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# A NOTE ON HUBER'S ROBUST ESTIMATION OF A LOCATION PARAMETER

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Huber, in his fundamental paper [1] and in [2], has considered the robust estimation of a location parameter and has obtained results which he applied to some examples including the $\varepsilon$-normal model, $\{F \mid \sup_x |F(x) - \Phi(x)| \leqq \varepsilon,\ F \text{ symmetric}\}$, when $\varepsilon$ is sufficiently small ($\varepsilon \leqq \varepsilon_0 \sim .03$). In this note we show how his methods work for the family of distributions $\{F \mid \int_{-A}^{A} dF \geqq p,\ F \text{ symmetric}\}$ and then use this to solve the $\varepsilon$-normal problem when $\varepsilon > \varepsilon_0$.

**0. Introduction and summary.** Let $\{X_i\}$ be a sequence of i.i.d. random variables with distribution function $F(x - \theta)$. Here $\theta$ is an unknown location parameter and $F$ is assumed to be in a convex class $\mathscr{F}$ of distribution functions which are symmetric and have absolutely continuous densities $f$ satisfying $E_F(f'/f)^2 = I(F) < \infty$. Huber proved (see Theorem 2 of [1]) that if $F_0 \in \mathscr{F}$ is sufficiently regular and $I(F_0) \leqq I(F)$ for all $F \in \mathscr{F}$, the maximum likelihood estimator, $\hat{\theta}$, of $\theta$ computed as if $F_0$ is the underlying distribution is robust in the sense that it "minimaxes" asymptotic variance (max over $\mathscr{F}$, min over a wide class of estimates). The maximum asymptotic variance of $\hat{\theta}$ is $1/I(F_0)$.

One of Huber's examples is the $\varepsilon$-contaminated normal model $\mathscr{F} = \{F \mid F = (1 - \varepsilon)\Phi + \varepsilon H\}$ where $\varepsilon$ is fixed, $\Phi$ is the standard normal distribution function and $H$ is arbitrary. The distribution $F_0$ having minimum information in $\mathscr{F}$ is given in Section 6 of [1]. Since $F_0$ is sufficiently regular, the theorem mentioned above applies to this example. For this model it has also been observed that there is a linear function of order statistics (LFO) which is robust in the same sense. In particular (cf. [2]) an appropriate $\alpha$-trimmed mean has asymptotic variance bounded on $\mathscr{F}$ by $1/I(F_0)$.

A second example is the $\varepsilon$-normal model $\mathscr{F}_\varepsilon = \{F \mid \sup_x |F(x) - \Phi(x)| \leqq \varepsilon\}$. For small $\varepsilon$ ($\varepsilon \leqq \varepsilon_0 \cong .03$) the $F_0$ which minimizes information is given in Section 9 of [1] and again Huber's Theorem produces a robust estimate. It is not known for this setting if there is a LFO which is robust.

In the first section of this paper, we show how Huber's methods work for the family of distributions $\mathscr{F} = \{F \mid \int_{-A}^{A} dF \geqq p\}$ and then use the results to solve the $\varepsilon$-normal problem for $\varepsilon > \varepsilon_0$. In Section 2, we show there is no robust LFO for a family of distributions of the above type and that this applies also to the $\varepsilon$-normal model when $\varepsilon$ is large enough. The tedium of calculations holds us

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to the observation that for $\varepsilon \geqq \varepsilon_1 \cong .07$, there is no robust LFO for the $\varepsilon$-normal model.

**1. Robust estimates.** It will be assumed throughout that the distributions under consideration are symmetric and have finite information. Thus each distribution $F$ has an absolutely continuous density $f$ satisfying $E_F(f'/f)^2 < \infty$.

Huber has shown (Theorem 2 of [1] and [2]) that if $f_0'/f_0$ is absolutely continuous then $I(F_0)$ is minimized over a convex family $\mathscr{F}$ at $F_0$ if, and only if,

$$\int u_0 f \leqq 0 \qquad \text{for } f \in \mathscr{F} \text{ where} \tag{1.1}$$

$$u_0 = I(F_0) + 4\frac{(f_0^{\frac{1}{2}})''}{f_0^{\frac{1}{2}}}\,. \tag{1.2}$$

Suppose now that $u_0$ is a given symmetric function and that $f_0$ is a density function satisfying (1.2). Then $F_0$ minimizes information over $\mathscr{M} = \{F \mid \int u_0 f \leqq 0\}$. Thus, by solving (1.2) for a fixed $u_0$, one might find robust estimates for models specified by an integral condition of the form $\int u_0 f \leqq 0$. We have done this for the simple integral condition $\int_{-A}^{A} f \geqq p$ (which comes from taking $u_0$ to be a negative constant on $|x| \leqq A$ and a positive constant on $|x| > A$). Instead of reproducing the (straight-forward) details involved in the choice of $u_0$ and the solution of (1.2), we proceed directly to the relevant densities.

For $0 < \alpha < \pi/2$, let

$$\begin{aligned} f_\alpha(x) &= \frac{\beta(\alpha)}{1+\beta(\alpha)}\cos^2 \alpha x && \text{if } |x| \leqq 1 \\ &= \frac{\beta(\alpha)}{1+\beta(\alpha)}\cos^2 \alpha e^{2\beta} e^{-2\beta|x|} && \text{if } |x| > 1 \end{aligned} \tag{1.3}$$

where $\beta(\alpha) = \alpha \tan \alpha$. In what follows we usually suppress the dependence of $\beta$ on $\alpha$. We note that $f_\alpha$ is a density with the following properties:

$$\begin{aligned} -f_\alpha'/f_\alpha &= (2\beta)\frac{\tan \alpha x}{\tan \alpha} && \text{if } |x| \leqq 1 \\ &= (2\beta)\operatorname{sgn} x && \text{if } |x| > 1\,, \end{aligned} \tag{1.4}$$

$$I(F_\alpha) = 4\alpha^2 \frac{\beta}{1+\beta}\,, \tag{1.5}$$

$$\begin{aligned} I(F_\alpha) + 4\frac{(f_\alpha^{\frac{1}{2}})''}{f_\alpha^{\frac{1}{2}}} = u_\alpha &= -\frac{I(F_\alpha)}{\beta} && \text{if } |x| \leqq 1 \\ &= (2\beta)^2 + I(F_\alpha) && \text{if } |x| > 1\,. \end{aligned} \tag{1.6}$$

Accordingly, $F_\alpha$ is the minimum information distribution in the class

$$\mathscr{M}_\alpha = \left\{F \,\middle|\, \int_{-1}^{1} f \geqq 1 - \frac{\cos^2 \alpha}{1+\beta}\right\}. \tag{1.7}$$

If we now set $f_{\alpha,A}(x) = A^{-1} f_\alpha(x/A)$ and take $\mathscr{M}_{\alpha,A} = \{F \mid \int_{-A}^{A} f \geqq 1 - \cos^2\alpha/(1+\beta)\}$, it follows that $F_{\alpha,A}$ is the minimum information distribution in $\mathscr{M}_{\alpha,A}$. Thus

PROPOSITION 1. *The maximum likelihood estimate of $\theta$ computed when the underlying distribution is $f_{\alpha,A}$ is robust in the sense described in the introduction with $\mathscr{F} = \mathscr{M}_{\alpha,A}$.*

Before proceeding we remark that $1 - \cos^2\alpha/(1+\beta)$ increases from 0 to 1 as $\alpha$ goes from 0 to $\pi/2$. Here is a short tabulation of this dependence together with the minimum information numbers in the particular case $A = 1$:

| $1 - \dfrac{\cos^2\alpha}{1+\beta}$ | $\alpha$ | $I(F_\alpha)$ |
|---|---|---|
| .4 | .5 | .21 |
| .5 | .59 | .39 |
| .6 | .67 | .62 |
| .7 | .76 | .99 |
| .8 | .88 | 1.6 |
| .9 | 1.02 | 2.59 . |

We turn now to the $\varepsilon$-normal example and show

PROPOSITION 2. *If $\Phi$ is the standard normal distribution and $\mathscr{F}_\varepsilon = \{F \mid \sup_x |F(x) - \Phi(x)| \leqq \varepsilon\}$ then there is a $G_\varepsilon$ in $\mathscr{F}_\varepsilon$ which minimizes information. The maximum likelihood estimate of $\theta$ when $G_\varepsilon$ is the underlying distribution is robust in the sense described in the introduction with $\mathscr{F} = \mathscr{F}_\varepsilon$. For $\varepsilon > \varepsilon_0$ ($\sim .03$) the density of $G_\varepsilon$ is $f_{\alpha_\varepsilon, A_\varepsilon}$ ($f_{\alpha,A}$ is defined following (1.7)) where $\alpha_\varepsilon$, $A_\varepsilon$ satisfy (1.8), (1.9), (1.10) below. For $\varepsilon < \varepsilon_0$, $G_\varepsilon$ is given by Huber in Section 9 of [1].*

PROOF. We wish to find $G_\varepsilon \in \mathscr{F}_\varepsilon$ which minimizes $I(F)$. Let $p(A) = \Phi(A) - \Phi(-A) - 2\varepsilon$. Then if $F \in \mathscr{F}_\varepsilon$, $F(A) - F(-A) \geqq p(A)$, i.e., $F \in \mathscr{M}_{\alpha,A}$ if $\alpha$ is now chosen so that $\Phi(A) - \Phi(-A) - 2\varepsilon = 1 - \cos^2\alpha/(1+\beta)$ or

$$\Phi(A) = \varepsilon + 1 - \tfrac{1}{2}\cos^2\alpha/(1+\beta)\,. \tag{1.8}$$

If we can find, for given $\varepsilon$, $\alpha_\varepsilon$, $A_\varepsilon$ *to satisfy* (1.8) *and such that* $G_\varepsilon = F_{\alpha_\varepsilon, A_\varepsilon} \in \mathscr{F}_\varepsilon$ we would be finished. It is enough to find $\alpha$, $A$ to satisfy (1.8) and, in addition, to satisfy $f_{\alpha,A} \leqq \varphi$ on $[0, A]$, $f_{\alpha,A} \geqq \varphi$ on $[A, \infty)$ ($\varphi = \Phi'$). This implies that $f_{\alpha,A}(A) = \varphi(A)$ or

$$A\varphi(A) = \cos^2\alpha \frac{\beta(\alpha)}{1+\beta(\alpha)}\,. \tag{1.9}$$

Now $f_{\alpha,A} \leqq \varphi$ on $[0, A]$ if $x \geqq 2\alpha/A \tan(\alpha x/A)$ on $[0, A]$ which, from convexity of tan, is equivalent to

$$A^2 \geqq 2\beta(\alpha)\,. \tag{1.10}$$

It is easy to verify that (1.10) implies $f_{\alpha,A} \geqq \varphi$ on $[A, \infty)$.

Our problem then is to find, a pair $\alpha$, $A$ which satisfies (1.8), (1.9), (1.10). To do so it is convenient to go backwards and for given $\alpha$ find $A(\alpha)$, $\varepsilon(\alpha)$ which satisfies (1.8), (1.9), (1.10) and then observe that $\varepsilon$ is a decreasing function of $\alpha$.

We will be able to carry out this argument for $\alpha \leqq \alpha_0$ ($\alpha_0$ is defined later in the proof) which will give the result for $\varepsilon \geqq \varepsilon(\alpha_0) = \varepsilon_0$.

The first step is to note that, if we let $g(\alpha)$ equal the right-hand side of (1.9),

$$(1.11) \qquad \sup_{0<\alpha<\pi/2} g(\alpha) < \sup_A A\phi(A) = \phi(1) = .24197 .$$

It follows from (1.11) and the fact that $A\phi(A)$ decreases to 0 on $[1, \infty)$ that

(1.12) for each $\alpha \in (0, \pi/2)$ there is an $A(\alpha) > 1$ which satisfies (1.9).

The solution $A$ on $[0, 1]$ is useless to us and we ignore it.

To establish (1.11) we note that

$$g'(\alpha) = -2 \sin \alpha \cos \alpha \frac{\beta}{1+\beta} + \frac{(\tan \alpha(1+\beta) + \alpha) \cos^2 \alpha}{(1+\beta)^2} \leqq 0$$

if, and only if, $(2\beta^2 + \beta - 1) \tan \alpha \geqq \alpha$. On $[\pi/4, \pi/2]$ $\tan \alpha \geqq 1$, $\beta(\alpha) \geqq \alpha$ and, consequently, $(2\beta^2 + \beta - 1) \tan \alpha \geqq 2\alpha^2 + \alpha - 1 \geqq \alpha + \pi^2/8 - 1 \geqq \alpha$ if $\alpha \geqq \pi/4$. Thus $g$ is decreasing on $[\pi/4, \pi/2]$ and $g(\pi/4) = \frac{1}{2}\pi/(4+\pi) < \phi(1)$.

On $[0, 2^{\frac{1}{2}}/2]$, $\tan \alpha \leqq 1$ so that $\beta(\alpha) \leqq \alpha$ and then $(2\beta^2 + \beta - 1) \tan \alpha \leqq \alpha$. Thus $g$ is increasing on $[0, 2^{\frac{1}{2}}/2]$ and it is easy to calculate that $g(2^{\frac{1}{2}}/2) < \phi(1)$.

On $[2^{\frac{1}{2}}/2, .74]$, $g(\alpha) \leqq \cos^2(.70).74 \tan(.74)/(1 + .74 \tan(.74)) \sim .237 < \phi(1)$.

On $[.74, \pi/4]$, $g(\alpha) \leqq \cos^2(.74)\pi/(4+\pi) < \phi(1)$.

(1.11) is now established. The next step is to discover those $\alpha$'s for which the solution $A(\alpha)$ in (1.12) satisfies (1.10). (1.9) and (1.10) are equivalent to (1.9) and

$$(1.13) \qquad (1+\beta)(\pi\beta)^{\frac{1}{2}} \exp(\beta)/\cos^2 \alpha \leqq 1 .$$

Let $H$ be the logarithm of the left-hand side of (1.13) and note that $H(0) = -\infty$. Let $Q = H'$. We will show that $Q \geqq 0$ and this implies that (1.13) is satisfied on an interval $[0, \alpha_0]$ where $\alpha_0$ is the $\alpha$ for which there is equality in (1.13).

Now

$$Q(\alpha) = \tan \alpha \frac{2\beta^2 + \beta + 1}{2\beta} + \frac{2\beta^2 + \beta + 1}{\beta(1+\beta)} - 2 \tan \alpha .$$

If $\beta \leqq \frac{1}{2}$ or $\beta \geqq 1$ we have $(2\beta^2 + \beta + 1) \geqq 4\beta$ which implies $Q(\alpha) \geqq 0$. In any case $(2\beta^2 + \beta + 1)/2\beta \geqq \frac{1}{2} + 2^{\frac{1}{2}}$ so that, if $\frac{1}{2} \leqq \beta \leqq 1$,

$$Q(\alpha) \geqq (2^{\frac{1}{2}} - 1.5) \tan \alpha + \alpha \inf_{\frac{1}{2}<\beta<1} \frac{2\beta^2 + \beta + 1}{2\beta(1+\beta)}$$
$$= (2^{\frac{1}{2}} - 1.5) \tan \alpha + \alpha$$

or, since $\beta \geqq \frac{1}{2}$ implies $\alpha \geqq \frac{1}{2}$,

$$\alpha Q(\alpha) \geqq \alpha^2 + (2^{\frac{1}{2}} - 1.5)\beta \geqq 2^{\frac{1}{2}} - 1.25 > 0 .$$

Thus $Q(\alpha) \geqq 0$ for all $\alpha$.

We next show that $\varepsilon(\alpha)$, defined by (1.8) where $A = A(\alpha)$ satisfies (1.9), is a decreasing function on $[0, \alpha_0]$. This comes from differentiating (1.8) and obtaining

$$\varepsilon' = A'\phi(A) + \frac{1}{1+\beta}\left(\frac{\beta'}{2(1+\beta)}\cos^2\alpha - \cos\alpha\sin\alpha\right). \tag{1.14}$$

Differentiating (1.9) we get

$$A'\phi(A) = \frac{1}{1-A^2}\frac{\beta}{1+\beta}\cos^2\alpha\left(-2\tan\alpha + \frac{\beta'}{\beta(1+\beta)}\right). \tag{1.15}$$

(1.14) and (1.15) imply that $\varepsilon' \leqq 0$ if, and only if,

$$\frac{\beta'}{2(1+\beta)} - \tan\alpha - \frac{\beta}{A^2-1}\left(\frac{\beta'}{\beta(1+\beta)} - 2\tan\alpha\right) \leqq 0. \tag{1.16}$$

Since $\beta' = \alpha + (1+\beta)\tan\alpha$ and $\alpha - (1+\beta)\tan\alpha \leqq \alpha - \tan\alpha \leqq 0$ we obtain from (1.16) that $\varepsilon' \leqq 0$ if and only if

$$A^2 \geqq \frac{(4\beta^2 + 3\beta - 1)\tan\alpha - 3\alpha}{(3+3\beta)\tan\alpha - \alpha}. \tag{1.17}$$

Since we are on $[0, \alpha_0]$ we need only show, in view of (1.10), that the right-hand side of (1.12) is no greater than $2\beta$ which is easy to do if we use $\tan\alpha \geqq \alpha$.

Let $\varepsilon_0 = \varepsilon(\alpha_0)$. Then since $\varepsilon(0) = \frac{1}{2}$ ((1.8) and (1.9)) for any $0 \leqq \varepsilon \leqq \varepsilon_0$ there is an $\alpha_\varepsilon \in [0, \alpha_0]$ such that $\varepsilon = \varepsilon(\alpha_\varepsilon)$ and by then taking $A_\varepsilon$ to satisfy (1.9) for $\alpha = \alpha_\varepsilon$ we will have found $G_\varepsilon$.

When $\varepsilon = \varepsilon_0$, equality holds in (1.10) and the solution $G_{\varepsilon_0}$ is the same as Huber's solution (see Section 9 of [1]) when his $a = b$. Since Huber has obtained the solution when $\varepsilon < \varepsilon_0$ the above argument gives the solution when $\varepsilon > \varepsilon_0$.

Here is a tabulation of some values of $\varepsilon$, $\alpha$, $A$ and $I$:

| $\varepsilon$ | $\alpha$ | $A$ | $I$ |
|---|---|---|---|
| .25 | .507 | 1.655 | .08 |
| .20 | .58 | 1.511 | .16 |
| .15 | .625 | 1.436 | .23 |
| .10 | .693 | 1.354 | .38 |
| .065 | .75 | 1.320 | .53 |
| .05 | .779 | 1.322 | .6 |
| .031 | .83 | 1.35 | .72 |

$\alpha_0 \sim .83$, $\varepsilon_0 \sim .03$.

**2. Non-robust estimates.** In the present section we will show that there is no linear function of order statistics (LFO) which is robust for the family

$$\mathscr{M}_\alpha = \left\{F \,\middle|\, \textstyle\int_{-1}^{1} f \geqq 1 - \frac{\cos^2\alpha}{1+\beta}, f > 0\right\}. \tag{2.1}$$

(Note that (2.1) differs from (1.7) by virtue of a positivity requirement—this allows us to avoid some dull details.) The same result is then carried over to the $\varepsilon$-normal model for $\varepsilon$ sufficiently large.

The distribution $F_\alpha$ given by (1.3) minimizes information over $\mathscr{M}_\alpha$. When

$F_\alpha$ is the underlying distribution the "best LFO" for estimating the location parameter (see [2]) is determined by the weight function

$$w(t) = -(\log f_\alpha)''F_\alpha^{-1}(t)/I(F_\alpha) = \frac{1+\beta}{2\beta}\sec^2(\alpha F_\alpha^{-1}(t)) \quad \text{if } F_\alpha(-1) \leqq t \leqq F_\alpha(1)$$
$$= 0 \quad \text{otherwise}. \tag{2.2}$$

This estimate has asymptotic variance $1/I(F_\alpha)$ at $F_\alpha$. We will show that the asymptotic variance takes values larger than $1/I(F_\alpha)$ on $\mathscr{M}_\alpha$.

Suppose $F \in \mathscr{M}_\alpha$ is the underlying distribution. Then the asymptotic variance of $1/n \sum_1^n w(i/(n+1))X_{(i)}$ is given by

$$V(F) = \int_{F_\alpha(-1)}^{F_\alpha(1)} \int_{F_\alpha(-1)}^{F_\alpha(1)} B(s,t)\frac{w(s)w(t)}{f(F^{-1}(s))f(F^{-1}(t))}\,ds\,dt \tag{2.3}$$

where $B(s,t) = \min(s,t) - st$. Changing variables in (2.3), we get

$$V(F) = \frac{1}{4}\int_{-1}^{1}\int_{-1}^{1} B(F_\alpha(x), F_\alpha(y))\frac{1}{f(F^{-1}(F_\alpha(x)))}\,\frac{1}{f(F^{-1}(F_\alpha(y)))}\,dx\,dy\,.$$

For $|x| \leqq 1$, we have $F_\alpha(x) = \frac{1}{2} + L(x)$ where (from (1.3))

$$L(x) = \frac{\beta}{2(1+\beta)}\left[x + \frac{\sin 2\alpha x}{2\alpha}\right]$$

is an odd function. Let $g$ be an even function and set $G(x) = \int_0^x g(v)\,dv$, $\lambda_g(x) = \int_{-1}^x L(v)g(v)\,dv$. Using this we can write, after some manipulation,

$$\begin{aligned}\rho(g) &= \int_{-1}^{1}\int_{-1}^{1} B(F_\alpha(x), F_\alpha(y))g(x)g(y)\,dx\,dy \\ &= G^2(1) + 4\int_0^1 G(x)\lambda_g(x)\,dx\,.\end{aligned} \tag{2.4}$$

If $g_0(x) = 1$ on $(z_0, z_1)$ with $0 < z_0 < z_1 < 1$, is symmetric, and is 0 where it is not 1, then (2.4) and some calculation yields

$$\begin{aligned}\rho(g_0) = (z_1 - z_0)^2 + \frac{2\beta}{1+\beta}\Big[&\frac{z_1^3 - z_0^3}{6} - \frac{z_1 - z_0}{2} \\ &- \frac{(\sin 2\alpha z_1 - \sin 2\alpha z_0)}{8\alpha^3} + \frac{(z_1 - z_0)\cos 2\alpha}{4\alpha^2}\Big]\,.\end{aligned} \tag{2.5}$$

Let $f$ be a density such that $f(x) = f_\alpha(x)$, $x > 1$, $f$ is symmetric, $I(F) < \infty$, $f(x) = a$ on $(c_0, c_1)$ where $0 < c_0 < c_1 < 1$, and $F \in \mathscr{M}_\alpha$. Let $g_f(x) = 1/f(F^{-1}(F_\alpha(x)))$ and note that $g_f(x) = 1/a$ if $F(c_0) < F_\alpha(x) < F(c_1)$. Put $z_0 = F_\alpha^{-1}(F(c_0))$, $z_1 = F_\alpha^{-1}(F(c_1))$ and get, from the middle term of (2.4),

$$4V(F) = \rho(g_f) \geqq 1/a^2\rho(g_0)\,. \tag{2.6}$$

Also,

$$\begin{aligned}a(c_1 - c_0) &= F(c_1) - F(c_0) = F_\alpha(z_1) - F_\alpha(z_0) \\ &= \frac{\beta}{2(1+\beta)}(z_1 - z_0) - \frac{1}{2\alpha}(\sin 2\alpha z_1 - \sin 2\alpha z_0)\,.\end{aligned}$$

Some more calculation produces, as $a \to 0$,

$$(z_1 - z_0) = \frac{(1+\beta)}{\beta \cos^2 \alpha}(c_1 - c_0)a[1 + o(1)] . \tag{2.7}$$

Using this in (2.5) and then using (2.6), we have

$$V(F)I(F_\alpha) \geqq \frac{\alpha^2}{\cos^4 \alpha}(c_1 - c_0)^2\left[\frac{1+\beta}{\beta} + 1 - \frac{\sin 2\alpha}{2\alpha}\right]\cdot[1 + o(1)] \tag{2.8}$$

where $o(1)$ goes to 0 as $a \to 0$. We remind the reader that $F$ depends on $a$ as well as $c_0$, $c_1$. It is clear that if we can show that

$$\frac{\alpha^2}{\cos^4 \alpha}\left[\frac{1+\beta}{\beta} + 1 - \frac{\sin 2\alpha}{2\alpha}\right] > 1 \tag{2.9}$$

then there exists $a$, $c_0$, $c_1$ such that $V(F)I(F_\alpha) > 1$. For $0 < \alpha < \pi/2$, $\sin 2\alpha < 2\alpha$, $\cos^4 \alpha < 1$ and (2.9) would therefore be satisfied if $\alpha + \alpha^2 \tan \alpha > \tan \alpha$ which is obviously true for $\alpha \geqq 1$ and is easy to check if $\alpha < 1$ by using $\sin \alpha > \alpha$ and $\cos \alpha > 1 - \alpha^2$. We have then shown that for each $\alpha$ there is an $F \in \mathscr{M}_\alpha$ with $V(F)I(F_\alpha) > 1$. A robust estimate for $\mathscr{M}_\alpha$ (as in Section 1, for example) has asymptotic variance under $F \leqq 1/I(F_\alpha)$ for all $F \in \mathscr{M}_\alpha$. Hence there is no LFO which is robust for $\mathscr{M}_\alpha$.

We are also interested in $\mathscr{M}_{\alpha,A} = \{F \mid \int_{-A}^{A} f \geqq 1 - \cos^2 \alpha/(1+\beta), f > 0\}$. The examples obtained above carry over to $\mathscr{M}_{\alpha,A}$ as follows: If $F \in \mathscr{M}_\alpha (= \mathscr{M}_{\alpha,1})$ has density $f$ then $f_A(x) = A^{-1}f(x/A)$ defines a distribution $F_A \in \mathscr{M}_{\alpha,A}$. It is easy to verify that $I(F_{\alpha,A}) = A^{-2}4\alpha^2\beta/(1+\beta)$ and by noting that $Af_A(F_A^{-1}(u)) = f(F^{-1}(u))$ we can obtain, for $F \in \mathscr{M}_\alpha$,

$$V(F_A)I(F_{\alpha,A}) = V(F)I(F_\alpha) . \tag{2.10}$$

Our previous examples can now be used for $\mathscr{M}_{\alpha,A}$ by the obvious transformation.

Let

$$\mathscr{F}_\varepsilon = \{F \mid \sup_x |F(x) - \Phi(x)| \leqq \varepsilon, f > 0\} \tag{2.11}$$

where $\Phi$ = standard normal distribution. In Section 1 we found a robust estimate for this model when $\varepsilon \gtrsim .03$ by use of the families $\mathscr{M}_{\alpha,A}$. For given $\varepsilon$ we found $\alpha_\varepsilon$, $A_\varepsilon$ such that $F_{\alpha_\varepsilon,A_\varepsilon} \in \mathscr{F}_\varepsilon \subset \mathscr{M}_{\alpha_\varepsilon,A_\varepsilon}$ so that $I(F_{\alpha_\varepsilon,A_\varepsilon})$ is the minimum information over $\mathscr{M}_{\alpha_\varepsilon,A_\varepsilon}$ and therefore, over $\mathscr{F}_\varepsilon$. Note that $\sup_x |F_{\alpha_\varepsilon,A_\varepsilon}(x) - \Phi(x)| \leqq \varepsilon$ is equivalent to saying $\sup_x |F_{\alpha_\varepsilon}(x) - \Phi^{A_\varepsilon}(x)| \leqq \varepsilon$ where $\Phi^{A_\varepsilon}$ = normal distribution with mean 0 and standard deviation $1/A_\varepsilon$. Let $F$ be a distribution function depending on $a$, $c_0$, $c_1$ which led to (2.8) when $\alpha = \alpha_\varepsilon$. We would like to show that we can choose $a$, $c_0$, $c_1$ so that the right side of (2.8) is $>1$, and, in addition, that such a choice gives an $F$ satisfying $\sup_x |F(x) - \Phi^{A_\varepsilon}(x)| \leqq \varepsilon$. If we can do so we will have shown that there is no LFO which is robust for $\mathscr{F}_\varepsilon$. We are able to do this for $\varepsilon \gtrsim .07$ and surmise that there are examples for $.07 \gtrsim \varepsilon \gtrsim .03$, but we have been hindered in finding them by the tedium of the calculations. Here are the pertinent numbers when $\varepsilon = .1$: from the table at the end of Section 1, we have $\alpha_\varepsilon = .69$, $A_\varepsilon = 1.35$. $c_1$ will be taken almost $=1$ and $a$ will be taken close to 0 so $c_0$ will have to satisfy

$$(2.12) \qquad \frac{(.69)^2}{\cos^4(.69)}\left[\frac{1 + .69 \tan .69}{.69 \tan .69} + 1 - \frac{\sin 1.38}{1.38}\right](1 - c_0)^2 > 1$$

in order for $V(F)I(F_\alpha) > 1$. This means $c_0 < .506$. To choose $a$, $c_0$, $c_1$ so that $\sup_x |F(x) - \Phi^{1.35}(x)| \leqq .1$ let us find $\gamma_0$ so that $\Phi^{1.35}(1) - \Phi^{1.35}(\gamma_0) = .2$, i.e., $\gamma_0 = .415$. From the definition of $f$ following (2.5), $F = F_\alpha$ on $(1, \infty)$ and from (1.8), we know that $(F_{\alpha_\varepsilon} - \Phi^{A_\varepsilon})(1) = -\varepsilon$. Thus, if we take $c_0 > \gamma_0$, $c_1$ close to 1, and $a$ close to 0, we can get $F$ to be within $\varepsilon$ of $\Phi^{A_\varepsilon}$. For $\varepsilon = .1$ this means $c_0 > .415$. Since (2.12) is satisfied for $c_0 < .506$, we can use (2.10) and conclude that there is an $F_{1.35} \in \mathscr{F}_{.1}$ with $V(F_{1.35})I(F_{.69,1.35}) > 1$ which means that no LFO is robust for $\mathscr{F}_{.1}$.

## REFERENCES

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