Z implies the existence of a probability vector r with $r = r\Pi$ and each component strictly positive. Thus we must have $\Psi = \Pi\Psi$, else $r\Psi < r\Pi\Psi = r\Psi$. But this implies $0 \leq \Psi = c\mathbf{1}$, $\mathbf{1}$ a column of ones, and the second inequality in (*) implies $0 \leq c \leq ce^{-q_i K}$. Thus $c = 0$ and $\Psi = 0$ and $\varphi_1 \equiv \varphi_2$. (e) This can be checked directly if the change of variable $s \rightarrow k(s, y, i)$ is made in (4.6). It is also a corollary of the fact that the infinitesimal generator and characteristic operator coincide on a certain class of functions, see Dynkin ([6] pages 46 and 143) and our Theorem (2.6). \square

Using the notation $D = D_{E \setminus U^-}$, and other notation above, define

$$L_\lambda(y, i) = E^{yi}(e^{-\lambda D}), \quad \lambda > 0$$

and, for g nonnegative bounded measurable on U^- ,

$$F(y, i) = E^{yi} \int_0^D g(X_s) ds.$$

We content ourselves with displaying the integral equations satisfied by L_λ and F and a remark on uniqueness. Statements and proofs of other results paralleling those given in the preceding theorem are easy and left to the reader.

(4.9) THEOREM. For each $\lambda > 0$, L_λ is the unique solution of the integral equation

$$L_\lambda(y, i) = \exp[-t_{y_i}(\lambda + q_i)] + \sum_j \pi_{ij} \int_0^{t_{y_i}} L_\lambda(k(s, y, i), j) q_j \exp[-(\lambda + q_j)s] ds.$$

(4.10) THEOREM. *The function F is the smallest nonnegative solution of the integral equation*

$$\begin{aligned} \Phi(y, i) = & \exp[-q_i t_{y_i}] \int_0^{t_{y_i}} g(k(s, y, i), i) ds \\ & + \int_0^{t_{y_i}} [\int_0^s g(k(r, y, i), i) dr] q_i e^{-q_i s} ds \\ & + \sum_j \pi_{ij} \int_0^{t_{y_i}} \Phi(k(s, y, i), i) q_j e^{-q_j s} ds. \end{aligned}$$

5. Invariant measures for X . Azéma, Kaplan-Duflo and Revuz [1], [2], [3] study recurrence and transience properties of standard processes, a class which includes Hunt processes. (Here we are using the terminology of ([4] page 45); c.f. ([1] page 187).) They then prove the existence and partial uniqueness of invariant measures for such processes having recurrent states. We review their terminology first.

Recall that $x \in E$ is *finitely recurrent* iff $P^x(X_t \in V \text{ i.o.}) = P^x(\limsup_{t \rightarrow \infty} I_{(X_t \in V)} = 1) = 1$ for every $V \in \mathcal{V}_f(x) = \{V \supset \{x\} \mid V \text{ is nearly Borel and finely open}\}$, while otherwise x is *finely transient* and then there is a $V \in \mathcal{V}_f(x)$ such that $P^x(X_t \in V \text{ i.o.}) = 0$ ([1] page 188). Further x leads to y (write $x \rightarrow y$) iff $P^x(T_V < \infty) > 0$ for all $V \in \mathcal{V}_f(y)$, and x *communicates with* y iff $x \rightarrow y$ and $y \rightarrow x$ (write $x \leftrightarrow y$). Finally, x is *essential* if for each y for which $x \rightarrow y$ we have $y \rightarrow x$.

It is shown that " \leftrightarrow " is an equivalence relation ([1] page 198), and that fine recurrence and essential-ness are class properties ([1] page 198, 199). Also, an essential class is finely closed ([1] page 199) and a recurrence class is essential. An almost Borel set $A \subset E$ is *stable* if $P^x(X_t \in A \text{ for all } t > 0) = 1$ for each $x \in A$ ([1] page 200).

It is shown ([1] page 202) that disjoint recurrent classes are contained in disjoint finely closed, stable sets (called *conservative classes*) when hypothesis (L) is satisfied. This and additional reasoning imply that each conservative class supports a unique invariant measure ([2] page 162).

For our processes, each essential class is stable and finely closed when there are no holding lines. Thus the invariant measure is supported by the recurrent class itself.

From now on assume X has no holding lines and Z is irreducible. The second assumption is harmless for each recurrent class K for Z leads to a family of lines $\bigcup_{i \in K} E_i$ in which essential classes of X must be contained, and each family may be studied separately.

For each i with $a_i \neq 0$ define $h_i = -b_i/a_i$, and when $a_i = 0$ set $h_i = +\infty$. Say h_i or $\mathbf{h}_i = (h_i, i)$ is *attracting* if $a_i < 0$ or if $a_i = 0 < b_i$. Otherwise h_i is *repelling*.

(5.1) THEOREM. *For the process X each essential class is finely closed and stable.*

Since Z is irreducible, X has at most three essential classes, each easily determined in any specific case.

PROOF. We have already noted that in general essential classes are finely closed. A proof of the remaining statements is given by an exhaustive study of cases. For simplicity we restrict ourselves to the cases when $-\infty < \alpha < \beta < \infty$ and when there is i and j with $h_i \neq h_j$. Let $\mathcal{H}_\alpha = \{h_i | h_i < \alpha\}$, $\mathcal{H}_\beta = \{h_i | h_i > \beta\}$ and $\mathcal{H} = \{h_i | \alpha \leq h_i \leq \beta\}$. Say \mathcal{H}_α is *at.* (*rep.*, *both*) if $H_\alpha \neq \emptyset$ and every h_i is attracting (repelling; $\exists h_i, h_j \in \mathcal{H}_\alpha, h_i$ attracting, h_j repelling). Similar statements are made about \mathcal{H}_β and \mathcal{H} . The unique essential class for X is E when \mathcal{H}_α is *both*, when \mathcal{H}_β is *both*, and when \mathcal{H}_α and \mathcal{H}_β are each *at.* or *rep.* When \mathcal{H}_α is *at.* or \mathcal{H}_β is *rep.* (1) and \mathcal{H} is void or *rep.* the unique essential class is $\alpha \times M$, (2) and \mathcal{H} is *at.* the unique essential class is the fine closure of $[\alpha, h) \times M, h = h_{\max.at.} = \max\{h_i \in \mathcal{H} | h_i \text{ at.}\}$, (3) and \mathcal{H} is *both*, the unique essential class is the fine closure of $[\alpha, h) \times M$, where $h = h_{\max.at.}$ if every repelling $h_j \in \mathcal{H}$ is greater equal $h_{\max.at.}$, and the unique essential class is E otherwise. The case \mathcal{H}_α *rep.* or \mathcal{H}_β *at.* is similar. Finally suppose $\mathcal{H}_\alpha \cup \mathcal{H}_\beta = \emptyset$ and $\mathcal{H} \neq \{h\}$: (1) If \mathcal{H} is *rep.* then $\alpha \times M$ and $\beta \times M$ are the only essential classes. (2) If \mathcal{H} is *at.* then the unique essential class is the fine closure of $(h_{\min.at.}, h_{\max.at.}) \times M, h_{\min.at.} = \min\{h_i | h_i \in \mathcal{H}\}$. (3) If \mathcal{H} is *both* the unique essential class is E unless either $h_{\max.rep.} \leq h_{\min.at.}$ or $h_{\max.at.} \leq h_{\min.rep.}$ when it is the fine closure of $(h_{\min.at.}, \beta] \times M$ or $[\alpha, h_{\max.at.}) \times M$, respectively. It is easy to verify the above statements and also to check that the given essential classes are stable. \square

Unfortunately we are unable to determine in general when an essential class is recurrent. We do have the following result which should cover many applications.

(5.2) THEOREM. Suppose either (a) $-\infty < \alpha < \beta < \infty$ and there is an $h_i \notin [\alpha, \beta]$, or (b) $-\infty < \alpha < \beta = \infty$, every $a_i = 0$ and either (i) every $b_i < 0$ or (ii) there are $b_i < 0 < b_j$ and the matrix $-B^{-1}Q, B = \text{diag}[b_1, \dots, b_m]$, has Jordan form with blocks $J_i = \lambda_i + N_i$ and $\lambda_1 = 0, \text{Re } \lambda_i > 0$ for $i \neq 1$. Then the unique essential class K is recurrent and stable and X has a unique (up to a multiplicative constant) invariant measure supported by K .

PROOF. The first case considered is $H_\alpha \cup H_\beta \neq \emptyset$, in the notation of Theorem 5.1. Suppose, without loss of generality, $h_i < \alpha$ is repelling. We show that (β, i) is finely recurrent and in K which implies that K is a recurrent class: For $T_{\beta i} = \inf\{t > 0 | X_t = (\beta, i)\}$, $f(y, j) = P^{yj}(T_{\beta i} < \infty)$ is the unique solution of (4.6) with $h \equiv 1$, for the hypothesis of Theorem 4.5. (c) holds. Since the function identically equal one also satisfies (4.6) with $h \equiv 1$, we have

$$P^{yj}(T_{\beta i} < \infty) = 1 \quad \text{for all } (y, j) \in E.$$

But, for $U(x, V) = E^x \int_0^\infty I_V(X_t) dt$, we have by the strong Markov property (c.f. [4] page 69), for $V = \{\beta, i\}$,

$$\begin{aligned}
 U(\beta, i, V) &= \frac{1}{q_i} + \sum_j \pi_{ij} U(\beta, j, V) \\
 &= \frac{1}{q_i} + \sum_j \pi_{ij} U(\beta, i, V) P^{\beta j}(T_{\beta i} < \infty) \\
 &= \frac{1}{q_i} + U(\beta, i, V).
 \end{aligned}$$

Thus (β, i) is finely recurrent ([1] page 188). Case (b i) is trivial for then $K = \alpha \times M$. In case (b ii) the vector function $f(y)$ with components $f(y, j) = P^{yj}(T_{\alpha i} < \infty)$ satisfies the differential equation $Bf' + Qf = 0$ (see (4.6), (4.8)), and the conditions guarantee that the only bounded solution of this is the constant function. Now $f \equiv 1$ and argue as above. The existence and uniqueness of an invariant measure μ for X with $\mu(E \setminus K) = 0$ is proved in ([2] page 162). \square

We give an example to show that condition (b ii) is not vacuous in Section 6.

Define the potential measures $U^\alpha(x, A) = E^x \int_0^\infty e^{-\alpha t} I_A(X_t) dt$, $\alpha \geq 0$. It is shown ([2] page 165) that the invariant measure is equivalent to $U^\alpha(x, \cdot)$ for each x in the recurrent class K and $\alpha > 0$. But each $U^\alpha(x, \cdot)$ is absolutely continuous with respect to the reference measure

$$\rho = \lambda + \nu$$

where λ is Lebesgue on E and ν is counting measure on H (which is finite since X has no holding lines). This follows since $U(x, A) = 0$ iff $\rho(A) = 0$ and $U^\alpha(x, \cdot) \leq U^0(x, \cdot)$.

(5.3) THEOREM. *In general, if μ is an invariant measure for X on a recurrence class, then μ is absolutely continuous with respect to the reference measure ρ for X and $\mu(E) < \infty$.*

PROOF. The first statement is contained in the above remarks. To see that μ is finite, define the additive functionals $A_t^i = \int_0^t I_{E_i}(X_s) ds = \int_0^t I_{\{i\}}(Z_s) ds$. $A_t^i \leq t$ so that A^i is integrable ([2] page 166). But almost surely $\lim_{t \rightarrow \infty} A_t^i/t = r_i > 0$, where $r = (r_1, \dots, r_m)$ is the unique positive invariant vector for the finite state Markov chain Z . Now $\mu(E) < \infty$ follows from ([2] page 173). \square

6. Conclusion. There may exist a non-finite invariant measure when X has no recurrent states: take $-\alpha = \beta = +\infty$, $M = \{1\}$, $a_1 = 0 < b_1$ (uniform motion to the right) and $\mu =$ Lebesgue measure on $E = R$. On the other hand, when X is as above but $\alpha = 0$, X has no invariant measure.

There may exist non-compact recurrent classes: take $\alpha = 0$, $\beta = +\infty$, $M = \{1, 2\}$, $|q_{ij}| = 1$, all i, j , and $a_1 = 0 = a_2$, $0 < b_1 \leq -b_2$. The vector function $f(y)$ with components $f(y, i) = P^{yi}(T_{(0,2)} < \infty)$ satisfies the differential equation $Bf' + Qf = 0$ (see (4.6), (4.8), $h \equiv 1$) and $-BQ$ has eigenvalues $0, (b_2 + b_1)/b_1 b_2$; the only probabilistic solution is $f \equiv 1$ and as in Theorem 5.1, $(0, 2)$ is recurrent, and E is a recurrent class.

We believe that condition (b ii) of Theorem 5.1 is too strong, but we are

unable to relax it or treat in general the question of recurrence when no line E_i can be deterministically traversed in a finite time.

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