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SOME SAMPLE FUNCTION PROPERTIES OF THE TWO-PARAMETER GAUSSIAN PROCESS

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Let $\{X(s, t; \omega): (s, t) \in [0, \infty) \times [0, \infty)\}$ be a two parameter Gaussian process with mean function zero and covariance function $R(s_1, t_1; s_2, t_2) = \min(s_1, s_2) \min(t_1, t_2)$. This paper derives a multiparameter law of the iterated logarithm and modulus of continuity for the process $X(s, t; \omega)$. Estimates are also given which enable the author to define an Itô type integral for a suitable class of functions and to solve a diffusion equation involving the process.

1. Introduction. A number of papers have appeared in the literature defining multiparameter analogs of the Brownian motion process. Čencov [3] and Yeh [12] have shown that a multiparameter process with parameter space the p-dimensional unit cube A and covariance function $R[(u_1, \cdots, u_p), (v_1, \cdots, v_p)] = \min(u_1, v_1) \cdots \min(u_p, v_p)$ can be realized in the space of continuous functions on A which vanish on $A_0 = \{(u_1, \cdots, u_p) \in A: u_j = 0 \text{ for some } j, 1 \leqq j \leqq p\}$. Delporte [5] and W. Park [11] construct such a process on the unit cube using a Haar function expansion and W. Park generalizes some results of C. Park, Shepp and Yeh.

In the present paper a Haar function construction was used with an arctangent transformation (Ciesielski [4]) to define a Gaussian process $\{X(s, t; \omega): (s, t) \in [0, \infty) \times [0, \infty)\}$ with mean function $m(s, t) \equiv 0$ and covariance function $R(s_1, t_1; s_2, t_2) = \min(s_1, s_2) \min(t_1, t_2)$. The sample functions of this process are continuous and the process has independent increments (i.e. if $0 = s_0 < \cdots < s_m = S$, $0 = t_0 < \cdots < t_n = T$ partitions $[0, S] \times [0, T]$, the random variables $\{\Delta X[(s_i, t_j), (s_{i-1}, t_{j-1}); \omega]: i = 1, \cdots, m, j = 1, \cdots, n\}$ where $\Delta X[(s_i, t_j), (s_{i-1}, t_{j-1}); \omega] = X(s_i, t_j; \omega) - X(s_i, t_{j-1}; \omega) - X(s_{i-1}, t_j; \omega) + X(s_{i-1}, t_{j-1}; \omega)$, are mutually independent).

Sample function properties of the process $X(s, t; \omega)$ are examined, an Itô type integral is defined for a suitable class of functions and a diffusion equation is solved. Some properties of the integral and the solution of the diffusion equation are also investigated.

A different generalization of Brownian motion to a p-dimensional parameter space has been discussed by Lévy [9].

2. Sample function properties of $X(s, t; \omega)$. For fixed $t = t_0$, $X(s, t; \omega)$ is a one-dimensional Brownian motion process with mean function zero and covariance function $R(s_1, s_2) = t_0 \min(s_1, s_2)$.

We define a partial ordering on $[0, \infty) \times [0, \infty)$ by $(s', t') < (s, t)$ if $s' \leqq s$,

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$t' \leqq t$. Let $\mathscr{F}_{st}$ be the σ-field generated by the random variables $\{X(u, v; \omega): (u, v) < (s, t)\}$.

LEMMA 2.1. *Let* $0 = s_0 < \cdots < s_m = S$, $0 = t_0 < \cdots < t_n = T$ *be a partition of* $[0, S] \times [0, T]$. *Let* $(s, t) \in [0, S] \times [0, T]$. *If* $s_{i-1} \geqq s$ *or* $t_{j-1} \geqq t$, *then* $\Delta X[(s_i, t_j), (s_{i-1}, t_{j-1}); \omega]$ *is independent of* $\mathscr{F}_{st}$.

PROOF. Since $X(s, t; \omega)$ is a Gaussian process and since $\Delta X[(s_i, t_j), (s_{i-1}, t_{j-1})] = \Delta X_{ij}$ independent of every finite linear combination of $X(u_1, v_1), \cdots, X(u_n, v_n)$, $(u_i, v_i) < (s, t)$, $i = 1, \cdots, n$, implies ΔX_{ij} is independent of $\mathscr{F}_{st}$, it is sufficient to show that $E[\Delta X_{ij} X(u, v)] = 0$ for $(u, v) < (s, t)$.

$$\begin{aligned} E[\{X(s_i, t_j) &- X(s_{i-1}, t_j) - X(s_i, t_{j-1}) + X(s_{i-1}, t_{j-1})\}X(u, v)] \\ &= u[\min(t_j, v) - \min(t_j, v) - \min(t_{j-1}, v) + \min(t_{j-1}, v)] = 0 \\ &\qquad\qquad \text{if} \quad s_{i-1} \geqq s \geqq u, \\ &= v[\min(s_i, u) - \min(s_{i-1}, u) - \min(s_i, u) + \min(s_{i-1}, u)] = 0 \\ &\qquad\qquad \text{if} \quad t_{j-1} \geqq t \geqq v. \end{aligned}$$

LEMMA 2.2. *Let* $(s_1, t_1), (s_2, t_2) \in [0, \infty) \times [0, \infty)$. *The random variable* $\Delta X[(s_2, t_2), (s_1, t_1); \omega]$ *is Gaussian with mean zero and covariance* $|s_2 - s_1||t_2 - t_1|$.

PROOF. Clearly $\Delta X[(s_2, t_2), (s_1, t_1); \omega]$ is a Gaussian random variable with mean zero. The covariance can be verified by a simple computation.

THEOREM 1. *For* $\lambda \geqq 0$,

$$P[\omega: \sup_{(s,t) \in [0,S]\times[0,T]} X(s, t; \omega) \geqq \lambda] \leqq 4P[X(S, T; \omega) \geqq \lambda]. \tag{2.1}$$

PROOF. Consider $X(s, t; \omega)$ at the points $\{(iS2^{-m}, jT2^{-n}): i = 0, 1, \cdots, 2^m; j = 0, \cdots, 2^n\}$. Let $Z_i(\omega) = \max_{0 \leqq j \leqq 2^n} X(iS2^{-m}, jT2^{-n}; \omega)$ and let

$$I(\omega) = \inf\{i: Z_i(\omega) \geqq \lambda\} \qquad (I(\omega) = +\infty \text{ if } \{i: Z_i(\omega) \geqq \lambda\} = \emptyset),$$

$$J(\omega) = \inf\{j: X(IS2^{-m}, jT2^{-n}; \omega) \geqq \lambda\} \qquad (J(\omega) = +\infty \text{ if } I(\omega) = +\infty).$$

Then

$$\begin{aligned} P[\max_{i,j} X(iS2^{-m}, jT2^{-n}) \geqq \lambda] &= P[X(IS2^{-m}, JT2^{-n}) \geqq \lambda] \\ &= P[X(IS2^{-m}, JT2^{-n}) \geqq \lambda,\ X(S, JT2^{-n}) \geqq \lambda] \\ &\quad + P[X(IS2^{-m}, JT2^{-n}) \geqq \lambda,\ X(S, JT2^{-n}) < \lambda]. \end{aligned} \tag{2.2}$$

Now, using Lemma 2.1, the symmetry of the increments of $X(s, t; \omega)$ and the fact that $X(s, 0; \omega) = 0$ a.s. for $s \in [0, \infty)$, we have

$$\begin{aligned} P[X(IS2^{-m}, &JT2^{-n}) \geqq \lambda,\ X(S, JT2^{-n}) < \lambda] \\ &= \textstyle\sum_{i,j} P[I(\omega) = i,\ J(\omega) = j,\ X(S, JT2^{-n}) < \lambda] \\ &\leqq \textstyle\sum_{i,j} P[I(\omega) = i,\ J(\omega) = j,\ X(S, jT2^{-n}) - X(iS2^{-m}, jT2^{-n}) < 0] \\ &= \textstyle\sum_{i,j} P[I(\omega) = i,\ J(\omega) = j]P[\Delta X[(S, jT2^{-n}), (iS2^{-m}, 0)] < 0] \\ &= \textstyle\sum_{i,j} P[I(\omega) = i,\ J(\omega) = j]P[\Delta X[(S, jT2^{-n}), (iS2^{-m}, 0)] > 0] \\ &\leqq \textstyle\sum_{i,j} P[I(\omega) = i,\ J(\omega) = j,\ X(S, jT2^{-n}) \geqq \lambda] \\ &\leqq P[X(S, JT2^{-n}) \geqq \lambda]. \end{aligned} \tag{2.3}$$

Combining this result with (2.2) gives

$$P[\max_{i,j} X(iS2^{-m}, jT2^{-n}) \geqq \lambda] \leqq 2P[X(S, JT2^{-n}) \geqq \lambda] \leqq 2P[\sup_j X(S, jT2^{-n}) \geqq \lambda] \leqq 4P[X(S, T) \geqq \lambda] \tag{2.4}$$

by the corresponding theorem for one-dimensional Brownian motion. Using the continuity of the sample paths of the process $X(s, t; \omega)$ and letting $n \to \infty$ gives (2.1).

Let $f(x, y)$ be a function defined on $[0, \infty) \times [0, \infty)$. By $\limsup_{s,t\to\infty} f(s, t)$ we shall mean $\lim_{s,t\to\infty} \sup_{(u,v)>(s,t)} f(u, v)$. Theorem 2 is a multiparameter version of the law of the iterated logarithm. If the lim sup is taken as $s, t \to \infty$, the constant in the multiparameter version is equal to 4, however if the lim sup is taken as $s \to \infty$ while t remains in some bounded interval $0 < a \leqq t \leqq b < \infty$, the result has constant equal to 2 as in the one parameter version of the theorem. This is shown in Theorem 3.

THEOREM 2.
$$P\left[\omega : \limsup_{s,t\to\infty} \frac{X(s, t; \omega)}{[4st \log_2 st]^{\frac{1}{2}}} = 1\right] = 1 .$$

PROOF. The proof is an analog of the standard proof of the one-dimensional theorem and will be omitted.

THEOREM 3. *Let* $0 < a \leqq b < \infty$, *then*

(i)
$$P\left[\omega : \limsup_{s\to\infty} \sup_{a\leqq t\leqq b} \frac{X(s, t; \omega)}{[2st \log_2 st]^{\frac{1}{2}}} = 1\right] = 1 ,$$

(ii)
$$P\left[\omega : \limsup_{s\to\infty} \frac{X(s, t; \omega)}{[2st \log_2 st]^{\frac{1}{2}}} \geqq 1 \ \textit{for all} \ t \in [a, b]\right] = 1 .$$

PROOF. Let $0 < \varepsilon < 1$ and suppose that

$$P\left[\omega : \limsup_{s\to\infty} \sup_{a\leqq t\leqq b} \frac{X(s, t; \omega)}{[2st \log_2 st]^{\frac{1}{2}}} > 1 + \varepsilon\right] > 0 . \tag{2.5}$$

Divide the interval $[a, b]$ into m equal parts each of length $\delta = (b - a)/m$. If (2.5) is true then for each m there exists some subinterval $[a_m, b_m]$ contained in $[a, b]$ such that

$$P\left[\omega : \limsup_{s\to\infty} \sup_{a_m\leqq t\leqq b_m} \frac{X(s, t; \omega)}{[2st \log_2 st]^{\frac{1}{2}}} > 1 + \varepsilon\right] > 0 . \tag{2.6}$$

Let m be chosen so that $\delta = (b - a)/m < (a\varepsilon)/2$ and let $1 < q < 1/(1 - \varepsilon/2)$. Let $G(x, y) = [2xy \log_2 xy]^{\frac{1}{2}}$ and $A_{k\delta} = [\omega : \sup_{0<s\leqq q^k} \sup_{a_m\leqq t\leqq b_m} X(s, t; \omega) > (1 + \varepsilon)G(q^{k-1}, a_m)]$. Then

$$\begin{aligned} P[A_{k\delta}] &\leqq P[\omega : \sup_{0<s\leqq q^k;\, 0<t\leqq b_m} X(s, t; \omega) > (1 + \varepsilon)G(q^{k-1}, a_m)] \\ &\leqq 4P[\omega : X(q^k, a_m + \delta) > (1 + \varepsilon)G(q^{k-1}, a_m)] \\ &\leqq \frac{2q^{\frac{1}{2}}[(a_m + \delta)/a_m]^{\frac{1}{2}}}{(1 + \varepsilon)[\pi \log_2 (q^{k-1}a_m)]^{\frac{1}{2}}} [(k - 1) \log q + \log a_m]^{-\gamma} \end{aligned}$$

where

$$\gamma = \frac{(1+\varepsilon)^2 a_m}{q(a_m+\delta)} > \frac{(1+\varepsilon)^2(1-\varepsilon/2)}{(1+\varepsilon/2)} > 1 .$$

Hence $\sum_{k=1}^{\infty} P[A_{k\delta}] < \infty$ and by the Borel-Cantelli lemma $P[A_{k\delta} \text{ i.o.}] = 0$ which contradicts (2.6). Thus

$$P\left[\omega : \limsup_{s\to\infty} \sup_{a\le t\le b} \frac{X(s, t; \omega)}{[2st \log_2 st]^{\frac{1}{2}}} < 1+\varepsilon\right] = 1 . \tag{2.7}$$

For any fixed t, $a \leqq t \leqq b$, using the one-dimensional theorem

$$P\left[\omega : \limsup_{s\to\infty} \frac{X(s, t; \omega)}{[2st \log_2 st]^{\frac{1}{2}}} > 1-\varepsilon\right] = 1 .$$

Hence

$$P\left[\omega : \limsup_{s\to\infty} \sup_{a\le t\le b} \frac{X(s, t; \omega)}{[2st \log_2 st]^{\frac{1}{2}}} > 1-\varepsilon\right] = 1 . \tag{2.8}$$

Combining (2.7) and (2.8) we have (i).

We now prove (ii). Without loss of generality we may assume that $b \leqq 1$, since $X(bs, b^{-1}t; \omega)$ has the same distribution as $X(s, t; \omega)$ and

$$\left[\omega : \limsup_{s\to\infty} \frac{X(s, t; \omega)}{[2st \log_2 st]^{\frac{1}{2}}} > 1-\varepsilon \quad \text{for all} \quad t \in [a, b]\right]$$
$$= \left[\omega : \limsup_{u\to\infty} \frac{X(u, v; \omega)}{[2uv \log_2 uv]^{\frac{1}{2}}} > 1-\varepsilon \quad \text{for all} \quad v \in [ab^{-1}, 1]\right] .$$

Suppose

$$P\left[\omega : \limsup_{s\to\infty} \frac{X(s, t; \omega)}{[2st \log_2 st]^{\frac{1}{2}}} \leqq 1-\varepsilon \quad \text{for some} \quad t \in [a, b]\right] > 0 . \tag{2.9}$$

As above, for each m divide the interval $[a, b]$ into m subintervals each of length $\delta = (b-a)/m$. Then (2.9) implies that for each m there exists some subinterval $[a_m, b_m]$ of length δ such that

$$P\left[\omega : \limsup_{s\to\infty} \frac{X(s, t; \omega)}{[2st \log_2 st]^{\frac{1}{2}}} \leqq 1-\varepsilon \quad \text{for some} \quad t \in [a_m, b_m]\right] > 0 . \tag{2.10}$$

Let m be chosen so that, $\delta < (a\varepsilon^4)/4$ and let $q > 1$ be such that $2/[(q-1)^{\frac{1}{2}}] < \varepsilon/4$, $[q/(q-1)]^{\frac{1}{2}} < 1 + \varepsilon/2$. From the proof of the law of the iterated logarithm for one-dimensional Brownian motion (Loève [10] page 560), we have

$$X(q^n, b_m) - X(q^{n-1}, b_m) > \{(1-\varepsilon/4) b_m^{\frac{1}{2}}[(q-1)/q]^{\frac{1}{2}}[2q^n \log_2 q^n]^{\frac{1}{2}}\} \text{ i.o.} \tag{2.11}$$

The first half of the theorem implies that for $a \leqq t \leqq b$, $n \geqq n_1(q, \omega)$,

$$|X(q^{n-1}, t)| \leqq 2(t/q)^{\frac{1}{2}}[2q^n \log_2 q^n]^{\frac{1}{2}} . \tag{2.12}$$

For $t \in [a_m, b_m]$, let $A_n(t) = [\omega : \Delta X[(q^n, b_m), (q^{n-1}, t)] \geqq [2q^n \log_2 q^n]^{\frac{1}{2}}\gamma(t)]$, where $\gamma(t) = \{(1-\varepsilon/4) b_m^{\frac{1}{2}}[(q-1)/q]^{\frac{1}{2}} - 2(t/q)^{\frac{1}{2}} - (1-\varepsilon)t^{\frac{1}{2}}\}$ and let $A_n =$

$[\omega : \sup_{a_m \leqq t \leqq b_m} \Delta X[(q^n, b_m), (q^{n-1}, t)] \geqq [2q^n \log_2 q^n]^{\frac{1}{2}} \gamma(b_m)]$. Then $A_n(t)$ is a subset of A_n, $a_m \leqq t \leqq b_m$, $n = 1, 2, \cdots$, and $P[A_n] = P[\omega : \sup_{a_m \leqq t \leqq b_m} \Delta X[(q^n, b_m), (q^{n-1}, t)] \geqq [2q^n \log_2 q^n]^{\frac{1}{2}} \gamma(b_m)] \leqq P[\omega : X(1, 1; \omega) \geqq \{[2q^n \log_2 q^n]^{\frac{1}{2}} \gamma(b_m)\}/[(q^n - q^{n-1})\delta]^{\frac{1}{2}}] \leqq \text{Constant}\, [n \log q]^{-\alpha}$ where $\alpha = [q/[(q-1)\delta]]\gamma^2(b_m) = (b_m/\delta)\{(1-\varepsilon/4) - 2/(q-1)^{\frac{1}{2}} - (1-\varepsilon)[q/(q-1)]^{\frac{1}{2}}\}^2 > (a/\delta)(\varepsilon^4/4) > 1$. Hence $\sum_{n=1}^{\infty} P[A_n] < \infty$ and by the Borel-Cantelli lemma $P[A_n \text{ i.o.}] = 0$. Thus if $n \geqq n_2(\omega)$

$$\Delta X[(q^n, b_m), (q^{n-1}, t)] < [2q^n \log_2 q^n]^{\frac{1}{2}} \gamma(t) \quad \text{for all} \quad t \in [a_m, b_m]\,. \tag{2.13}$$

If $n \geqq \max \{n_1(\omega), n_2(\omega)\}$, then (2.12) and (2.13) hold. If (2.11) also holds it follows that

$$X(q^n, t) > (1-\varepsilon)t^{\frac{1}{2}}[2q^n \log_2 q^n]^{\frac{1}{2}} > (1-\varepsilon)[2q^n t \log_2 (q^n t)]^{\frac{1}{2}} \tag{2.14}$$

for all $t \in [a_m, b_m]$. Hence (2.14) is true infinitely often contradicting (2.10). Thus (ii) follows.

COROLLARY. *There exists a set Ω_0 of probability zero such that if $\omega \notin \Omega_0$, $X(s, t; \omega)$ satisfies the one-parameter law of the iterated logarithm (as a function of s) simultaneously on the lines $t = T$, $0 < T < \infty$.*

PROOF. (i) and (ii) imply that for any finite interval $[a, b]$,

$$P\left[\omega : \limsup_{s\to\infty} \frac{X(s, t; \omega)}{[2st \log_2 st]^{\frac{1}{2}}} = 1 \quad \text{for all} \quad t \in [a, b]\right] = 1\,. \tag{2.15}$$

Let $\{a_n\}$, $\{b_n\}$ be sequences such that $a_n \downarrow 0$, $b_n \uparrow \infty$. Let

$$A_n = \left[\omega : \limsup_{s\to\infty} \frac{X(s, t; \omega)}{[2st \log_2 st]^{\frac{1}{2}}} = 1 \quad \text{for all} \quad t \in [a_n, b_n]\right].$$

Then $A_n \downarrow$, and by (2.15) $P[A_n] = 1$ for each n, hence $P[\lim_{n\to\infty} A_n] = P[\omega : \limsup_{s\to\infty} X(s, t; \omega)/[2st \log_2 st]^{\frac{1}{2}} = 1$ for all $t \in (0, \infty)] = \lim_{n\to\infty} P[A_n] = 1$.

THEOREM 4. *Let $S, T \geqq 1$, $(s_1, t_1), (s_2, t_2) \in [0, S] \times [0, T]$, then*

$$P\left[\omega : \limsup_{|t_2 - t_1| = \eta \downarrow 0;\, |s_2 - s_1| = \varepsilon \downarrow 0} \frac{|\Delta X[(s_2, t_2), (s_1, t_1); \omega]|}{[2\varepsilon\eta \log (1/\varepsilon\eta)]^{\frac{1}{2}}} = 1\right] = 1\,. \tag{2.16}$$

PROOF. The proof uses generalizations of the ideas and methods of the one-dimensional case as given by Itô and McKean [8].

3. Stochastic integrals and diffusion equations. Let $\{Z(s, t; \omega) : (s, t) \in [0, \infty) \times [0, \infty)\}$ be a stochastic process. $Z(s, t; \omega)$ is said to be a martingale with respect to $\mathscr{F}_{st} = \sigma\{Z(u, v; \omega) : (u, v) < (s, t)\}$ if $Z(s, t; \omega)$ is integrable for all $(s, t) \in [0, \infty) \times [0, \infty)$ and whenever $(s, t), (s', t') \in [0, \infty) \times [0, \infty)$ are such that $(s', t') < (s, t)$, $E[Z(s, t; \omega) \mid \mathscr{F}_{s't'}] = Z(s', t'; \omega)$ a.s.

THEOREM 5. *Let (s_0, t_0) be a fixed point in $[0, S] \times [0, T]$. The process $\{\Delta X[(s, t), (s_0, t_0); \omega] : s_0 \leqq s \leqq S,\ t_0 \leqq t \leqq T\}$ is a martingale.*

PROOF. Clearly $\Delta X[(s, t), (s_0, t_0); \omega]$ is integrable for all (s, t). Let $\mathscr{G}_{st} = \sigma\{X(u, v; \omega) : s_0 \leqq u \leqq s,\ t_0 \leqq v \leqq t\}$. If $(s_0, t_0) < (s', t') < (s, t)$, write $\Delta X[(s, t),$

$(s_0, t_0)] = \Delta X[(s, t), (s', t')] + \Delta X[(s', t'), (s_0, t_0)] + \Delta X[(s', t), (s_0, t')] + \Delta X[(s, t'), (s', t_0)]$. By Lemma 2.1, $\Delta X[(s, t), (s', t')]$, $\Delta X[(s, t'), (s', t_0)]$ and $\Delta X[(s', t), (s_0, t')]$ are independent of $\mathscr{G}_{s't'}$. $\Delta X[(s', t'), (s_0, t_0)]$ is measurable with respect to $\mathscr{G}_{s't'}$. Hence $E[\Delta X[(s, t), (s_0, t_0)] \mid \mathscr{G}_{s't'}] = \Delta X[(s', t'), (s_0, t_0)]$.

We want to define an Itô type integral $I(s, t, \omega; f)$ for the class $\mathscr{M}$ of square integrable functions $f(s, t; \omega)$ defined on $[0, S] \times [0, T] \times \Omega$ having the property that $f(s, t; \omega)$ is measurable with respect to $\mathscr{F}_{st} = \sigma\{X(u, v; \omega) : (u, v) < (s, t)\}$ for all $(s, t) \in [0, S] \times [0, T]$. The following lemmas give the estimates needed to construct such an integral and to solve a differential equation involving $X(s, t; \omega)$.

LEMMA 3.1. *Let $X(\omega)$ and $Y(\omega)$ be random variables on a probability space $(\Omega, \mathscr{F}, P)$. Let $\mathscr{F}_0$ be a sub-σ-field of $\mathscr{F}$. If* (i) *X is independent of Y,* (ii) *X is independent of $\mathscr{F}_0$,* (iii) $E[X] = 0$, (iv) $E[XY] < \infty$, *then* $E[XY \mid \mathscr{F}_0] = 0$ a.s.

LEMMA 3.2. *Let $X_{ij}(\omega)$ and $Y_{ij}(\omega)$ $i = 1, \cdots, m; j = 1, \cdots, n$ be random variables such that*

(i) $E[X_{ij}(\omega)] = 0$ *for all i and j,*

(ii) $E[X_{ij}(\omega) Y_{ij}(\omega)] < \infty$ *for all i and j*

(iii) *$Y_{uv}(\omega)$ $u \leqq i$, $v \leqq j$ and $X_{uv}(\omega)$ $u \leqq i-1$, $v \leqq j-1$ are independent of $X_{uv}(\omega)$ $u \geqq i$ or $v \geqq j$. Let $S_{ij}(\omega) = \sum_{u=1}^{i} \sum_{v=1}^{j} X_{uv}(\omega) Y_{uv}(\omega)$, then*

$$P[\omega : \max_{i,j} |S_{ij}| \geqq b] \leqq (4/b^2) E[S_{mn}]^2 \tag{3.1}$$

$$E[\max_{i,j} |S_{ij}|^2] \leqq 16 E[S_{mn}]^2 . \tag{3.2}$$

PROOF. We first show that $\{S_{mj} : 1 \leqq j \leqq n\}$ is a martingale with respect to $\mathscr{F}_{mj}$. If $j' \leqq j$, using Lemma 3.1 and the fact that $S_{mj'}$ is $\mathscr{F}_{mj'}$ measurable, we have

$$\begin{aligned} E[S_{mj} \mid \mathscr{F}_{mj'}] &= \sum_{u=1}^{m} \sum_{v=1}^{j'} E[X_{uv} Y_{uv} \mid \mathscr{F}_{mj'}] + \sum_{u=1}^{m} \sum_{v=j'+1}^{n} E[X_{uv} Y_{uv} \mid \mathscr{F}_{mj'}] \\ &= \sum_{u=1}^{m} \sum_{v=1}^{j'} X_{uv} Y_{uv} = S_{mj'} . \end{aligned}$$

Similarly $E[S_{mj} \mid \mathscr{F}_{in}] = S_{ij}$ and thus using Jensen's inequality $E[|S_{mj}| \mid \mathscr{F}_{in}] \geqq |S_{ij}|$.

Let $Z_i(\omega) = \max_{1 \leqq j \leqq n} |S_{ij}(\omega)|$ and let $I(\omega) = \min\{i : Z_i(\omega) \geqq b\}$ ($I(\omega) = +\infty$ if $\{i : Z_i(\omega) \geqq b\} = \emptyset$), $J(\omega) = \min\{j : |S_{Ij}| \geqq b\}$ ($J(\omega) = +\infty$ if $I(\omega) = +\infty$). Now $\Lambda = [\omega : \max_{i,j} |S_{ij}| \geqq b] = \bigcup_{i=1}^{m} \bigcup_{j=1}^{n} [\omega : I(\omega) = i, J(\omega) = j] = \bigcup_{i=1}^{m} \bigcup_{j=1}^{n} B_{ij}$ and the sets B_{ij} are disjoint.

For any (i, j), $(0, 0) < (i, j) < (m, n)$ $\int_{B_{ij}} S_{mj}^2 \, dP = E[S_{ij}^2 I_{B_{ij}}] + E[(S_{mj} - S_{ij})^2 I_{B_{ij}}] \geqq \int_{B_{ij}} S_{ij}^2 \, dP$ since $E[S_{ij}(S_{mj} - S_{ij}) I_{B_{ij}}] = 0$. Thus

$$\begin{aligned} E[S_{mJ}^2] \geqq \int_\Lambda S_{mJ}^2 \, dP = \sum_{i=1}^{m} \sum_{j=1}^{n} \int_{B_{ij}} S_{mj}^2 \, dP &\geqq \sum_{i=1}^{m} \sum_{j=1}^{n} \int_{B_{ij}} S_{ij}^2 \, dP \\ &\geqq b^2 P[\Lambda] \end{aligned} \tag{3.3}$$

and since $\{S_{mj} : 1 \leqq j \leqq n\}$ is a martingale with respect to $\mathscr{F}_{mj}$, we have

$$E[S_{mJ}^2] \leqq E[\max_{1 \leqq j \leqq n} S_{mj}^2] \leqq 4 E[S_{mn}^2] \tag{3.4}$$

(Doob [6] Chapter VII, Theorem 3.4) .

(3.1) follows from (3.3) and (3.4).

To prove (3.2), we note that

$$\begin{aligned} P[\omega : \max_{i,j} |S_{ij}| \geqq b] &= \textstyle\sum_{i=1}^{m} \sum_{j=1}^{n} P[B_{ij}] \leqq (1/b) \sum_{i=1}^{m} \sum_{j=1}^{n} \int_{B_{ij}} |S_{ij}|\, dP \\ &\leqq (1/b) \textstyle\sum_{i=1}^{m} \sum_{j=1}^{n} \int_{B_{ij}} E[|S_{mj}| \,|\, \mathscr{F}_{in}]\, dP = (1/b) \sum_{i=1}^{m} \sum_{j=1}^{n} \int_{B_{ij}} |S_{mj}|\, dP \\ &= (1/b) \textstyle\int_{\Lambda} |S_{mJ}|\, dP \end{aligned}$$

since $B_{ij} \in \mathscr{F}_{in}$. Using Theorem 3.4′ (Chapter VII [6]) and (3.4), we have $E[\max_{i,j} |S_{ij}|^2] = E[(\max_{i,j} |S_{ij}|)^2] \leqq 4E[|S_{mJ}|^2] \leqq 16E[S_{mn}^2]$.

Let $f(s, t; \omega)$ be an $\mathscr{F}_{st}$ measurable function for all s and t. Let $0 = s_0 < s_1 < \cdots < s_m = S$, $0 = t_0 < t_1 < \cdots < t_n = T$ partition $[0, S] \times [0, T]$. Consider the product $f(s_{i-1}, t_{j-1}; \omega) f(s_{u-1}, t_{v-1}; \omega) \Delta X_{ij} \Delta X_{uv}$. Without loss of generality we may assume that $s_{u-1} \leqq s_{i-1}$. Then $f(s_{u-1}, t_{v-1}; \omega)$ and $f(s_{i-1}, t_{j-1}; \omega)$ are measurable with respect to the σ-field $\mathscr{F}_{s_{i-1}\tau}$ where $\tau = \max(t_{v-1}, t_{j-1})$ and either (i) ΔX_{uv} is measurable with respect to $\mathscr{F}_{s_{i-1}\tau}$ and ΔX_{ij} is independent of $\mathscr{F}_{s_{i-1}\tau}$ or (ii) ΔX_{uv} and ΔX_{ij} are independent of $\mathscr{F}_{s_{i-1}\tau}$ and also independent of each other.

LEMMA 3.3. *Let $f(s, t; \omega)$, $g(s, t, \omega) \in L^2[[0, S] \times [0, T] \times \Omega]$ be functions which are $\mathscr{F}_{st}$ measurable for all s and t. Then*

(i) $E[|f(s_{i-1}, t_{j-1}; \omega) g(s_{u-1}, t_{v-1}; \omega) \Delta X_{ij} \Delta X_{uv}|] < \infty$,

(ii) $E[f(s_{i-1}, t_{j-1}; \omega) g(s_{u-1}, t_{v-1}; \omega) \Delta X_{ij} \Delta X_{uv}] = 0$ *$i \neq u$ or $t \neq v$*

(iii) $E[f(s_{i-1}, t_{j-1}; \omega) g(s_{i-1}, t_{j-1}; \omega) (\Delta X_{ij})^2] = E[f(s_{i-1}, t_{j-1}; \omega) g(s_{i-1}, t_{j-1}; \omega)] \times (s_i - s_{i-1})(t_j - t_{j-1})$.

PROOF. These results follow using Lemma 2.1 and the remark preceding the lemma.

Let $\mathscr{G}$ be the class of measurable functions $f(s, t; \omega)$ defined on $[0, S] \times [0, T] \times \Omega$ having the property that $f(s, t; \omega)$ is measurable with respect to $\mathscr{F}_{st} = \sigma\{X(u, v; \omega) : (u, v) < (s, t)\}$ for all $(s, t) \in [0, S] \times [0, T]$. Let $H(s_0, \cdots, s_m; t_0, \cdots, t_n)$ $0 = s_0 < \cdots < s_m = S$, $0 = t_0 < \cdots < t_n = T$, be the class of functions $f(s, t; \omega) \in \mathscr{M} = \mathscr{G} \cap L^2[[0, S] \times [0, T] \times \Omega]$ which have $f(s, t; \omega) = f(s_{i-1}, t_{j-1}; \omega)$ for $s_{i-1} \leqq s < s_i$, $t_{j-1} \leqq t < t_j$, $H = \bigcup H(s_0, \cdots, s_m; t_0, \cdots, t_n)$. The integral is first defined for functions in H as follows: if $s_k \leqq s \leqq s_{k+1}$, $t_p \leqq t \leqq t_{p+1}$,

$$\begin{aligned} I(s, t, \omega; f) = {} & \textstyle\sum_{i=1}^{k} \sum_{j=1}^{p} f(s_{i-1}, t_{j-1}; \omega) \Delta X[(s_i, t_j), (s_{i-1}, t_{j-1})] \\ & + \textstyle\sum_{i=1}^{k} f(s_{i-1}, t_p; \omega) \Delta X[(s_i, t), (s_{i-1}, t_p)] \\ & + \textstyle\sum_{j=1}^{p} f(s_k, t_{j-1}; \omega) \Delta X[(s_i, t_j), (s_k, t_{j-1})] \\ & + f(s_k, t_p; \omega) \Delta X[(s, t), (s_k, t_p)] \,. \end{aligned}$$

This definition can easily be shown to be independent of the choice of partition used in defining the integral.

The integral so defined has the following properties. For $f, g \in H$ ($L^2 = L^2[[0, S] \times [0, T] \times \Omega]$)

(i) $E[I(s, t, \omega; f)] = 0$ for all $(s, t) \in [0, S] \times [0, T]$,

(ii) $I(s, t, \omega; \alpha f + \beta g) = \alpha I(s, t, \omega; f) + \beta I(s, t, \omega; g)$,

(iii) $I(s, t, \omega; 1) = \Delta X[(s, t), (0, 0); \omega] = X(s, t; \omega)$,

(iv) $I(s, t, \omega; f)$ is a continuous function of (s, t) with probability one,

(v) $E[I(s, t, \omega; f)I(s, t, \omega; g)] = (f, g)_{L^2}$

(vi) $P[\omega : \sup_{0 \leqq s \leqq S;\, 0 \leqq t \leqq T} |I(s, t, \omega; f)| \geqq b] \leqq (4/b^2)||f(s, t; \omega)||^2_{L^2}$,

(vii) if $f(s, t; \omega) = h(s, t; \omega)$ for all $\omega \in \Omega_1$ where Ω_1 is any P-measurable subset of Ω, then $I(s, t, \omega; f) = I(s, t, \omega; h)$ $0 \leqq s \leqq S$, $0 \leqq t \leqq T$ a.e. in Ω_1.

These properties follow using standard techniques and the results of Lemmas 3.2 and 3.3.

As in the one-dimensional case, the integral so defined is a linear isometric operator from H to $L^2(\Omega)$ and can be extended to a linear operator from the closure of H to $L^2(\Omega)$. It can be shown that the closure of the class of $(s, t; \omega)$ step functions in $\mathscr{G}$ includes all functions in $\mathscr{M} = \mathscr{G} \cap L^2[[0, S] \times [0, T] \times \Omega]$. The extension satisfies (i)—(vii) for $f, g \in \mathscr{M}$ (Itô [7]).

LEMMA 3.4. *Let $f \in \mathscr{M}$ and let $b > 0$, then*

(i) $E[\sup_{(s,t) \in [0,S] \times [0,T]} [I(s, t, \omega; f)]^2] \leqq 16E[I(S, T, \omega; f)]^2$

(ii) $bP[\omega : \sup_{(s,t) \in [0,S] \times [0,T]} [I(s, t, \omega; f)]^2 \geqq b] \leqq 16E[I(S, T, \omega; f)]^2$.

PROOF. The lemma follows using Lemma 3.2.

THEOREM 6. *Let $Y(s, t; \omega) = I(s, t, \omega; f) = \int_0^t \int_0^s f(u, v; \omega)\, dX(u, v; \omega)$. Then $\{Y(s, t; \omega) : (s, t) \in [0, S] \times [0, T]\}$ is a martingale.*

PROOF. Let $\{f_n(s, t; \omega)\}$ be a sequence in H such that $||f_n - f||_{L^2} \to 0$ as $n \to \infty$. Let $Y_n(s, t; \omega) = I(s, t, \omega; f_n)$. $Y_n(s, t; \omega)$ $n = 1, 2, \cdots$, is measurable with respect to $\mathscr{F}_{st}$ for all $(s, t) \in [0, S] \times [0, T]$ and $||Y_n - Y||_\Omega \to 0$ as $n \to \infty$. Hence $Y(s, t; \omega)$ is measurable with respect to $\mathscr{F}_{st}$ for all $(s, t) \in [0, S] \times [0, T]$.

Let $(s', t') < (s, t)$. Without loss of generality we may assume that $f_n \in H(s_0, \cdots, s_m; t_0, \cdots, t_n)$ where $s' = s_\alpha$, $t' = t_\beta$, $s = s_r$, $t = t_p$ are partition points. We can write

$$\begin{aligned} E[Y_n(s, t; \omega) \,|\, \mathscr{F}_{s't'}] &= \textstyle\sum_{i=1}^{\alpha} \sum_{j=1}^{\beta} E[f_n(s_{i-1}, t_{j-1}; \omega)\Delta X_{ij} \,|\, \mathscr{F}_{s't'}] \\ &\quad + \textstyle\sum_{i=1}^{r} \sum_{j=\beta+1}^{p} E[f_n(s_{i-1}, t_{j-1}; \omega)\Delta X_{ij} \,|\, \mathscr{F}_{s't'}] \\ &\quad + \textstyle\sum_{i=\alpha+1}^{r} \sum_{j=1}^{\beta} E[f_n(s_{i-1}, t_{j-1}; \omega)\Delta X_{ij} \,|\, \mathscr{F}_{s't'}] \\ &= \mathrm{I} + \mathrm{II} + \mathrm{III}\,. \end{aligned}$$

In I both $f_n(s_{i-1}, t_{j-1}; \omega)$ and ΔX_{ij} are measurable with respect to $\mathscr{F}_{s't'}$, hence $\mathrm{I} = Y_n(s', t'; \omega)$ a.s. To show that II and III are zero, for each term in II and III we consider the σ-field $\mathscr{F}_{\gamma\tau}$ where $\gamma = \max[s', s_{i-1}]$, $\tau = \max[t', t_{j-1}]$. $\mathscr{F}_{s't'}$ is a sub-σ-field of $\mathscr{F}_{\gamma\tau}$, $f_n(s_{i-1}, t_{j-1}; \omega)$ is measurable with respect to $\mathscr{F}_{\gamma\tau}$ and ΔX_{ij} is independent of $\mathscr{F}_{\gamma\tau}$. Thus, for a typical term in II or III, $E[f_n(s_{i-1}, t_{j-1}; \omega)\Delta X_{ij} \,|\, \mathscr{F}_{s't'}] = E\{E[f_n(s_{i-1}, t_{j-1}; \omega)\Delta X_{ij} \,|\, \mathscr{F}_{\gamma\tau}] \,|\, \mathscr{F}_{s't'}\} = E[f_n(s_{i-1}, t_{j-1}; \omega)E[\Delta X_{ij}] \,|\, \mathscr{F}_{s't'}] = 0$ a.s. Hence $E[Y_n(s, t; \omega) \,|\, \mathscr{F}_{s't'}] = Y_n(s', t'; \omega)$ a.s. Letting $n \to \infty$ gives the desired result.

As an application of the stochastic integral we find a solution of the diffusion equation

$$\Delta Y[(s, t), (0, 0); \omega] = \int_0^t \int_0^s m[u, v; Y(u, v; \omega)]\, du\, dv + \int_0^t \int_0^s \sigma[u, v; Y(u, v; \omega)]\, dX(u, v; \omega) . \tag{3.5}$$

The following hypotheses are made.

(i) $m(\cdot, \cdot; \cdot)$ and $\sigma(\cdot, \cdot; \cdot)$ are Baire functions of $(s, t; \eta)$ for $(s, t) \in [0, S] \times [0, T]$, $-\infty < \eta < \infty$.

(ii) There is a constant K such that $|m(s, t; \eta)| \leqq K(1 + \eta^2)^{\frac{1}{2}}$, $0 \leqq \sigma(s, t; \eta) \leqq K(1 + \eta^2)^{\frac{1}{2}}$.

(iii) $m(\cdot, \cdot; \cdot)$ and $\sigma(\cdot, \cdot; \cdot)$ satisfy a uniform Lipschitz condition in η i.e. $|m(s, t; \eta_2) - m(s, t, \eta_1)| \leqq K|\eta_2 - \eta_1|$, $|\sigma(s, t; \eta_2) - \sigma(s, t; \eta_1)| \leqq K|\eta_2 - \eta_1|$ where K is independent of s, t and η.

Assuming hypotheses (i), (ii) and (iii) it is possible using Lemma 3.4 and an iterative process as in the one-dimensional case (Doob [6] page 277–282) to construct a process $Y(s, t; \omega)$ having the following properties.

(A) The $Y(s, t; \omega)$ sample functions are almost all continuous in $[0, S] \times [0, T]$.

(B) $\int_0^T \int_0^S E[Y(s, t; \omega)]^2\, ds\, dt < \infty$.

(C) For each $(s_0, t_0) \in [0, S] \times [0, T]$, $Y(s_0, t_0; \omega)$ is independent of $\Delta X[(s_i, t_j), (s_{i-1}, t_{j-1}); \omega]$ for $s_{i-1} \geqq s_0$ or $t_{j-1} \geqq t_0$.

(D) $Y(s, t; \omega)$ is measurable with respect to $\mathscr{B}_{st} = \sigma\{X(u, v; \omega) : (u, v) < (s, t)\} \vee \sigma\{Y(0, 0; \omega), Y(u, 0; \omega), Y(0, v; \omega) : (u, v) < (s, t)\}$.

(E) For each $(s, t) \in [0, S] \times [0, T]$, $Y(s, t; \omega)$ satisfies (3.5). The process $Y(s, t; \omega)$ is essentially uniquely determined.

4. Sample variation. Let $\{X(s, t: \omega) : (s, t) \in [0, S] \times [0, T]\}$ be a Gaussian stochastic process of real-valued random variables with mean function $m(s, t)$ and covariance function $R(s_1, t_1; s_2, t_2) = E[X(s_1, t_1; \omega)X(s_2, t_2; \omega)] - m(s_1, t_1)m(s_2, t_2)$ satisfying the following conditions:

(i) $m(s, t)$ has bounded second order partial derivatives on $[0, S] \times [0, T]$.

(ii) $R(s_1, t_1; s_2, t_2)$ is continuous for $0 \leqq s_1, s_2 \leqq S$, $0 \leqq t_1, t_2 \leqq T$ and has uniformly bounded third order partial derivatives for $s_1 \neq s_2$, $t_1 \neq t_2$.

Define

$$D_-{}^+(s, t) = \lim_{u \to s+;\, v \to t-} \frac{R(s, t; s, t) - R(s, t; u, t) - R(s, t; s, v) + R(s, t; u, v)}{(s - u)(t - v)} .$$

Similarly define $D_+{}^-(s, t)$, $D_+{}^+(s, t)$ and $D_-{}^-(s, t)$. Let $f(s, t) = D_-{}^-(s, t) - D_-{}^+(s, t) - D_+{}^-(s, t) + D_+{}^+(s, t)$.

Note. Unless otherwise indicated the sums in this section will be on $j, k = 1, \cdots, 2^n$ and the indices will be omitted.

THEOREM 7. *If $X(s, t; \omega)$ satisfies the above conditions then with probability one,*

$$\lim_{n\to\infty} \sum [\Delta X[(jS2^{-n}, kT2^{-n}), ((j-1)S2^{-n}, (k-1)T2^{-n}); \omega]]^2 = \int_0^T \int_0^S f(s, t)\, ds\, dt . \tag{4.1}$$

PROOF. The proof uses Taylor's series expansions and follows the methods used by Baxter [1].

COROLLARY 1. *If $\{X(s, t; \omega) : (s, t) \in [0, S] \times [0, T]\}$ satisfies the assumptions of Theorem 7 and if*

$$\begin{aligned} R(s_1, t_1; s_2, t_2) &= u(s_1, t_1)v(s_2, t_2) && s_1 \leqq s_2, \quad t_1 \leqq t_2 \\ &= u(s_1, t_2)v(s_2, t_1) && s_1 \leqq s_2, \quad t_1 \geqq t_2 \\ &= u(s_2, t_1)v(s_1, t_2) && s_1 \geqq s_2, \quad t_1 \leqq t_2 \\ &= u(s_2, t_2)v(s_1, t_1) && s_1 \geqq s_2, \quad t_1 \geqq t_2 \end{aligned}$$

then with probability one,

$$\begin{aligned} \lim_{n\to\infty} &\sum [\Delta X[(jS2^{-n}, kT2^{-n}), ((j-1)S2^{-n}, (k-1)T2^{-n}); \omega]]^2 \\ &= \int_0^T \int_0^S \left[u(s, t)\frac{\partial^2 v(s, t)}{\partial s\, \partial t} - \frac{\partial u(s, t)}{\partial s}\frac{\partial v(s, t)}{\partial t} \right. \\ &\qquad \left. - \frac{\partial u(s, t)}{\partial t}\frac{\partial v(s, t)}{\partial s} + v(s, t)\frac{\partial^2 u(s, t)}{\partial s\, \partial t} \right] ds\, dt . \end{aligned}$$

COROLLARY 2. *Let $\{X(s, t; \omega) : (s, t) \in [a, b] \times [c, d]\}$ be a Gaussian process satisfying the conditions of Theorem 7. Then with probability one,*

$$\begin{aligned} \lim_{n\to\infty} &\sum [\Delta X[(a + j(c-a)2^{-n}, b + k(d-b)2^{-n}), \\ &\quad \times (a + (j-1)(c-a)2^{-n}, b + (k-1)(d-b)2^{-n}); \omega]]^2 \\ &= \int_b^d \int_a^c f(s, t)\, ds\, dt . \end{aligned} \tag{4.2}$$

COROLLARY 3. *Let $\{X(s, t; \omega) : (s, t) \in [0, S] \times [0, T]\}$ be a Gaussian process with mean function zero and covariance function $R(s_1, t_1; s_2, t_2) = \min(s_1, s_2) \min(t_1, t_2)$. Then with probability one,*

$$\lim_{n\to\infty} \sum [\Delta X[(jS2^{-n}, kT2^{-n}), ((j-1)S2^{-n}, (k-1)T2^{-n}); \omega]]^2 = ST .$$

THEOREM 8. *Let $\{Y(s, t; \omega) : (s, t) \in [0, S] \times [0, T]\}$ be a solution of the diffusion equation* (3.5), *then with probability one,*

$$\lim_{n\to\infty} \sum [\Delta Y[(jS2^{-n}, kT2^{-n}), ((j-1)S2^{-n}, (k-1)T2^{-n}); \omega]]^2 = \int_0^T \int_0^S \sigma^2[u, v; Y(u, v; \omega)]\, du\, dv . \tag{4.3}$$

PROOF. Let $K = kT2^{-n}$, $K - 1 = (k-1)T2^{-n}$, $J = jS2^{-n}$, $J - 1 = (j-1)S2^{-n}$, then

$$\begin{aligned} \sum [\Delta Y[(J, K), (J-1, K-1)]]^2 &= \sum \left[\int_{K-1}^K \int_{J-1}^J m[u, v; Y(u, v)]\, du\, dv\right]^2 \\ &\quad + 2 \sum \left[\int_{K-1}^K \int_{J-1}^J m[u, v; Y(u, v)]\, du\, dv\right]\left[\int_{K-1}^K \int_{J-1}^J \sigma[u, v; Y(u, v)]\, dX(u, v)\right] \\ &\quad + \sum \left[\int_{K-1}^K \int_{J-1}^J \sigma[u, v; Y(u, v)]\, dX(u, v)\right]^2 = A_n + 2B_n + C_n . \end{aligned}$$

It can be shown that $A_n \to 0$, $|B_n| \to 0$ a.s. as $n \to \infty$. Hence the existence of the limit on the left-hand side is an event whose probability is the same for all functions $m(s, t; \eta)$ satisfying (i), (ii) and (iii) and the value of the limit when it exists is independent of $m(s, t; \eta)$. We shall use $m(s, t; \eta) \equiv 0$ in our calculations.

The sum of squares of the increments of the process can be decomposed in the following manner.

$$\sum [\int_{K-1}^{K} \int_{J-1}^{J} \sigma[u, v; Y(u, v)]\, dX(u, v)]^2 = \tfrac{1}{4} \sum [\int_{K-1}^{K} \int_{J-1}^{J} F(\sigma, Y, u, v, K, J)\, dX(u, v)]^2$$

where $F(\sigma, Y, u, v, K, J) = \{\sigma[u, v; Y(u, v)] - \sigma[u, K-1; Y(u, K-1)]\} + \{\sigma[u, v; Y(u, v)] - \sigma[J-1, v; Y(J-1, v)]\} + \{\sigma[u, K-1; Y(u, K-1)] - \sigma[J-1, K-1; Y(J-1, K-1)]\} + \{\sigma[J-1, v; Y(J-1, v)] - \sigma[J-1, K-1; Y(J-1, K-1)]\} + 2\sigma[J-1, K-1; Y(J-1, K-1)]$. When expanded this sum of squares gives fifteen terms. It can be shown, using methods similar to those employed by Berman [2], that the absolute values of all terms in the sum converge to zero almost surely as $n \to \infty$ with the exception of the term

$$\begin{aligned} S_n &= \sum [\int_{K-1}^{K} \int_{J-1}^{J} \sigma[J-1, K-1; Y(J-1, K-1)]\, dX(u, v)]^2 \\ &= \sum \sigma^2[(j-1)S2^{-n}, (k-1)T2^{-n}; Y((j-1)S2^{-n}, (k-1)T2^{-n})] \\ &\qquad \times [\Delta X[(jS2^{-n}, kT2^{-n}), ((j-1)S2^{-n}, (k-1)T2^{-n})]]^2 . \end{aligned}$$

Let m be an arbitrary fixed positive integer. Then for $n > m$, $\Gamma = \{(s, t): (\alpha-1)S2^{-m} \leqq s \leqq \alpha S2^{-m}, (\beta-1)T2^{-m} \leqq t \leqq \beta T2^{-m}\}$, $\sum_{\alpha,\beta=1}^{2^m} \min_\Gamma \sigma^2[s, t; Y(s, t)] \sum_{j,k=1}^{2^{n-m}} [\Delta X[(\{(\alpha-1)2^{-m} + j2^{-n}\}S, \{(\beta-1)2^{-m} + k2^{-n}\}T), (\{(\alpha-1)2^{-m} + (j-1)2^{-n}\}S, \{(\beta-1)2^{-m} + (k-1)2^{-n}\}T)]]^2 \leqq S_n \leqq \sum_{\alpha,\beta=1}^{2^m} \max_\Gamma \sigma^2[s, t; Y(s, t)] \sum_{j,k=1}^{2^{n-m}} [\Delta X[(\{(\alpha-1)2^{-m} + j2^{-n}\}S, \{(\beta-1)2^{-m} + k2^{-n}\}T), (\{(\alpha-1)2^{-m} + (j-1)2^{-n}\}S, \{(\beta-1)2^{-m} + (k-1)2^{-n}\}T)]]^2$. Applying Corollary 2 of Theorem 7, we see that the lower bound converges a.s. as $n \to \infty$ to

$$\sum_{\alpha,\beta=1}^{2^m} \min_\Gamma \sigma^2[s, t; Y(s, t)][\alpha 2^{-m} - (\alpha-1)2^{-m}]S[\beta 2^{-m} - (\beta-1)2^{-m}]T = \sum_{\alpha,\beta=1}^{2^m} \min_\Gamma \sigma^2[s, t; Y(s, t)](S2^{-m})(T2^{-m})$$

and the upper bound converges a.s. as $n \to \infty$ to $\sum_{\alpha,\beta=1}^{2^m} \max_\Gamma \sigma^2[s, t; Y(s, t)] \times (S2^{-m})(T2^{-m})$. As $m \to \infty$ these bounds converge a.s. to the common limit $\int_0^T \int_0^S \sigma^2[u, v; Y(u, v)]\, du\, dv$ since $\sigma^2[s, t; Y(s, t; \omega)]$ is a continuous function of (s, t) with probability one.

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