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MAXIMUM AND HIGH LEVEL EXCURSION OF A GAUSSIAN PROCESS WITH STATIONARY INCREMENTS

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Let $X(t)$, $t \geqq 0$, be a centered separable Gaussian process with stationary increments. Put $\sigma^2(t) = EX^2(t)$, and suppose that $\sigma(0) = 0$. Let $Y(t)$ be the normalized process $X(t)/\sigma(t)$, and B an arbitrary bounded closed subinterval of the positive real axis. Under general conditions on σ we find (1) an explicit asymptotic formula for $P\{\max_B Y > u\}$ for $u \to \infty$ in terms of σ and various functions derived from it, and (2) the limiting conditional distribution of the time spent above the level u (for $u \to \infty$) given that the time spent is positive. This limiting distribution is a scale-mixture of the corresponding distribution previously obtained under comparable conditions in the case of the stationary Gaussian process.

0. Introduction and discussion of the results. This paper represents an extension of certain results on the maxima and high level excursions obtained in [2] for the stationary Gaussian process to the more general process with stationary increments. A simple example is $W(t) - W(0)$ where W is stationary. This extension is built on the "local stationarity" of the normalized process $Y(t)$: If t is a fixed point in B, then, for all s near t, the correlation $EY(s)Y(t)$ is very nearly equal to $\exp(-\sigma^2(s - t)f(t))$, where f is a continuous positive function, uniquely determined by σ. When $\sigma^2(t)$ varies regularly for $t \to 0$—and we assume this—then there exists $f^*(t)$ such that $f(t)\sigma^2(s - t) \sim \sigma^2(f^*(t)(s - t))$ for $s \to t$; therefore, the correlation function is approximately $\exp[-\sigma^2(f^*(t)(t - s))]$ for s near t. This is the correlation obtained from $\exp[-\sigma^2(t - s)]$ by changing the time scale by a factor $f^*(t)$. It follows that if I is a small interval containing t, and if Z is a stationary process with the covariance function $\exp(-\sigma^2(s))$, then $\max_{s \in I} Y(s)$ has approximately the same distribution as $\max_{s \in I} Z(f^*(t)s)$. For this reason the asymptotic form of the tail of $\max_I Y$ differs from that of $\max_I Z$ by a simple alteration of the time interval.

The local stationarity of $Y(s)$ also explains the form of the limiting conditional distribution of the time spent above a high level. If Y exceeds such a level somewhere on B, then it does so *exactly* once and no more, remaining above the level for just a brief time period. For this reason the conditional distribution has, in the *stationary* case, a limit which is independent of the length of B.

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However, the length of the excursion of $Y(s)$ above u depends on the point t where it crosses above u; indeed, the length of the excursion is multiplied by a factor $f^*(t)$. By the total probability formula the limiting distribution of the time above u is a weighted average of the distribution—a scale mixture—obtained in the stationary case.

A trivial example of a process which we explicitly exclude is $X(t) = Xt$, where X is Gaussian with mean 0; here $Y(t) \equiv X$, $\max_B Y(t) = X$, and the conditional distribution of the excursion is degenerate.

1. Hypotheses. Let $X(t)$, $t \geqq 0$, be a separable Gaussian process with stationary increments. We suppose that $EX(t) \equiv 0$ and that $X(0) = 0$ almost surely. Put $\sigma^2(t) = EX^2(t)$; we assume that $\sigma^2(t)$ is continuous. In this work we study the distributions of the maximum and the high level occupation time of the normalized process

$$Y(t) = X(t)/\sigma(t) .$$

In most examples of theoretical interest Y is unbounded near $t = 0$; therefore, the maximum of Y over a closed interval containing the origin is equal to $+\infty$ almost surely. In the case of Brownian motion Lévy noted that $Y(t)$ has the same behavior for $t \to 0$ as it has for $t \to \infty$ ("projective invariance"). To exclude the case where $\max Y = \infty$, we shall consider the process only on a finite interval bounded away from the origin. This interval, which we denote by B, is fixed throughout the paper.

It follows from the definitions above that $Y(t)$ has mean 0, variance 1, and the covariance function

$$EY(s)Y(t) = \frac{\sigma^2(t) + \sigma^2(s) - \sigma^2(t-s)}{2\sigma(s)\sigma(t)} .$$

This follows from the relations $\sigma^2(t-s) = E(X(t) - X(s))^2 = EX^2(t) + EX^2(s) - 2EX(s)X(t) = \sigma^2(t) + \sigma^2(s) - 2EX(s)X(t)$. An equivalent form more convenient for our purposes is

$$EY(s)Y(t) = 1 - \frac{\sigma^2(t-s) - (\sigma(t) - \sigma(s))^2}{2\sigma(s)\sigma(t)} . \tag{1.1}$$

We make five assumptions about σ^2.

I. $\sigma(t-s) = \sigma(t) - \sigma(s)$ if and only if $s = t$; in particular, $\sigma^2(t)$ is not identically equal to a constant multiple of t^2.

II. $\min_B \sigma > 0$.

III. There exists a non-decreasing nonnegative slowly varying function $g(t)$, $t \geqq 0$, and a number α, $0 < \alpha \leqq 2$, such that

$$\sigma^2(t) \sim g(t)t^\alpha \quad \text{for} \quad t \to 0 .$$

IV. $\sigma^2(t)$ has a continuous derivative on B.

Assumption I states that the process has no singularities on B: in view of (1.1), this assumption is equivalent to

I′ $Y(s) \equiv Y(t)$ if and only if $s = t$.

Assumption II means that the process is not degenerate at any point of B. Assumption III was used in [2] in deriving the tail of the distribution of the excursion and the maximum in the stationary case. The assumption that g is non-decreasing was used in the proof of Theorem 2.1 of [2]; however, it may not be necessary. As noted there, if $\alpha = 2$ then g can be taken to be a positive constant. The existence of certain limits in calculations to follow requires Assumption IV. Assumption I eliminates the trivial case where $Y(t)$ is identically equal to a fixed random variable with a Gaussian distribution, multiplied by t.

We now study the behavior of $EY(s)Y(t)$ for small $|t - s|$. There are two cases to consider: $\alpha < 2$ and $\alpha = 2$ in Assumption III. The first case is covered in the following lemma.

LEMMA 1.1. *If* $\alpha < 2$, *then*

$$\frac{\sigma(t + h) - \sigma(t)}{\sigma(h)} \to 0 \quad \textit{for} \quad h \to 0 ,$$

uniformly for $t \in B$.

PROOF. The ratio above is equal to

$$\frac{\sigma^2(t + h) - \sigma^2(t)}{[\sigma(t + h) + \sigma(t)]\sigma(h)} .$$

By Assumption IV the numerator is of the order $|h|$ uniformly in $t \in B$; and, by Assumptions II and III, the denominator is asymptotic to a positive multiple of $g^{\frac{1}{2}}(|h|)|h|^{\alpha/2}$; hence, since $\alpha < 2$, the fraction converges to 0.

Now we consider the more complicated case $\alpha = 2$. First we recall that $\sigma^2(t)$ always has the spectral representation [5], page 552,

$$\sigma^2(t) = \int_{-\infty}^{\infty} |e^{i\lambda t} - 1|^2(1 + \lambda^2)\lambda^{-2}\, dH(\lambda) ,$$

or, equivalently,

$$\sigma^2(t) = 2 \int_{-\infty}^{\infty} (1 - \cos \lambda t)(1 + \lambda^2)\lambda^{-2}\, dH(\lambda) , \tag{1.2}$$

where H is a bounded monotonic function. This is used in the following two lemmas.

LEMMA 1.2. *If* $\alpha = 2$, *then*

$$\lim_{t \to 0} t^{-2}\sigma^2(t) = \int_{-\infty}^{\infty} (1 + \lambda^2)\, dH(\lambda) ; \tag{1.3}$$

the latter is positive and finite.

PROOF. Divide both sides of (1.2) by t^2, and let $t \to 0$. The right-hand side

converges to the right-hand side of (1.3), which might possibly be infinite or zero. We shall show that it is neither. Assumption III implies that

$$\limsup_{t\to 0} t^{-2}\sigma^2(t) \leqq g(\varepsilon)$$

for every $\varepsilon > 0$; hence the limit in (1.3) must be finite. The integral in (1.3) cannot be equal to 0: if it were so, then H would have all its mass at $\lambda = 0$, and so $\sigma^2(t)$ would be a constant multiple of t^2, and this would contradict Assumption I.

LEMMA 1.3. *If $\alpha = 2$, then there exists a continuous function $q(t)$ assuming values $|q| < 1$ such that*

$$\lim_{h\to 0} \frac{\sigma(t+h) - \sigma(t)}{\sigma(h)} = q(t), \tag{1.4}$$

uniformly for $t \in B$.

PROOF. Define $q(t)$ as

$$q(t) = \frac{d\sigma^2}{dt} \Big/ 2\sigma(t)(g(0))^{\frac{1}{2}}. \tag{1.5}$$

By Lemma 1.2 and Assumption III, $g(0)$ is positive because

$$g(0) = \int_{-\infty}^{\infty} (1 + \lambda^2)\, dH(\lambda). \tag{1.6}$$

The finiteness of the integral (1.6) permits one to compute the derivative of $\sigma^2(t)$ by differentiation under the integral sign in (1.2):

$$\frac{d\sigma^2}{dt} = 2\int_{-\infty}^{\infty} \lambda \sin \lambda t (1 + \lambda^2)\lambda^{-2}\, dH(\lambda). \tag{1.7}$$

Using the integral representations (1.2), (1.6) and (1.7), and the Cauchy-Schwarz inequality, we find that

$$\left|\frac{d\sigma^2}{dt}\right|^2 \leqq g(0) \cdot 2\int_{-\infty}^{\infty} \sin^2 \lambda t (1 + \lambda^2)\lambda^{-2}\, dH(\lambda). \tag{1.8}$$

From the elementary relations, $\sin^2 x = 1 - \cos^2 x \leqq 2(1 - \cos x)$, we find that the right-hand side is at most $g(0) \cdot 4\sigma^2(t)$; hence, the function $q(t)$, defined by (1.5), is of modulus at most 1.

We shall show that $q(t)$ does not assume the value 1. If it did, then (1.8) would be an equality; thus, by the Cauchy-Schwarz inequality and its converse, we would have

$$\sin \lambda t = \text{constant} \cdot \lambda \qquad \text{a.e. } dH(\lambda).$$

Such a relation holds only if H has its support at the origin. This would contradict Assumption I; hence, $|q(t)| < 1$.

The limit relation (1.4) is verified by writing the fraction as

$$\frac{\sigma^2(t+h)-\sigma^2(t)}{(\sigma(t+h)-\sigma(t))\sigma(h)},$$

dividing the numerator and denominator by h, letting $h \to 0$, and then applying Assumption IV, Lemma 1.2, and the relation (1.6). The uniformity of the convergence is demonstrated by using the Law of the Mean and the continuity of $d\sigma^2/dt$. The proof is complete.

We define the function $W_\alpha(t)$, $t \in B$, as

$$W_\alpha(t) = \frac{1}{2\sigma^2(t)} \quad \text{for} \quad \alpha < 2, \tag{1.9}$$

$$= \frac{1-q^2(t)}{2\sigma^2(t)} \quad \text{for} \quad \alpha = 2.$$

For convenience we will drop the subscript α, and write $W_\alpha = W$. It is positive and continuous on B.

As an immediate consequence of Lemmas 1.1 and 1.3, and the formulas (1.1) and (1.9) we obtain:

$$\lim_{h\to 0} \frac{1-EY(t+h)Y(t)}{\sigma^2(h)} = W(t) \tag{1.10}$$

uniformly for $t \in B$; indeed, by (1.1), we have

$$\frac{1-EY(t+h)Y(t)}{\sigma^2(h)} = \frac{1-\dfrac{(\sigma(t+h)-\sigma(t))^2}{\sigma^2(h)}}{2\sigma(t+h)\sigma(t)},$$

and, letting $h \to 0$, we get (1.10).

2. Estimate of the tail of the distribution of the maximum over a small interval. In this section we show that the covariance of the process Y is very close to that of certain stationary processes. Then we use the known properties of the distribution of the maximum for stationary processes to estimate that of the process Y.

We observe that, for any positive constant c, the function $\exp(-c\sigma^2(t))$ is the covariance function of a stationary Gaussian process with variance 1; indeed, it follows from the integral representation (1.2) of σ^2 that $\exp(-c\sigma^2(t))$ is the characteristic function of a symmetric, infinitely divisible distribution, and so is also a covariance function.

For arbitrary ε, $0 < \varepsilon < 1$, and subinterval A of B we define two stationary Gaussian processes with means 0 and variances 1. The first, denoted $U(t)$, has the covariance function

$$EU(s)U(t) = \exp\{-\sigma^2(s-t)(1+\varepsilon)\max_A W\}; \tag{2.1}$$

the second, $L(t)$, has the covariance

$$EL(s)L(t) = \exp\{-\sigma^2(s-t)(1-\varepsilon)\min_A W\}. \tag{2.2}$$

The covariances of Y, U and L are compared in the following:

LEMMA 2.1. *For every $\varepsilon > 0$ there exists $\delta > 0$ such that if A is a closed subinterval of B of length less than δ, then*

$$EU(s)U(t) \leqq EY(s)Y(t) \leqq EL(s)L(t) \tag{2.3}$$

for all s, t in A.

PROOF. In what follows we put

$$|A| = \text{length of } A .$$

By (1.10) and the continuity and positivity of W it follows that for every ε there exists δ such that

$$\begin{aligned} 1 - (1 + \varepsilon)\sigma^2(t - s) \max_A W \\ \leqq EY(s)Y(t) \leqq 1 - (1 - \varepsilon)\sigma^2(t - s) \min_A W \quad \text{for} \quad s, t \in A \end{aligned} \tag{2.4}$$

as long as $|A| < \delta$. Since $\sigma^2(t - s) \to 0$ for $|t - s| \to 0$, since (2.4) is stated for small values of $t - s$, and since ε is arbitrary, we may, by the elementary relation $1 - e^{-x} \sim x$, for $x \to 0$, replace the extreme members of (2.4) by exponentials;

$$\begin{aligned} \exp\{-\sigma^2(t - s)(1 + \varepsilon) \max_A W\} \leqq EY(s)Y(t) \\ \leqq \exp\{-\sigma^2(t - s)(1 - \varepsilon) \min_A W\} . \end{aligned}$$

This, by (2.1) and (2.2), is equivalent to (2.3).

It is now shown that $\max_A U$ is stochastically larger than $\max_A Y$, and the latter is stochastically larger than $\max_A L$; thus, U and L represent "upper" and "lower" processes for the estimation of $\max_A Y$.

LEMMA 2.2. *Given $\varepsilon > 0$, let δ be as in Lemma 2.1. If $|A| < \delta$, then, for all x.*

$$P\{\max_A L > x\} \leqq P\{\max_A Y > x\} \leqq P\{\max_A U > x\} . \tag{2.5}$$

PROOF. The processes Y, U and L have means 0 and variances 1. Their covariances satisfy (2.3); thus, the inequalities in (2.5) follow from the well-known inequality of Slepian [7].

In order to estimate the tail of the distribution of $\max_A L$ and $\max_A U$, we use the following result from [2] for the maximum of stationary Gaussian processes. Let $Z(t)$ be such a process with mean 0, variance 1, and covariance function $r(t)$ satisfying $1 - r(t) \sim g(t)|t|^\alpha$ for $t \to 0$, where g is as in Assumption III above. For $u > 0$, let v be defined in terms of u as the unique solution of

$$u^2 g(v^{-1})v^{-\alpha} = 1 . \tag{2.6}$$

Then there exists a constant V_α not depending on T such that

$$\lim_{u \to \infty} \frac{P\{\max_{[0,T]} Z > u\}}{Tv\varphi(u)/u} = V_\alpha , \tag{2.7}$$

where

$$\varphi(u) = (2\pi)^{-\frac{1}{2}} e^{-u^2/2} .$$

Now we show how the relation (2.7) is modified when the assumption on $r(t)$ is changed to

$$1 - r(t) \sim cg(t)|t|^{\alpha}, \tag{2.8}$$

where c is an arbitrary positive constant.

LEMMA 2.3. *If Z is a stationary Gaussian process satisfying* (2.8), *and if v is defined by* (2.6), *then*

$$\lim_{u\to\infty} \frac{P\{\max_{[0,T]} Z > u\}}{Tv\varphi(u)/u} = c^{1/\alpha} V_{\alpha}; \tag{2.9}$$

in other words, if g is multiplied by a constant, then V_{α} is multiplied in a corresponding way.

PROOF. Instead of the process $Z(t)$, $t \in [0, T]$, consider the process with the altered time parameter $Z(c^{-1\alpha/}t)$, $t \in [0, T]$. It has the covariance function $r(c^{-1/\alpha}t)$; hence, by (2.8), it satisfies

$$1 - r(c^{-1/\alpha}t) \sim g(c^{-1/\alpha}t)|t|^{\alpha};$$

therefore, by the slowly-varying property of g, it follows that

$$1 - r(c^{-1/\alpha}t) \sim g(t)|t|^{\alpha}.$$

The result (2.7) implies

$$\lim_{u\to\infty} \frac{P\{\max(Z(c^{-1/\alpha}t): t \in [0, T]) > u\}}{Tv\varphi(u)/u} = V_{\alpha}. \tag{2.10}$$

By definition:

$$\max(Z(c^{-1/\alpha}t): t \in [0, T]) = \max(Z(t): t \in [0, Tc^{-1/\alpha}]);$$

therefore, (2.10) is equivalent to

$$\lim_{u\to\infty} \frac{P\{\max_{[0,Tc^{-1/\alpha}]} Z(t) > u\}}{Tv\phi(u)/u} = V_{\alpha}. \tag{2.11}$$

Since T and c are arbitrary, we may replace T by $Tc^{1/\alpha}$ in (2.11), and obtain (2.9).

Using Lemmas 2.2 and 2.3 we find upper and lower asymptotic bounds for the tail of the distribution of $\max_A Y$.

LEMMA 2.4. *For $\varepsilon > 0$, let δ be as given in Lemma* 2.1. *If A is a subinterval of B of length less than δ, then*

$$\begin{aligned} &|A|V_{\alpha}(1 - \varepsilon)^{1/\alpha} \min_A W^{1/\alpha} \\ &\qquad \leqq \liminf_{u\to\infty} \frac{P\{\max_A Y > u\}}{v\phi(u)/u} \\ &\qquad \leqq \limsup_{u\to\infty} \frac{P\{\max_A Y > u\}}{v\phi(u)/u} \\ &\qquad \leqq |A|V_{\alpha}(1 + \varepsilon)^{1/\alpha} \max_A W^{1/\alpha}. \end{aligned} \tag{2.12}$$

PROOF. The covariance function of the stationary process U satisfies

$$1 - EU(t)U(0) \sim (1 + \varepsilon) \max_A Wg(t)|t|^\alpha .$$

Apply Lemma 2.3 with $c = (1 + \varepsilon) \max_A W$ and $A = [0, T]$; then

$$\lim_{u\to\infty} \frac{P\{\max_A U > u\}}{v\phi(u)/u} = |A|(1 + \varepsilon)^{1/\alpha} \max_A W^{1/\alpha} V_\alpha .$$

The last inequality in (2.12) now follows from Lemma 2.2. The first inequality is obtained in a similar manner, by using the process L and $c = (1 - \varepsilon) \min_A W$.

3. An exact asymptotic formula for the tail of the distribution of the maximum. To find the tail of the distribution of $\max_B Y$ we modify and extend the method in [3]. The interval B is decomposed into many small intervals $A_1, \cdots, A_m$ of equal length; then

$$\max_B Y = \max_j (\max_{A_j} Y) .$$

We show that the m events $\{\max_{A_j} Y > u\} j = 1, \cdots, m$, are "asymptotically disjoint": if $\max_{A_j} Y > u$, then it is relatively unlikely that $\max_{A_j} Y > u$ for any $j \neq i$. If follows that

$$P\{\max_B Y > u\} \sim \sum_{j=1}^m P\{\max_{A_j} Y > u\} .$$

Using Lemma 2.4, we estimate the terms on the right-hand side; finally, we let $m \to \infty$.

Our first step is showing that $\max_A Y$ is stochastically relatively small compared to $\max_B Y$ if the set A is small compared to B.

LEMMA 3.1. *For ε, $0 < \varepsilon < \frac{1}{2}$, there exists $\delta > 0$ such that if $A_1, \cdots, A_m$ are closed subintervals of B of lengths all less than δ, then*

$$\limsup_{u\to\infty} \frac{P\{\max_{\cup A_j} Y > u\}}{P\{\max_B Y > u\}} \leqq 2\left(\frac{1 + \varepsilon}{1 - \varepsilon}\right)^{1/\alpha} \delta^{-1} \left(\frac{\max_B W}{\min_B W}\right)^{1/\alpha} \sum_{j=1}^m |A_j| . \tag{3.1}$$

PROOF. First we estimate the probability in the numerator in (3.1). For arbitrary ε, $0 < \varepsilon < \frac{1}{2}$, choose δ as in Lemma 2.4; then

$$\begin{aligned} \limsup_{u\to\infty} \frac{P\{\max_{\cup A_j} Y > u\}}{v\phi(u)/u} &\leqq \sum_{j=1}^m \limsup_{u\to\infty} \frac{P\{\max_{A_j} Y > u\}}{v\phi(u)/u} \\ &\leqq V_\alpha (1 + \varepsilon)^{1/\alpha} \sum_{j=1}^m |A_j| \max_{A_j} W^{1/\alpha} \\ &\leqq V_\alpha (1 + \varepsilon)^{1/\alpha} \max_B W^{1/\alpha} \sum_{j=1}^m |A_j| . \end{aligned} \tag{3.2}$$

Now we find a lower asymptotic bound on the denominator in (3.1). If A is any subinterval of B of length less than δ, then, by Lemma 2.4,

$$\begin{aligned}
&\liminf_{u\to\infty} \frac{P\{\max_B Y > u\}}{v\phi(u)/u} \\
&\qquad \geqq \liminf_{u\to\infty} \frac{P\{\max_A Y > u\}}{v\phi(u)/u} \\
&\qquad \geqq |A| V_\alpha (1-\varepsilon)^{1/\alpha} \min_A W^{1/\alpha} \\
&\qquad \geqq |A| V_\alpha (1-\varepsilon)^{1/\alpha} \min_B W^{1/\alpha} .
\end{aligned} \tag{3.3}$$

It follows from (3.2) and (3.3) that the left-hand side of (3.1) is not more than

$$\frac{1}{|A|}\left(\frac{1+\varepsilon}{1-\varepsilon}\right)^{1/\alpha}\left(\frac{\max_B W}{\min_B W}\right)^{1/\alpha} \sum_{j=1}^{m} |A_j| .$$

Since A is an arbitrary subinterval of length less than $\delta/2$, we may replace $|A|$ by $\delta/2$ in the expression above; thus, we obtain the right-hand side of (3.1).

Now we estimate the difference between the distributions of the maximum over a linear set and the maximum over a finite subset. Let A be an arbitrary linear set and $D = \{t_1, \cdots, t_N\}$ a finite subset of A having N elements with order $t_1 < \cdots < t_N$. Put

$$S_j^2 = \max_{t_j \leqq s \leqq t_{j+1}} E(Y(s) - Y(t_j))^2, \qquad j = 1, \cdots, N-1 .$$

LEMMA 3.2. *There exist positive constants K_1, K_2 and K_3 such that for every $\varepsilon > 0$ the inequality*

$$\begin{aligned}
&|P\{\max_A Y > u\} - P\{\max_D Y > u\}| \\
&\qquad \leqq (K_1/K_2)(u/\varepsilon) \sum_{j=1}^{N-1} S_j \exp(-K_2^2\varepsilon^2/2u^2S_j^2) \\
&\qquad\quad + P\{u < \max_A Y \leqq u + \varepsilon/u\}
\end{aligned} \tag{3.4}$$

holds as long as $\max_j(uS_j) < \varepsilon K_3$.

PROOF. Since the event $\max_D Y > u$ implies $\max_A Y > u$, we have

$$P\{\max_A Y > u\} - P\{\max_D Y > u\} = P\{\max_A Y > u, \max_D Y \leqq u\} .$$

For arbitrary $\varepsilon > 0$ the latter is equal to

$$\begin{aligned}
&P\{\max_A Y > u + \varepsilon/u, \max_D Y \leqq u\} \\
&\qquad + P\{u < \max_A Y \leqq u + \varepsilon/u, \max_D Y \leqq u\} .
\end{aligned} \tag{3.5}$$

The first term in (3.5) is not more than

$$\sum_{j=1}^{N-1} P\{\max_{t_j \leqq s \leqq t_{j+1}} Y(s) - Y(t_j) > \varepsilon/u\} ; \tag{3.6}$$

in fact, if $\max_A Y > u + \varepsilon/u$ and $\max_D Y \leqq u$, then for some j,

$$\max_{t_j \leqq s \leqq t_{j+1}} Y(s) - Y(t_j) > \varepsilon/u .$$

Apply Fernique's inequality [6] to the jth term in (3.6): there exist constants K_1, K_2 and K_3 such that

$$P\left\{\max_{t_j \leqq s \leqq t_{j+1}} \frac{Y(s) - Y(t_j)}{S_j} > \frac{\varepsilon}{uS_j}\right\} \leqq K_1 \int_{K_2\varepsilon/uS_j}^{\infty} e^{-y^2/2}\, dy$$

as long as $uS_j < \varepsilon K_3$. Sum over $j = 1, \cdots, N-1$ and apply the well-known inequality

$$\int_x^{\infty} e^{-y^2/2}\, dy \leqq x^{-1} e^{-x^2/2}, \qquad x > 0;$$

then (3.6) is at most equal to the first term on the right-hand side of (3.4). The second term in (3.5) is evidently less than the corresponding term on the right-hand side (3.4). This completes the proof of (3.4).

Now we estimate the second term on the right-hand side of (3.4).

LEMMA 3.3. *There exists a constant $K > 0$ such that*

$$\limsup_{u\to\infty} \frac{P\{u < \max_A Y \leqq u + \varepsilon/u\}}{V_\alpha(v\phi(u)/u) \int_A W^{1/\alpha}(s)\, ds} \leqq K\varepsilon \tag{3.7}$$

for all ε, $0 < \varepsilon < \frac{1}{2}$, and all subintervals A of B. This holds also when A is a union of disjoint intervals.

PROOF. It is sufficient to consider the case where A is an interval. Let it be decomposed into m intervals $A_1, \cdots, A_m$ of equal length $|A|/m$, where m is an arbitrary positive integer. Let δ be the number in Lemma 2.1, and suppose that m is so large that $|A|/m < \delta$. If $u < \max_A Y \leqq u + \varepsilon/u$, then for some $j = 1, \cdots, m$, we have $u < \max_{A_j} Y \leqq u + \varepsilon/u$; therefore,

$$P\{u < \max_A Y \leqq u + \varepsilon/u\} \leqq \sum_{j=1}^{m} P\{u < \max_{A_j} Y \leqq u + \varepsilon/u\}. \tag{3.8}$$

The jth term in this sum is equal to

$$P\{\max_{A_j} Y > u\} - P\{\max_{A_j} Y > u + \varepsilon/u\}. \tag{3.9}$$

An upper asymptotic bound on the first term in (3.9) is furnished by Lemma 2.4:

$$\limsup_{u\to\infty} \frac{P\{\max_{A_j} Y > u\}}{v\phi(u)/u} \leqq V_\alpha |A_j| (1 + \varepsilon)^{1/\alpha} \max_{A_j} W^{1/\alpha}. \tag{3.10}$$

To obtain a lower asymptotic bound on the second term in (3.9), we apply Lemmas 2.3 and 2.4 with $u + \varepsilon/u$ in place of u. First we note that if v_ε is the solution of

$$(u + \varepsilon/u)^2 g(v_\varepsilon^{-1}) v_\varepsilon^{-\alpha} = 1,$$

i.e. (2.6) with $u + \varepsilon/u$ in place of u, then $v \sim v_\varepsilon$ for $u \to \infty$; therefore, v is still the "correct" function for Lemmas 2.3 and 2.4 when u is replaced by $u + \varepsilon/u$. (The proof is given in [2], Section 3.) Divide the second term in (3.9) by $v\phi(u)/u$ and let $u \to \infty$:

$$\begin{aligned}
&\liminf_{u\to\infty} \frac{P\{\max_{A_j} Y > u + \varepsilon/u\}}{v\phi(u)/u} \\
&= \liminf_{u\to\infty} \frac{\phi(u+\varepsilon/u)u}{\phi(u)(u+\varepsilon/u)} \cdot \frac{P\{\max_{A_j} Y > u + \varepsilon/u\}}{v\phi(u+\varepsilon/u)/(u+\varepsilon/u)} \qquad (3.11)\\
&= e^{-\varepsilon} \liminf_{u\to\infty} \frac{P\{\max_{A_j} Y > u\}}{v\phi(u)/u} \\
&\geqq V_\alpha |A_j| (1-\varepsilon)^{1/\alpha} e^{-\varepsilon} \min_{A_j} W^{1/\alpha} .
\end{aligned}$$

We now sum (3.10) and (3.11) over j; then (3.8) and (3.9) imply

$$\begin{aligned}
&\limsup_{u\to\infty} \frac{P\{u < \max_A Y \leqq u + \varepsilon/u\}}{v\phi(u)/u} \\
&\leqq V_\alpha \textstyle\sum_{j=1}^m |A_j| (1+\varepsilon)^{1/\alpha} \max_{A_j} W^{1/\alpha} \qquad (3.12)\\
&\quad - V_\alpha \textstyle\sum_{j=1}^m |A_j| (1-\varepsilon)^{1/\alpha} e^{-\varepsilon} \min_{A_j} W^{1/\alpha} .
\end{aligned}$$

Since m is arbitrary we let $m \to \infty$ on the right-hand side. As a continuous function on a closed interval, $W^{1/\alpha}$ is Riemann integrable; hence, the sums in (3.12) converge to integrals, and so the right-hand side converges to

$$V_\alpha \textstyle\int_A W^{1/\alpha}(s)\, ds \cdot [(1+\varepsilon)^{1/\alpha} - (1-\varepsilon)^{1/\alpha} e^{-\varepsilon}] .$$

By elementary calculus, there exists $K > 0$ such that $K\varepsilon$ dominates the bracketed function of ε given above. The proof of (3.7) is complete.

Using Lemmas 3.2 and 3.3, we show that the tail of the distribution of the maximum is asymptotically unchanged if the set A is replaced by an increasing sequence of finite subsets whose density grows sufficiently quickly.

LEMMA 3.4. *Let A be a union of disjoint closed subintervals of B and, for each positive integer N, let $D = D_N$ be a finite subset as in Lemma* 3.2. *If $N \to \infty$ with u in such a way that*

$$N \sim u^{5/\alpha} \qquad (3.13)$$

then

$$P\{\max_D Y > u\} \sim P\{\max_A Y > u\} .$$

PROOF. We write

$$1 - \frac{P\{\max_D Y > u\}}{P\{\max_A Y > u\}} = \frac{P\{\max_A Y > u\} - P\{\max_D Y > u\}}{P\{\max_A Y > u\}} ,$$

and show that the latter converges to 0. By (3.3), the denominator is asymptotically, at least a constant multiple of $v\phi(u)/u$; therefore, it suffices to show that

$$\frac{P\{\max_A Y > u\} - P\{\max_D Y > u\}}{v\phi(u)/u} \qquad (3.14)$$

tends to 0 as $u \to \infty$. Apply Lemma 3.2 to the numerator in (3.14); for $\varepsilon > 0$, the ratio is bounded above by the sum of two terms,

$$\frac{(K_1/K_2)(u/\varepsilon)\sum_{j=1}^{N-1} S_j \exp(-K_2^2\varepsilon^2/2u^2S_j^2)}{v\phi(u)/u} \tag{3.15}$$

and

$$\frac{P\{u < \max_A Y \leqq u + \varepsilon/u\}}{v\phi(u)/u} . \tag{3.16}$$

By Lemma 3.3, the lim sup of (3.16) is dominated by $V_\alpha(\int_B W^{1/\alpha}(s)\,ds)K\varepsilon$; therefore, this may be ignored because ε is arbitrary.

We now estimate (3.15). By the definition of S_j^2, we have

$$S_j^2 = 2 \max_{[t_j, t_{j+1}]}[1 - EY(s)Y(t_j)] .$$

If u is large, then $t_{j+1} - t_j$ is small; thus, (1.10) implies

$$S_j^2 \leqq \sigma^2(t_{j+1} - t_j) \max_{[t_j, t_{j+1}]} W$$

for all sufficiently large u; furthermore, Assumption III and (3.13) imply

$$\sigma^2(t_{j+1} - t_j) = O(|t_{j+1} - t_j|^\alpha) = O(u^{-5}) ;$$

therefore,

$$\max_j S_j^2 = O(u^{-5}) .$$

It follows that the product $u^2S_j^2$ in (3.15) may be replaced by u^{-3}; hence, the exponential factor is at most $\exp(-\text{constant } u^3)$. The function $\phi(u)$ in the denominator has the larger exponential factor $\exp(-\frac{1}{2}u^2)$. The number of terms in the sum in (3.15) is of the order $u^{5/\alpha}$; hence, the ratio converges to 0 as $u \to \infty$. The proof is now complete.

Our final lemma—before the statement and proof of the main theorem of this section—states that if $A_1, \cdots, A_m$ are disjoint intervals, then the events $\max_{A_j} Y > u$ are "asymptotically disjoint."

LEMMA 3.5. *For an arbitrary positive integer m, let B be decomposed into m disjoint subintervals $A_1, \cdots, A_m$, and let $\max_{A_j} Y$ refer to the maximum on the closure of A_j. Then, for $u \to \infty$*

$$\sum_{j=1}^m P\{\max_{A_j} Y > u\} \sim P\{\max_B Y > u\} . \tag{3.17}$$

PROOF. According to Lemma 3.1, the tail of the distribution of $\max_B Y$ is changed by at most a relatively small amount if relatively small pieces are removed from B. Given $\varepsilon > 0$, let us clip a small segment of length ε/m from the right endpoint of each interval $A_1, \cdots, A_m$; this changes the quantities in (3.17) by a relatively small amount. For this reason we may suppose in this proof that $A_1, \cdots, A_m$ are closed and disjoint, separated by successive intervals of a fixed positive length ε/m, and that B is the union of $A_1, \cdots, A_m$.

By Lemma 3.4, the quantities in (3.17) are asymptotically unchanged when

each A_j is replaced by its intersection with a finite set D of N equally spaced points, where N grows as in (3.13); thus, it suffices to prove

$$\sum_{j=1}^{m} P\{\max_{A_j \cap D} Y > u\} \sim P\{\max_{B \cap D} Y > u\} . \tag{3.18}$$

The event $\max_{B\cap D} Y > u$ is the union of $\max_{A_j \cap D} Y > u$; hence, by the elementary formula for the union of m events in terms of probabilities of their intersections, we get

$$P\{\max_{B\cap D} Y > u\} = \sum_{j=1}^{m} P\{\max_{A_j \cap D} Y > u\} + Q , \tag{3.19}$$

where

$$|Q| \leqq P\{\max_{A_j \cap D} Y > u \text{ for at least two distinct indices } j\} . \tag{3.20}$$

The event in (3.20) implies that for some $s \in A_i \cap D$ and $t \in A_j \cap D$, for some pair $i \neq j$,

$$Y(s) > u \qquad \text{and} \qquad Y(t) > u .$$

There are fewer than N^2 such pairs, and the intervals A_i and A_j are separated by a distance at least equal to ε/m; therefore, the event in (3.20) has the probability at most equal to

$$N^2 \sup_{|s-t| \geqq \varepsilon/m} P\{Y(s) > u, Y(t) > u\} .$$

According to a well-known formula in [4], page 27, this is equal to the sum of the two terms,

$$N^2\{\int_u^\infty \phi(y)\, dy\}^2 \tag{3.21}$$

and

$$N^2 \sup_{|s-t| \geqq \varepsilon/m} \int_0^{EY(s)Y(t)} \phi(u, u; y)\, dy , \tag{3.22}$$

where $\phi(x, y; \rho)$ is the standard bivariate Gaussian density with correlation coefficient ρ. By the well-known relation

$$\int_u^\infty \phi(y)\, dy \leqq \phi(u)/u ,$$

and the growth rate (3.13) of N, the term (3.21) is at most

$$u^{(10/\alpha)-2}\phi^2(u) . \tag{3.23}$$

To estimate (3.22) we note that, by Assumption I, or its equivalent (I′), and the continuity of σ, $EY(s)Y(t)$ is bounded away from 1 for $|s - t| \geqq \varepsilon/m$; thus, there exists $b > 0$ such that (3.22) is at most

$$N^2 \int_0^{1-b} \phi(u, u; y)\, dy ,$$

which, by the identity

$$\phi(x, y; \rho) = \frac{\phi(x)}{(1 - \rho^2)^{\frac{1}{2}}} \phi\left(\frac{y - \rho x}{(1 - \rho^2)^{\frac{1}{2}}}\right) ,$$

is equal to

$$N^2\phi(u)\int_0^{1-b}\phi\left(u\left(\frac{1-y}{1+y}\right)^{\frac{1}{2}}\right)\frac{dy}{(1-y^2)^{\frac{1}{2}}},$$

which is at most

$$N^2\phi(u)\phi\left(u\left(\frac{b}{2-b}\right)^{\frac{1}{2}}\right)\left(\frac{1-b}{1+b}\right)^{\frac{1}{2}} \sim u^{10/\alpha}\phi(u)\phi\left(u\left(\frac{b}{2-b}\right)^{\frac{1}{2}}\right)\left(\frac{1-b}{1+b}\right)^{\frac{1}{2}}. \tag{3.24}$$

To complete the proof of (3.18), we cite (3.19) and show that

$$Q/P\{\max_{B\cap D} Y > u\} \to 0 .$$

By Lemma 3.4 and (3.3), the denominator is at least of the asymptotic order $v\phi(u)/u$; and, by (3.20), (3.23), and (3.24) the numerator is of the order of the sum of the expressions in (3.23) and (3.24). An elementary calculation now shows that the quotient converges to 0. The proof is complete.

The main result of this section is

THEOREM 3.1.

$$\lim_{u\to\infty}\frac{P\{\max_B Y > u\}}{v\phi(u)/u} = V_\alpha \cdot \int_B W^{1/\alpha}(s)\,ds . \tag{3.25}$$

PROOF. Given ε, $0 < \varepsilon < 1$, choose δ according to Lemma 2.1. Choose the positive integer m so large that $|B|/m < \delta$; and decompose B into disjoint intervals $A_1, \cdots, A_m$ of equal lengths $|B|/m$. By Lemma 3.5, it suffices, for (3.25), to find the limit of

$$\frac{\sum_{j=1}^m P\{\max_{A_j} Y > u\}}{v\phi(u)/u} . \tag{3.26}$$

By Lemma 2.4, the lim sup of (3.26) is at most

$$V_\alpha(1+\varepsilon)^{1/\alpha}\sum_{j=1}^m |A_j| \max_{A_j} W^{1/\alpha} \tag{3.27}$$

and the lim inf is at least

$$V_\alpha(1-\varepsilon)^{1/\alpha}\sum_{j=1}^m |A_j| \min_{A_j} W^{1/\alpha} . \tag{3.28}$$

Since m is arbitrary we let $m \to \infty$. The sums in (3.27) and (3.28) converge to

$$\int_B W^{1/\alpha}(s)\,ds$$

because $W^{1/\alpha}$ is Riemann integrable. The number ε is also arbitrary, so we put $\varepsilon = 0$. In this way (3.27) and (3.28) are transformed into the expression on the right-hand side of (3.25).

4. Conditional limiting distribution of the occupation time above a high level. For $u > 0$ and the linear set A, put

$$\xi(u; Y, A) = \text{Lebesgue measure of } \{t: t \in A, Y(t) > u\} ,$$

that is, the time spent by Y above the level u on the time set A. In this section we explicitly determine the conditional limiting distribution of ξ, given that it is positive, i.e., the limit of

$$P\{v\xi \leqq x \mid \xi > 0\} \tag{4.1}$$

for $u \to \infty$ on a dense set of $x > 0$. We use the method of conditional moments in [1, 2, 3]. Since ξ is a nonnegative random variable the conditional rth moment of ξ given that it is positive is

$$E\xi^r / P\{\xi > 0\} .$$

Since Y has continuous sample functions, it spends positive time above u, i.e., $\xi > 0$, if and only if $\max Y > u$; therefore, the conditional rth moment is equal to

$$\frac{E\xi^r(u;\, Y,\, A)}{P\{\max_A Y > u\}} . \tag{4.2}$$

As in Section 2 we use some known results about stationary processes to get approximations for the nonstationary Y. Let Z be a stationary Gaussian process satisfying the conditions of Lemma 2.3 with $c = 1$. According to [2], Theorem 2.3, there exists a distribution function Ψ_α with support on the nonnegative axis such that the conditional rth moment of $v\xi$ converges:

$$\frac{E\{v\xi(u;\, Z,\, [0,\, T])\}^r}{P\{\max_{[0,T]} Z > u\}} \to \int_0^\infty x^r \, d\Psi_\alpha(x) , \tag{4.3}$$

for every $r = 1$ and $T > 0$. Recall that Ψ_α is the unique distribution on the nonnegative axis whose mth moment is given by the expression in (2.1) of [2], divided by V. The moment generating function of Ψ_α is finite everywhere, so that (4.3) implies the convergence of (4.1) to the limit Ψ_α. The latter distribution was shown to be completely determined by the constants α and V_α. Now, as in Lemma 2.3, we show how (4.3) is modified when c is not necessarily equal to 1.

LEMMA 4.1. *If* (2.8) *holds, and if* v *is the solution of* (2.6), *then*

$$\frac{E\{vc^{1/\alpha}\xi(u;\, Z,\, [0,\, T])\}^r}{P\{\max_{[0,T]} Z > u\}} \to \int_0^\infty x^r \, d\Psi_\alpha(x) , \tag{4.4}$$

that is, v *is replaced by* $vc^{1/\alpha}$ *in* (4.3).

PROOF. Let v_c be the solution of the equation

$$u^2 c g(v_c^{-1}) v_c^{-\alpha} = 1 ; \tag{4.5}$$

then (2.7) holds when v is replaced by v_c. Since Ψ_α is determined by α and V_α, the result (4.3) holds under (2.8) when v is replaced by v_c. We have

$$v_c \sim vc^{1/\alpha} ; \tag{4.6}$$

indeed, by Lemma 2.3, the relation (2.7) holds when v is replaced by $vc^{1/\alpha}$. The conclusion (4.4) now follows from (4.6).

Our main result is that (4.1) converges to the distribution function

$$\frac{\int_B \Psi_\alpha(xW^{1/\alpha}(s))W^{1/\alpha}(s)\,ds}{\int_B W^{1/\alpha}(s)\,ds}, \tag{4.7}$$

a mixture of Ψ_α over a scale parameter. Before going to the rigorous proof, we give an intuitive explanation of (4.7). Let the interval B be decomposed into many infinitesimally small intervals of the form $[t, t + dt]$; then by Theorem 3.1,

$$P\{\max_{[t,t+dt]} Y > u \mid \max_B Y > u\} = \frac{P\{\max_{[t,t+dt]} Y > u\}}{P\{\max_B Y > u\}} \to \frac{W^{1/\alpha}(t)\,dt}{\int_B W^{1/\alpha}(s)\,ds}. \tag{4.8}$$

Write the occupation time $\xi(u; Z, [0, T])$ as the sum over the various intervals,

$$\sum \xi(u; Z, [t, t + dt]). \tag{4.9}$$

According to the proof of Lemma 3.5, the probability that this sum is positive is asymptotic to the probability that *exactly one* of the summands is positive; therefore, the conditional distribution of the sum (4.9), given that it is positive, is asymptotic to the conditional distribution given that exactly one summand $\xi(u; Z, [t, t + dt])$ is positive. For this reason the conditional distribution of the sum is a mixture of the conditional distributions for the various summands; and the mixing density is given by the last member of (4.8):

$$P\{\sum \xi(u; Z, [t, t + dt]) \leqq x \mid \sum \xi > 0\} \approx \sum P\{\xi(u; Z, [t, t + dt]) \leqq x \mid \xi > 0\} \frac{W^{1/\alpha}(t)\,dt}{\int_B W^{1/\alpha}(s)\,ds}.$$

The proof of the convergence of the distribution (4.1) is based on an extension of the method of upper and lower moments used in [1].

LEMMA 4.2. *For every $\varepsilon > 0$ there exists $\delta > 0$ such that if A is a subinterval of B of length less than δ, then for every positive integer r,*

$$E\xi^r(u; U, A) \leqq E\xi^r(u; Y, A) \leqq E\xi^r(u; L, A),$$

where U and L are the "upper" *and* "lower" *stationary processes defined by* (2.1) *and* (2.2).

PROOF. If $Z(t)$, $t \in A$ is a measurable Gaussian process, then, by Fubini's theorem:

$$E\xi^r(u; Z, A) = \int_A \cdots \int_A P\{Z(s_i) > u, i = 1, \cdots, r\}\,ds_1 \cdots ds_r. \tag{4.10}$$

By Lemma 2.1 and the version of Slepian's inequality for the minimum of Gaussian random variables, we get

$$P\{U(s_i) > u,\ i = 1, \cdots, r\} \leqq P\{Y(s_i) > u,\ i = 1, \cdots, r\} \leqq P\{L(s_i) > u,\ i = 1, \cdots, r\}$$

for all $s_1, \cdots, s_r$ in A. Integrate the last double inequality over A^r, and apply (4.10); the assertion of the lemma follows.

Now we extend Lemma 3.5.

LEMMA 4.3. *Let B be decomposed into m subintervals $A_1, \cdots, A_m$; then for every $x > 0$:*

$$\frac{P\{0 < v\xi(u;\ Y,\ B) \leqq x\}}{P\{\max_B Y > u\}} \sim \sum_{j=1}^{m} \frac{P\{0 < v\xi(u;\ Y,\ A_j) \leqq x\}}{P\{\max_{A_j} Y > u\}}$$

for $u \to \infty$.

PROOF. Put

$$E_j = \text{event } \{\max_{A_j} Y > u\}, \qquad j = 1, \cdots, m;$$

then the event $\{0 < v\xi(u;\ Y,\ B) \leqq x\}$ is the union of

$$E_j \cap \{v\xi(u;\ Y,\ B) \leqq x\}, \qquad j = 1, \cdots, m.$$

By the formula for the probability of the union of events in terms of their intersections we have

$$P\{0 < v\xi(u;\ Y,\ B) \leqq x\} = \sum_{j=1}^{m} P(E_j \cap \{v\xi(u;\ Y,\ B) \leqq x\}) + Q \tag{4.11}$$

where

$$|Q| \leqq P(E_i \cap E_j \quad \text{for some pair} \quad i \neq j). \tag{4.12}$$

According to the proof of Lemma 3.5, the right-hand side of (4.12) is of smaller order than $P\{\max_B Y > u\}$ for $u \to \infty$; therefore, from (4.11) and (4.12) we obtain

$$\frac{P\{0 < v\xi(u;\ Y,\ B) \leqq x\}}{P\{\max_B Y > u\}} \sim \sum_{j=1}^{m} \frac{P(E_j \cap \{v\xi(u;\ Y,\ B) \leqq x\})}{P\{\max_B Y > u\}}. \tag{4.13}$$

In order to complete the proof we have to replace the set B on the right-hand side by A_j. If E_j occurs *but not* E_k for any other $k \neq j$, then $\xi(u;\ Y,\ B)$ is equal to $\xi(u;\ Y,\ A_j)$, that is, the time spent over u is consumed on A_j. The probability that E_j occurs *and* E_k *also* for some $k \neq j$ is not more than the right-hand side of (4.12). By the remark following (4.12), the latter probability may be ignored in passing to the limit in (4.13). For this reason B may be replaced by A_j on the right-hand side of (4.13), and the assertion of the lemma follows.

Our major result is:

THEOREM 4.1. *The conditional distribution*

$$P\{v\xi(u;\ Y,\ B) \leqq x \mid \xi > 0\} \tag{4.14}$$

converges to the limiting distribution (4.7).

PROOF. As noted following (4.3) the moment generating function of Ψ_α is finite everywhere. The moment generating function of (4.7) is a mixture of the corresponding functions for Ψ_α, namely,

$$\frac{\int_B \{\int_{-\infty}^{\infty} \exp[txW^{-1/\alpha}(s)]\, d\Psi_\alpha(x)\} W^{1/\alpha}(s)\, ds}{\int_B W^{1/\alpha}(s)\, ds}.$$

This is finite for every t because W is positive on B and so is bounded away from 0. As in [2] the finiteness of the moment generating function allows us to use the moment convergence theorem to prove the convergence of (4.14).

Let B be decomposed into intervals $A_1, \cdots, A_m$ as in Lemma 4.3. By definition, the conditional distribution (4.14) is equal to

$$\frac{P\{0 < v\xi(u; Y, B) \leqq x\}}{P\{\max_B Y > u\}}, \tag{4.15}$$

and, by Lemma 4.3, is asymptotic to

$$\sum_{j=1}^m \frac{P\{0 < v\xi(u; Y, A_j) \leqq x\}}{P\{\max_B Y > u\}}. \tag{4.16}$$

The rth moment of the monotone function (4.16) is equal to

$$\sum_{j=1}^m \frac{E[v\xi(u; Y, A_j)]^r}{P\{\max_B Y > u\}}. \tag{1.17}$$

Given ε, $0 < \varepsilon < 1$, choose $\delta > 0$ as in Lemma 4.2, and then choose the intervals $A_1, \cdots, A_m$ so that each is of length less than δ; thus, by Lemma 4.2, the moment (4.17) is at most

$$\frac{\sum_{j=1}^m E\{v\xi(u; L, A_j)\}^r}{P\{\max_B Y > u\}} \tag{4.18}$$

and at least

$$\frac{\sum_{j=1}^m E\{v\xi(u; U, A_j)\}^r}{P\{\max_B Y > u\}}. \tag{4.19}$$

Apply Lemma 4.1 with $Z = L$ and $c = (1 - \varepsilon) \min_{A_j} W$: the jth term in the numerator in (4.18) is asymptotic to

$$(1 - \varepsilon)^{-r/\alpha} \max_{A_j} W^{-r/\alpha} \cdot P(\max_{A_j} L > u) \int_0^{\infty} x^r\, d\Psi_\alpha(x),$$

and so (4.18) is asymptotic to

$$\frac{(1 - \varepsilon)^{-r/\alpha} \int_0^{\infty} x^r\, d\Psi_\alpha(x)}{P\{\max_B Y > u\}} \sum_{j=1}^m \max_{A_j} W^{-r/\alpha} P\{\max_{A_j} L > u\}.$$

By Lemma 2.3 (with $Z = L$ and c as before) and Theorem 3.1 this converges to

$$\frac{\int_0^{\infty} x^r\, d\Psi_\alpha(x) \sum_{j=1}^m \max_{A_j} W^{-r/\alpha} \min_{A_j} W^{1/\alpha} |A_j|}{(1 - \varepsilon)^{(r-1)/\alpha} \int_B W^{1/\alpha}(s)\, ds}. \tag{4.20}$$

By an analogous argument, with U in place of L, we find that (4.19) converges to

$$\frac{\int_0^\infty x^r \, d\Psi_\alpha(x) \sum_{j=1}^m \min_{A_j} W^{-r/\alpha} \max_{A_j} W^{1/\alpha} |A_j|}{(1+\varepsilon)^{(r-1)/\alpha} \int_B W^{1/\alpha}(s)\, ds}. \tag{4.21}$$

The sums appearing in (4.20) and (4.21) converge, as $m \to \infty$, to the common limit

$$\int_B W^{(1-r)/\alpha}(s)\, ds .$$

Letting $\varepsilon \to 0$, we find that (4.20) and (4.21) converge to the common limit

$$\frac{\int_B \{\int_0^\infty [xW^{-1/\alpha}(s)]^r \, d\Psi_\alpha(x)\} W^{1/\alpha}(s)\, ds}{\int_B W^{1/\alpha}(s)\, ds}, \tag{4.22}$$

which is the rth moment of the distribution function (4.7). From this we wish to conclude that the moment (4.17) converges to the moment (4.22); this would complete the proof of the theorem.

These last steps are taken with a compactness argument. Let $F_u(x)$ be the distribution function (4.15), and $F_{u,m}(x)$ the distribution (4.16); then, as shown above,

$$F_u(x) \sim F_{u,m}(x) \tag{4.23}$$

for every $x > 0$ and $m \geqq 1$, as $u \to \infty$. Let $F_{u'}(x)$ be an arbitrary weakly convergent sequence with index $u' \to \infty$, and let F be the weak limit. By (4.23), the sequence $F_{u',m}$ also converges weakly to F. For each r, the moment

$$\int_0^\infty x^r \, dF_{u'm} \tag{4.24}$$

is bounded above by (4.18) (with $u = u'$), which converges; therefore (4.24) is bounded; thus, we can, by the diagonal process, extract a subsequence $F_{u'',m}$ such that

$$\int_0^\infty x^r \, dF_{u'',m} \tag{4.25}$$

converges for every $r \geqq 1$. The limit of (4.25) is necessarily the rth moment (for $r \geqq 1$) of a fixed distribution G_m,

$$\int_0^\infty x^r \, dG_m . \tag{4.26}$$

This is dominated by the moment sequence (4.20). Since the latter has an everywhere convergent generating function, so does the moment sequence (4.26). Apply the moment convergence theorem: the subsequence $F_{u'',m}$ converges completely to G_m. Since, by definition, F is the weak limit of $F_{u'',m}$, it follows from the uniqueness of the weak limit that $F = G_m$ at all continuity points for all $m \geqq 1$. Since m is arbitrary and since (4.26) is bounded above and below by (4.20) and (4.21), respectively, we let $m \to \infty$ and then $\varepsilon \to 0$; then (4.26) converges to (4.22); therefore, F has the moment sequence (4.22). Since F is the limit of an arbitrary weakly convergent subsequence, it follows that it is the limit of the original sequence, and indeed, it is also the complete limit.

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