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ON BAHADUR EFFICIENCY OF THE JOINT-RANKING PROCEDURE[1]

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1. Introduction. Consider the problem of testing the equality of several, say K, treatments on the basis of paired-observations, viz. (X_{il}, X_{jl}), $l = 1, 2, \cdots, N_{ij}$ $(1 \leqq i < j \leqq K)$ obtained by N_{ij} independent paired-comparisons for each pair (i, j) of treatments. If we assume that the N_{ij} differences $Z_l^{(i,j)} = X_{il} - X_{jl}$, $l = 1, 2, \cdots, N_{ij}$ have a common continuous cdf $G_{ij}(z)$ $(1 \leqq i < j \leqq K)$, the hypothesis of no-difference among the treatments can be formally expressed as

$$H_0: G_{ij}(z) + G_{ij}(-z) = 1 \quad \text{and} \quad G_{ij}(z) = G_{i'j'}(z) \quad \text{for any two pairs } (i, j) \text{ and } (i', j') .$$

In [7] Mehra and in [8] Mehra and Puri had proposed and investigated a family of rank-order tests for the above problem based on a generalization of the Wilcoxon-one-sample ranking procedure: Let $R_{N,l}^{(i,j)}$ denote the rank of $|Z_l^{(i,j)}|$ when the $N = \sum_{i=1}^{K-1} \sum_{j>i} N_{ij}$ absolute values of the observed differences $Z_l^{(i,j)}$, $l = 1, 2, \cdots, N_{ij}$, $(1 \leqq i < j \leqq K)$ are arranged in ascending order of magnitude in a *combined ranking*. For a given set of rank-scores $\xi_{N,\alpha}$, $\alpha = 1, 2, \cdots, N$, define a step function $\xi_N(u)$ over $(0, 1)$, with $\xi_N(u) = \xi_{N,\alpha} = \xi_N(\alpha/(N+1))$ for $(a-1)/N < u \leqq \alpha/N$, $\alpha = 1, 2, \cdots, N$ and set

$$V_N^{(i,j)} = \sum_{l=1}^{N_{ij}} \xi_N(R_{N,l}^{(i,j)}/(N+1)) \operatorname{sign} Z_l^{(i,j)} . \tag{1.1}$$

Assume further the existence of a function $\xi(u)$, $0 < u < 1$, such that

$$\int_0^1 \xi^2(u)\, du < \infty \tag{1.2}$$

and

$$\lim_{N\to\infty} \int_0^1 \{\xi_N(u) - \xi(u)\}^2\, du = 0 . \tag{1.3}$$

For testing the hypothesis H_0, rank-order statistics of the form

$$L_N = L_N(\xi_N, \xi) = \sum_{i=1}^K \{\sum_{j\neq i} (V_N^{(i,j)}/(N_{ij})^{\frac{1}{2}})\}^2 \Big/ \left(\frac{1}{N} \sum_{\alpha=1}^N \xi_{N,\alpha}^2\right) K \tag{1.4}$$

(with the test consisting in rejecting H_0 when L_N is too large) were considered in [8]. It was shown that if the hypothesis H_0 were true and the conditions (1.2) and (1.3) were satisfied, L_N is distributed in the limit, as $N \to \infty$, as a χ^2-variable with $(K-1)$ df provided $\lim_{N\to\infty} (N_{ij}/N) = \eta_{ij} > 0$ for all (i, j), and that against shift alternatives its asymptotic Pitman-efficiency relative to the normal theory

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$\mathscr{F}$-statistic does not depend on K. Since for $K = 2$ a statistic belonging to the family L_N reduces to the corresponding statistic for testing symmetry in the underlying distribution of the observed differences, e.g., Wilcoxon, Fisher-Yates-Fraser etc. one-sample statistics, the family L_N provides an appropriate generalization of these statistics from 2 to K treatments.

From the Pitman-efficiency standpoint, however, the "joint-ranking" procedure proposed above—despite its greater intuitive appeal—seems redundant. For given any member of the family L_N, one can construct a statistic in exactly the same manner but with $V_N^{(i,j)}$ now based on "separate-rankings" of the absolute Z's for each pair (i, j) $(1 \leqq i < j \leqq K)$ which in the Pitman sense, is as efficient as the given statistic. This latter family of statistics is constructed as follows: Let $R^{*(i,j)}_{N_{ij},l} = R_l{}^{*(i,j)}$ be the rank of $|Z_l{}^{(i,j)}|$ when the N_{ij} absolute values $|Z_l{}^{(i,j)}|$, $l = 1, 2, \cdots, N_{ij}$, are ranked separately for each pair (i, j) $(1 \leqq i < j \leqq K)$ and set

$$L_N{}^* = L_N{}^*(\xi_N, \xi) = \sum_{i=1}^{K} \{\sum_{j \neq i} (V_N{}^{*(i,j)}/(Kd^2_{N_{ij}})^{\frac{1}{2}})\}^2 , \tag{1.5}$$

where

$$d^2_{N_{ij}} = \sum_{\alpha=1}^{N_{ij}} \xi^2_{N_{ij},\alpha}$$

and

$$V^{*(ij)}_{N_{ij}} = \sum_{l=1}^{N_{ij}} \xi_{N_{ij}}(R_l{}^{*(i,j)}/(N_{ij} + 1)) \cdot \operatorname{sign} Z_l{}^{(i,j)} .$$

The question as to which of the two procedures—the "joint-ranking" or the "separate-rankings"—is preferable was partially investigated in [8] by comparing the local powers of L_N and $L_N{}^*$ as the number of comparisons N_{ij} for each pair (i, j) is kept fixed and K, the number of treatments, approaches infinity. The heuristic results obtained therein (see pages 538—539 of [8]), despite the restricted sense in which they hold, nevertheless indicate the superiority of the "joint-ranking" procedure over the "separate-rankings". These results, however, are based on the comparison of local powers, as $K \to \infty$. The object of this paper is to demonstrate that, for the nearby alternatives, the "joint-ranking" procedure is relatively superior from the standpoint of Bahadur-efficiency as well.

As remarked above, it was shown in [8] that for testing against a shift in location, the asymptotic Pitman-efficiency E^P_{L,L^*} of L relative to L^* satisfies $E^P_{L,L^*} = 1$. Since Pitman-efficiency pertains to the asymptotic comparison of the first order terms of the local powers of the statistics under consideration, the question arises naturally as to how L and L^* compare with regard to the next higher order terms of the local power (cf., Witting [10]). This is precisely our approach. It is shown that for testing against small shifts in location, the "joint-ranking" procedure is superior to the "separate-ranking" procedure from the standpoint of Bahadur's criterion, provided the underlying distribution is unimodal and certain regularity conditions are satisfied.

2. Standard sequences: $\{L_N{}^{\frac{1}{2}}\}$ and $\{L_N{}^{*\frac{1}{2}}\}$. It will be shown that $\{L_N{}^{\frac{1}{2}}\}$ and $\{L_N{}^{*\frac{1}{2}}\}$ are "standard sequences" of statistics for testing H_0 in the sense of [1] and, consequently, an explicit expression for their relative asymptotic Bahadur-efficiency will be obtained.

Let S be an abstract sample space, with $\mathscr{B}$ a σ-algebra of sets in S; and let $\mathscr{P} = \{P_\theta: \theta \in \Theta\}$ be a family of probability measures defined on $(S, \mathscr{B})$, where Θ is an abstract parameter space. Let H_0 denote the hypothesis that $\theta \in \Theta_0$, where Θ_0 is a proper subset of Θ. Then a sequence $\{T_N\}$ of real-valued statistics is said to be a *standard sequence* (for testing H_0) if (I) there exists a continuous cdf F such that, for each $\theta \in \Theta_0$, $\lim_{N\to\infty} P_\theta[T_N \leqq x] = F(x)$ for every x, (II) $\log(1 - F(x)) = -(\frac{1}{2}ax^2)[1 + o(1)]$, as $x \to \infty$, where a is a positive constant, and (III) there exists a function b on $\Theta - \Theta_0$, with $0 < b < \infty$ such that for each $\Theta - \Theta_0$, $T_N/N^{\frac{1}{2}}$ converges in P_θ-probability to $b(\theta)$. Define $c(\theta) = a[b(\theta)]^2$ for $\theta \in \Theta_0$ and $c(\theta) = 0$ otherwise. $c(\theta)$ is called the asymptotic slope of $\{T_N\}$ when $\theta \in \Theta$ is the true parameter point. Consider now two standard sequences $\{T_N^{(1)}\}$, $\{T_N^{(2)}\}$ for testing H_0 and define $c^{(i)}(\theta)$ $(i = 1, 2)$ as above. It was shown by Bahadur [1] that $E_{1,2} = c^{(1)}(\theta)/c^{(2)}(\theta)$ can be regarded as a measure of asymptotic efficiency of $T_N^{(1)}$ relative to $T_N^{(2)}$ for $\theta \in \Theta - \Theta_0$ as the true parameter point.

Let Θ denote the parameter set consisting of all sets $\{G_{ij}: 1 \leqq i < j \leqq K\}$ of continuous distribution functions G_{ij} and $\Theta_0 \subset \Theta$ consist of those sets $\{G_{ij}: 1 \leqq i < j \leqq K\}$ for which H_0 is satisfied. Now it was shown in [8] (Theorem 2.1) that if the hypothesis H_0 and the conditions (1.2), (1.3) are true and $\lim_{N\to\infty}(N_{ij}/N) = \eta_{ij} > 0$ for each pair (i, j), both statistics L_N and L_N^* are distributed in the limit as χ^2_{K-1}-variables. Thus conditions I and II above are satisfied with $a = 1$ (see Section 5 of Bahadur [1]). Theorems 2.1 and 2.2 below show that if some additional conditions are imposed on the functions $\xi_N(u)$ and $\xi(u)$, $L_N^{\frac{1}{2}}$ and $L_N^{*\frac{1}{2}}$ also satisfy condition III: Let $\xi(u)$ and $\xi_N(u)$ satisfy the Chernoff-Savage [2] conditions

$$\text{(2.1)} \quad \text{(i)} \ \lim_{N\to\infty} \xi_N(u) = \xi(u) \quad \text{for} \quad 0 < u < 1\,,$$
$$\text{(ii)} \ |\xi^{(i)}(u)| = |d^{(i)}/du^{(i)}| \leqq K^*[u(1-u)]^{-\frac{1}{2}+\delta}$$
$$i = 0, 1 \quad \text{for some constants} \quad K^* > 0, \delta > 0\,.$$

(Cf. assumptions (1) and (3) of Puri and Sen [9].) For proving the preceding statement concerning condition III, we need Lemmas 2.1 and 2.2 for which we set out some notation: Let C stand for $\binom{K}{2}$, the number of all possible pairs (i, j), $1 \leqq i < j \leqq K$, and label the pair (i, j) by $\alpha = (i-1)K + j - \binom{i+1}{2}$. Then $G_\alpha(x)(H_\alpha(x) = G_\alpha(x) - G_\alpha(-x))$ denotes the cdf of $Z_l^{(\alpha)}(|Z_l^{(\alpha)}|)$, $\alpha = 1, 2, \cdots, C$, and $V_N^{(\alpha)} = V_N^{(i,j)}$, etc. Let $G_{N_\alpha}^{(\alpha)}(x)(H_{N_\alpha}^{(\alpha)}(x) = G_{N_\alpha}^{(\alpha)}(x) - G_{N_\alpha}^{(\alpha)}(-x-))$ denote the sample cdf of the N_α observations $Z_l^{(\alpha)}$, $l = 1, 2, \cdots, N_\alpha(|Z_l^{(\alpha)}|, l = 1, 2, \cdots, N_\alpha)$, $1 \leqq \alpha \leqq C$, and $H_N(x) = \sum_{\alpha=1}^C \lambda_{\alpha,N} H_{N_\alpha}^{(\alpha)}(x)$, $H^*(x) = \sum_{\alpha=1}^C \lambda_{\alpha,N} H_\alpha(x)$, where $\lambda_{\alpha,N} = (N_\alpha/N)$, $\alpha = 1, 2, \cdots, C$. If we now define $T_{N,\alpha} = \sum_{k=1}^N \xi_N(k/(N+1))t_{N,k}^{(\alpha)}/N_\alpha$, where $t_{N,k}^{(\alpha)} = 1(-1)$ if the kth smallest of N ordered absolute observations $\{|Z_l^{(\beta)}|, 1 \leqq l \leqq N_\beta, 1 \leqq \beta \leqq C\}$ is the absolute value of a positive (negative)

$Z_l^{(\alpha)}$ observation, then $T_{N,\alpha} = V_N^{(\alpha)}/N_\alpha$ and can be expressed (as in [9]) as

$$T_{N,\alpha} = \int_0^\infty \xi_N\left(\frac{NH_N(x)}{N+1}\right) d[G_{N\alpha}^{(\alpha)}(x) + G_{N\alpha}^{(\alpha)}(-x-)]\,, \quad 1 \leqq \alpha \leqq C\,. \tag{2.2}$$

Assume further the existence of a λ_0, $0 < \lambda_0 \leqq \frac{1}{2}$, such that

$$0 < \lambda_0 \leqq \lambda_{\alpha,N} \leqq 1 - \lambda_0 < 1 \quad \text{for all} \quad \alpha = 1, 2, \cdots, C\,. \tag{2.3}$$

LEMMA 2.1. *If the function $\xi(u)$ is monotone or continuous bounded and the conditions* (1.2), (1.3), (2.1) *and* (2.3) *hold, then for fixed $G_\alpha(x)$, $\alpha = 1, 2, \cdots, C$, the statistics $T_{N,\alpha}$, $\alpha = 1, 2, \cdots, C$, satisfy $p - \lim_{N\to\infty}(T_{N,\alpha} - A_{N,\alpha}) = 0$, where $A_{N,\alpha} = \int_0^\infty \xi(H^*(x))d[G_\alpha(x) + G_\alpha(-x)]$.*

PROOF. First note that (as for (2.2))

$$\begin{aligned}
&\left|\int_0^\infty \left[\xi_N\left(\frac{N}{N+1}H_N(x)\right) - \xi\left(\frac{N}{N+1}H_N(x)\right)\right] d[G_{N\alpha}^{(\alpha)}(x) + G_{N\alpha}^{(\alpha)}(-x-)]\right| \\
&\quad = \left|\frac{1}{N_\alpha}\sum_{k=1}^N \left[\xi_N\left(\frac{k}{N+1}\right) - \xi\left(\frac{k}{N+1}\right)\right] t_{N,k}^{(\alpha)}\right| \\
&\quad \leqq \frac{1}{\lambda_0}\frac{1}{N}\sum_{k=1}^N \left|\xi_N\left(\frac{k}{N+1}\right) - \xi\left(\frac{k}{N+1}\right)\right| \qquad (2.4)\\
&\quad = \frac{1}{\lambda_0}\int_0^1 |\xi_N(u) - \xi_N^*(u)|\,du \\
&\quad \leqq \frac{1}{\lambda_0}\left[\int_0^1 |\xi_N(u) - \xi(u)|\,du + \int_0^1 |\xi_N^*(u) - \xi(u)|\,du\right],
\end{aligned}$$

where $\xi_N^*(u) = \xi(k/(N+1))$ for $(k - 1/N) < u \leqq (k/N)$, $k = 1, 2, \cdots, N$. As $N \to \infty$, the last two integrals on the right tend to zero on account of, respectively, (1.3) and Lemma 2.2 of Hájek [4]. Thus the first integral in (2.4) is $o_p(1)$. Decomposing $T_{N,\alpha}$ as in (2.8) of [9], it follows in view of (2.4), assumptions (2.1) and the arguments of Theorem 2.1 of [9] that $p - \lim_{N\to\infty}(T_{N,\alpha} - A_{N,\alpha}) = 0$, $\alpha = 1, 2, \cdots, C$. This is because for the present lemma, we simply need that the B– and the C–terms in the decomposition of $T_{N,\alpha}$ are $o_p(1)$. For the C–terms to be so, (2.4) suffices in place of assumption 2 of [9], and the B–terms obviously satisfy this property. The proof is complete.

LEMMA 2.2. *Under the assumptions* (2.1) *and* (2.3) *$A_\alpha(\lambda_N) = A_{N,\alpha}$ is continuous in $\lambda_N = (\lambda_{1N}, \cdots, \lambda_{CN})'$.*

PROOF. It is easily seen that $A_\alpha(\lambda_N) = E[\xi(\sum_{\beta=1}^C \lambda_\beta H_\beta(|Z_1^{(\alpha)}|))\,\text{sign}\,Z_1^{(\alpha)}]$, with $\lambda_\beta = \lambda_{\beta,N}$ and the absolute value of the integrand not exceeding (see (2.1) (ii))

$$\begin{aligned}
&K^*[\textstyle\sum_{\beta=1}^C \lambda_\beta H_\beta(|Z_1^{(\alpha)}|)]^{-\frac{1}{2}+\delta}[1 - \sum_{\beta=1}^C \lambda_\beta H_\beta(|Z_1^{(\alpha)}|)]^{-\frac{1}{2}+\delta} \\
&\quad \leqq K^*\lambda_0^{-\frac{1}{2}+\delta}[H_\alpha(|Z_1^{(\alpha)}|)]^{-\frac{1}{2}+\delta}[(1 - \textstyle\sum_{\beta\neq\alpha}\lambda_\beta) - \lambda_\alpha H_\alpha(|Z_1^{(\alpha)}|]^{-\frac{1}{2}+\delta} \\
&\quad \leqq K^*\lambda_0^{-1+2\delta}\{H_\alpha(|Z_1^{(\alpha)}|)[1 - H_\alpha(|Z_1^{(\alpha)}|)]\}^{-\frac{1}{2}+\delta}\,.
\end{aligned}$$

Since the last expression is independent of λ_N and has finite expectation (note that $H_\alpha(|Z_1^{(\alpha)}|)$ has uniform distribution over (0, 1)) and $\xi(u)$ is continuous in u, the dominated convergence theorem gives the result. The proof is complete.

THEOREM 2.1. *Suppose* $\xi(u)$ *is monotone or continuous bounded and that* (1.2), (1.3), (2.1) *and* (2.3) *are satisfied; then* $p - \lim_{N\to\infty} (N^{-\frac{1}{2}}L_N^{\frac{1}{2}}) = b^{\frac{1}{2}}$, *where*

$$b = \sum_{i=1}^K \{\sum_{j\neq i} (\eta_{ij})^{\frac{1}{2}} b_{ij}\}^2 / K(\int_0^1 \xi^2(u)\, du) \qquad \text{and} \tag{2.5}$$

$$b_{ij} = \int_{x=0}^\infty \xi(\sum_{i'<j'} \eta_{i'j'} H_{i'j'}(x)) d(G_{ij}(x) + G_{ij}(-x))$$

with $H_{ij} = G_{ij}(x) - G_{ij}(-x)$, *provided* $\eta_{ij} = \lim_{N\to\infty} (N_{ij}/N) > 0$.

PROOF. From Lemmas 2.1 and 2.2, it follows by replacing α by (i, j), etc. that $p - \lim_{N\to\infty} [V_N^{(i,j)}/N^{\frac{1}{2}}N_{ij}^{\frac{1}{2}}] = \eta_{ij}^{\frac{1}{2}} b_{ij}$. By using Theorem 5 in Mann and Wald [6] and the result that $\lim_{N\to\infty} (\sum_{k=1}^N \xi_{N,k}^2/N) = \int_0^1 \xi^2(u)\, du$, a consequence of (1.3), we obtain the desired result. The proof is complete.

THEOREM 2.2. *Under the conditions of Theorem* 2.1 $p - \lim_{N\to\infty} (N^{-\frac{1}{2}}L_N^{*\frac{1}{2}}) = b^{*\frac{1}{2}}$, *where*

$$b^* = \sum_{i=1}^K \{\sum_{j\neq i} (\eta_{ij})^{\frac{1}{2}} b_{ij}^*\}^2 / K(\int_0^1 \xi^2(u)\, du) \qquad \text{and} \tag{2.6}$$

$$b_{ij}^* = \int_{x=0}^\infty \xi(H_{ij}(x)) d[G_{ij}(x) + G_{ij}(-x)] .$$

PROOF. Using Lemma 2.1 with $C = 1$ for each specified pair (i, j), we obtain $p - \lim_{N\to\infty} (V_N^{*(i,j)}/N^{\frac{1}{2}}N_{ij}^{\frac{1}{2}}) = \eta_{ij}^{\frac{1}{2}} b_{ij}^*$. The rest of the argument is the same as for Theorem 2.1. The proof is complete.

On account of the preceding discussion and Theorems 2.1 and 2.2, $\{L_N^{\frac{1}{2}}\}$ and $\{L_N^{*\frac{1}{2}}\}$ are "standard" sequences for testing H_0, with slopes b and b^*, respectively, which gives the Bahadur-efficiency E_{L,L^*} of L with respect to L^* as

$$E_{L,L^*} = [\sum_{i=1}^K \{\sum_{j\neq i} (\eta_{ij})^{\frac{1}{2}} b_{ij}\}^2 / \sum_{i=1}^K \{\sum_{j\neq i} (\eta_{ij})^{\frac{1}{2}} b_{ij}^*\}^2] . \tag{2.7}$$

We conclude that the "joint-ranking" procedure is preferable or not preferable to the "separate-ranking" procedure for a specified alternative, according to $E_{L,L^*} > 1$ or $\leqq 1$.

3. Local efficiency. Since the above test-statistics are intended for alternatives where the underlying distributions of the observed differences differ only in location parameters, we shall consider only such alternatives in this section (for details see [7]): Let

$$K\colon G_{ij}(x) = G(x - \mu_{ij}) \tag{3.1}$$

where $G(x)$ is a continuous cdf satisfying $G(x) + G(-x) = 1$. It will follow from the considerations below that as each μ_{ij}, $1 \leqq i < j \leqq K$, converges to zero, the efficiency E_{L,L^*} defined by (2.7) converges to 1, provided certain conditions are satisfied. This was precisely the Pitman-efficiency obtained in the earlier paper [8]. To resolve the question of preference between the *joint-ranking* and the *separate-ranking* statistics, we shall compare below the behavior of the 'asymptotic-slopes' of the two statistics L and L^*, as $\max_{i<j} |\mu_{ij}|$ converges to zero, by comparing the next higher order terms.

We first note that for alternatives (3.1) $H_{ij}(x) = G(x + \mu_{ij}) - G(-x + \mu_{ij})$ and $H(x) = 2G(x) - 1$. Assume that

(3.2) $G(x)$ possesses a density $g(x)$ such that $|g'(x)|$, $|g''(x)|$ and $|g'''(x)|$ exist in each of the open intervals $(-\infty, a_1)$, (a_1, a_2), $\cdots$, (a_r, ∞) and are continuous bounded therein,

and that the sample sizes N_{ij} are all equal, so that $\eta_{ij} = (1/C)$. Using Taylor's expansion as each μ_{ij} converges to zero, one obtains

$$
\begin{aligned}
(3.3)\quad b_{i'j'} = &-\textstyle\int_0^\infty \xi[H(x) + \sum_{i<j} (\mu_{ij}^2/2C)(g'(x-\theta_1\mu_{ij}) + g'(x+\theta_1\mu_{ij}))] \\
&\times [2\mu_{i'j'}g'(x) + (\mu_{i'j'}^3/6)(g'''(x-\theta_2\mu_{i'j'}) + g'''(x+\theta_2\mu_{i'j'}))]\, dx\,.
\end{aligned}
$$

Further since $g'(x)$ is bounded, the second term in the argument of ξ can be made uniformly (in x) small, as $\max_{i<j}|\mu_{ij}|$ tends to zero, so that again applying Taylor's formula to ξ in (3.3), we obtain

$$
\begin{aligned}
b_{i'j'} = &-\Big\{2\mu_{i'j'} \textstyle\int_0^\infty \xi(H(x))g'(x)\,dx + \frac{\mu_{i'j'}^3}{6} \int_0^\infty \xi(H(x))[g'''(x-\theta_2\mu_{i'j'}) \\
&+ g'''(x+\theta_2\mu_{i'j'})]\,dx \\
&+ \frac{\mu_{i'j'}}{C} \textstyle\sum_{i<j} \mu_{ij}^2 \int_0^\infty (g'(x-\theta_1\mu_{ij}) + g'(x+\theta_1\mu_{ij}))g'(x) \\
(3.3\text{A})\qquad &\times \xi'\Big[H(x) + \frac{\theta_3}{2C} \textstyle\sum_{i<j} \mu_{ij}^2(g'(x-\theta_1\mu_{ij}) + g'(x+\theta_1\mu_{ij}))\Big]\,dx \\
&+ \frac{\mu_{i'j'}^3}{12C} \textstyle\sum_{i<j} \mu_{ij}^2 \int_0^\infty [g'''(x-\theta_2\mu_{i'j'}) + g'''(x+\theta_2\mu_{i'j'})] \\
&\times [g'(x-\theta_1\mu_{ij}) + g'(x+\theta_1\mu_{ij})] \\
&\times \xi'\Big[H(x) + \frac{\theta_3}{2C} \textstyle\sum_{i<j} \mu_{ij}^2(g'(x-\theta_1\mu_{ij}) + g'(x+\theta_1\mu_{ij}))\Big]\,dx\Big\}\,,
\end{aligned}
$$

where $\theta_1, \theta_2, \theta_3$ are constants (depending on x) belonging to (0, 1). Let us suppose that the distribution $G(x)$ is such that the right-hand side expression yields

$$
\begin{aligned}
b_{i'j'} = &-\Big[2\mu_{i'j'} \textstyle\int_0^\infty \xi(H(x))g'(x)\,dx + (\mu_{i'j'}^3/3) \int_0^\infty \xi(H(x))g'''(x)\,dx \\
(3.3\text{B})\qquad &+ \frac{2\mu_{i'j'}}{C} (\textstyle\sum_{i<j} \mu_{ij}^2) \int_0^\infty \xi'(H(x))[g'(x)]^2\,dx \\
&+ \text{fourth and higher order terms in } \mu_{ij}\ (1 \leqq i < j \leqq K)\Big]\,.
\end{aligned}
$$

We give below sufficient conditions for (3.3A) and (3.3B) to be true: Let the density $g(x)$ be such that

(3.4) $g'(x) \leqq 0$ for $x > 0$ and each of $|g'(x)|$, $|g''(x)|$ and $|g'''(x)|$ converge to zero monotonically, as $x \to \infty$, after a certain x_0;

(3.4A) There exist functions $A(x)$, $B(x)$ with $|g'((x-t)| \leqq A(x)$, $|g'''(x-t)| \leqq B(x)$ for sufficiently small $|t|$, such that $\int_0^\infty |\xi(H(x))| B(x)\,dx$, $\int_0^\infty |\xi'(H(x))| \cdot A(x) \cdot |g'(x)|\,dx$, $\int_0^\infty |\xi'(H(x))| \cdot A(x)B(x)\,dx$ are all finite and nonzero.

THEOREM 3.1. *Let $\xi(u)$ and $\xi'(u)$, $0 < u < 1$, be non-decreasing and the conditions* (2.1), (3.2), (3.4), (3.4A) *be satisfied. Then for the alternatives* (3.1), *statement* (3.3B) *holds for sufficiently small $\mu_{ij}(1 \leqq i < j \leqq K)$.*

PROOF. It is well known that under the condition (2.1) (ii), the first integral on the right of (3.3A) is finite. Since $g'(x) \leqq 0$ and $\xi(\mu)$, $\xi'(\mu)$ are non-decreasing the absolute values of the last three integrands in (3.3A) are bounded, respectively, by the three integrands in (3.4A). As $\max_{i<j} |\mu_{ij}| \to 0$, therefore, the dominated convergence theorem and arguments similar to those of Lemma 3 of [5] yield the result forthwith.

In view of the Chernoff-Savage conditions (2.1), the condition (3.4A) is not too restrictive. They are easily checked to be satisfied by Normal, Logistic, Double exponential distributions when the function $\xi(\mu)$ corresponds to normal-score or Wilcoxon statistics. Other conditions imposed are clearly satisfied by these score functions and a large class of distributions including the ones mentioned above. It should be possible to improve these conditions. However, we shall not pursue this question further.

Now consider the expression b_{ij}^*. By following similar arguments one obtains under the same conditions

$$b_{ij}^* = -\Big[2\mu_{ij} \int_0^\infty \xi(H(x))g'(x)\,dx + 2\mu_{ij}^3 \int_0^\infty \xi'(H(x))[g'(x)]^2\,dx + \frac{\mu_{ij}^3}{3} \int_0^\infty \xi(H(x))g'''(x)\,dx + \text{higher order terms}\Big]. \tag{3.4B}$$

Thus (3.3B) and (3.4B) yield, for the case when $\eta_{ij} = 1/C$ for all $1 \leqq i < j \leqq K$, the following expansions:

$$\begin{aligned}\textstyle\sum_{i=1}^K \{\sum_{j\neq i} b_{ij}(\eta_{ij})^{\frac12}\}^2 = \frac{2}{K(K-1)} \sum_{i=1}^K \Big[& 4(\textstyle\sum_{j\neq i} \mu_{ij})^2\{\int_0^\infty \xi(H(x))g'(x)\,dx\}^2 \\ &+ \tfrac{4}{3}(\textstyle\sum_{j\neq i} \mu_{ij})(\sum_{j\neq i} \mu_{ij}^3)(\int_0^\infty \xi(H(x))g'(x)\,dx) \cdot (\int_0^\infty \xi(H(x))g'''(x)\,dx) \\ &+ \frac{16}{K(K-1)}(\textstyle\sum_{j\neq i} \mu_{ij})^2(\sum_{i<j} \mu_{ij}^2)(\int_0^\infty \xi(H(x))g'(x)\,dx) \\ &\times (\textstyle\int_0^\infty \xi'(H(x))\{g'(x)\}^2\,dx) + (\text{terms of higher order than 4th})\Big].\end{aligned} \tag{3.5A}$$

and

$$\begin{aligned}\textstyle\sum_{i=1}^K \{\sum_{j\neq i} b_{ij}^*(\eta_{ij})^{\frac12}\}^2 = \frac{2}{K(K-1)} \sum_{i=1}^K [& 4(\textstyle\sum_{j\neq i} \mu_{ij})^2\{\int_0^\infty \xi(H(x))g'(x)\,dx\}^2 \\ &+ \tfrac{4}{3}(\textstyle\sum_{j\neq i} \mu_{ij})(\sum_{j\neq i} \mu_{ij}^3)(\int_0^\infty \xi(H(x))g'(x)dx)(\int_0^\infty \xi(H(x))g'''(x)\,dx) \\ &+ 8(\textstyle\sum_{j\neq i} \mu_{ij})(\sum_{j\neq i} \mu_{ij}^3)(\int_0^\infty \xi'(H(x))\{g'(x)\}^2\,dx) \cdot (\int_0^\infty \xi(H(x))g'(x)\,dx) \\ &+ (\text{terms of order higher than 4th})]\end{aligned} \tag{3.5B}$$

The expressions (3.5A) and (3.5B), coupled with Theorems 2.1 and 2.2, immediately yield the "asymptotic slopes" in the local region. It is clear from these expressions that the limiting value of the asymptotic (Bahadur) efficiency E_{L,L^*}, defined by (2.7), as $\max_{i<j} |\mu_{ij}| \to 0$ is unity. That is, the comparison of the first order terms alone gives us no clue as to which of the two procedures—"joint-ranking" or "separate-rankings"—to prefer. However, a comparison of the next order terms does throw some light on this question, as is shown below.

Consider now those distributions $G(x)$ for which $g'(x) \leqq 0$ for $x > 0$ and the statement (3.2) holds, and suppose that $\xi(u)$, $0 < u < 1$, is nonnegative and non-decreasing. Then it follows from the expressions (3.5A) and (3.5B) that the "joint-ranking" procedure would be locally more efficient than "separate-ranking" procedure provided

$$(3.6) \quad \sum_{i=1}^{K} (\sum_{j\neq i} \mu_{ij})(\sum_{j\neq i} \mu_{ij}^3) - \frac{2}{K(K-1)} \{\sum_{i=1}^{K} (\sum_{j\neq i} \mu_{ij})^2\}\{\sum_{i<j} \mu_{ij}^2\} > 0 .$$

This is true under certain conditions as is shown by the following

LEMMA 3.1. *Let μ_{ij}, $1 \leqq i, j \leqq K$ and $i \neq j$, be $K(K-1)$ numbers satisfying the conditions*

$$\text{(i)} \qquad \mu_{ij} = -\mu_{ji}$$

$$\text{(ii)} \qquad \mu_{ij} + \mu_{jk} = \mu_{ik} ;$$

then the inequality (3.6) *is true, unless all μ_{ij} are zero, in which case we have equality.*

PROOF. The conditions (i) and (ii) imply the existence of numbers θ_i, $i = 1, 2, \cdots, K$ such that $\mu_{ij} = \theta_i - \theta_j$ $(1 \leqq i < j \leqq K)$. Substituting these expressions, the left-hand side of (3.6) reduces to

$$K^2(\sum_i \theta_i'^4) + \left(3K - \frac{2K^2}{K-1}\right)(\sum_i \theta_i'^2)^2$$

where $\theta_i' = (\theta_i - \bar{\theta})$, with $\bar{\theta} = (\sum_i \theta_i)/K$, which is clearly positive unless all θ's are equal. The proof is complete.

The condition (ii) in the above lemma seems rather strong, and can presumably be weakened. It seems that the inequality (3.6) will remain true if the condition (ii) is replaced by

(ii′) Assuming (without loss of generality) a labelling of treatments such that $\mu_{ij} \geqq 0$ for all pairs (i, j) with $i < j$, let $\mu_{ij} \leqq \mu_{i'j'}$, for any two pairs (i, j) and (i', j') with $i' \leqq i < j \leqq j'$.

This possibility was suggested during discussions with Dr. Willett. The inequality remains true for the case $K = 3$. The proof (or disproof) of this statement in the general case is open.

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