

ON LIMITING DISTRIBUTIONS OF A RANDOM NUMBER OF DEPENDENT RANDOM VARIABLES¹

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Let $\{X_n, n \geq 1\}$ be a sequence of random variables such that for suitably chosen constants $a_n > 0$ and $b_n, n \geq 1$, $\{(X_n - b_n)/a_n\}$ converges in distribution to a nondegenerate random variable X . Let $\{N_m, m \geq 1\}$ be a sequence of positive, integer-valued random variables distributed independently of the sequence $\{X_n\}$ and converging to infinity in probability as $m \rightarrow \infty$. If $\{a_n\}$ and $\{b_n\}$ are the normalizing constants computed from a cdf F which is in the domain of attraction of one of the extreme value distributions and if the cdf of X satisfies a condition determined by the domain of attraction to which F belongs, then conditions on the limiting distribution of $\{N_m/m\}$ are obtained which are necessary and sufficient for the convergence in distribution of the sequence $\{(X_{N_m} - b_m)/a_m\}$ to a nondegenerate random variable Y . The cdf of Y is either a location or a scale mixture of the cdf of X ; and the cdf F is often unrelated to the distribution of $\{X_n\}$. These results extend a theorem stated by Berman; however, the method of proof is conceptually simpler.

1. Introduction. Let $\{X_n, n \geq 1\}$ be a sequence of random variables defined on a probability space (Ω, \mathcal{A}, P) such that for suitably chosen sequences $\{a_n > 0\}$ and $\{b_n\}$ the sequence $\{(X_n - b_n)/a_n\}$ converges in distribution to a nonconstant random variable X . Let $\{N_m, m \geq 1\}$ be a sequence of positive, integer-valued random variables defined on the same space, distributed independently of the sequence $\{X_n\}$, and converging to infinity in probability. Define $G_m(t) = P[N_m \leq mt]$. With the assumption that $\{a_n\}$ and $\{b_n\}$ are the normalizing constants computed from a cumulative distribution function (cdf) $F(\cdot)$ which is in the domain of attraction of one of the extreme value distributions (Gnedenko, 1943; Smirnov, 1952), this paper investigates conditions on the weak limit of $\{G_m(\cdot)\}$ which are necessary and sufficient for the sequence $\{(X_{N_m} - b_m)/a_m\}$ to converge in distribution to a nonconstant random variable Y . As in Berman's Theorem 3.1 (1962), the cdf of Y is either a location or a scale mixture of the cdf of X ; however, the results given here are more general in that X_n need not be the maximum term in a sequence of independent and identically distributed random variables with common cdf $F(\cdot)$. In fact, Example 2.1 below demonstrates why Berman's result was similar to that obtained by Robbins (1948) for the limiting distribution of the sum of a random number of random variables.

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2. Main results and examples. Throughout this paper, any extended-valued monotone function $h(\cdot)$ will be assumed to be right continuous and \mathcal{C}_h will denote its set of continuity points. Such a function will be said to be non-degenerate in case there does not exist an x_0 in the extended real numbers such that $h(x)$ equals $h(-\infty)$ for $x < x_0$ and $h(+\infty)$ for $x \geq x_0$. A cdf $F(\cdot)$ will be considered to be a proper distribution function (df); i.e., a df such that $\lim_{x \rightarrow \infty} F(x) = 1$ and $\lim_{x \rightarrow -\infty} F(x) = 0$. A random variable will always have a proper df.

In order to emphasize that the normalizing constants must have certain properties essentially unrelated to the extreme value problem, the first proposition states Smirnov's (1952) basic result in a slightly more general framework.

PROPOSITION 2.1. *Suppose $f(\cdot)$ is a monotone function and $g(\cdot)$ is a nondegenerate extended-valued monotone function such that $\{g(-\infty), g(+\infty)\} \subset \{0, +\infty\}$. If for real constants $a_n > 0$, b_n*

$$(2.1) \quad \lim_{n \rightarrow \infty} n f(a_n x + b_n) = g(x)$$

for every x in \mathcal{C}_g , then $g(\cdot)$ has one of the following forms:

$$\begin{aligned} g_1(x; \delta) &= \alpha_1(x - x_0)^\delta && \text{for } x > x_0 \\ &= \alpha_2 && \text{for } x \leq x_0, \\ g_2(x; \delta) &= \beta_1 && \text{for } x \geq x_0 \\ &= \beta_2(-x + x_0)^\delta && \text{for } x < x_0, \end{aligned}$$

or

$$g_3(x; \delta) = \gamma e^{\delta x} \quad \text{for every } x,$$

where x_0 is finite, $\delta \neq 0$ ($\alpha_2 = \beta_1 = 0$ whenever $\delta > 0$; otherwise, $\alpha_2 = \beta_1 = +\infty$), and $\alpha_1, \beta_2, \gamma > 0$.

Henceforth, the normalizing constants, $\{a_n\}$ and $\{b_n\}$, will be said to be of type k ($k = 1, 2, 3$) with exponent δ in case there exists a monotone function $f(\cdot)$ with $\{f(-\infty), f(+\infty)\} \subset \{0, +\infty\}$ such that (2.1) holds with $g(x) = g_k(x; \delta)$. This definition deemphasizes the role of $f(\cdot)$ —the corresponding statement in extreme value theory being: $f(\cdot)$ is said to belong to the domain of attraction of type k with exponent δ in case normalizing constants exist such that (2.1) holds. . . Gnedenko (1943) succeeded in characterizing these domains of attraction (see Proposition 3.1 for a statement of his results when k equals 1 or 2) and further properties of the normalizing constants result from his proofs:

PROPOSITION 2.2. *Suppose that $\{a_n\}$ and $\{b_n\}$ are of type k ($k = 1, 2$) with exponent δ . Then,*

$$(2.2) \quad \lim_{n \rightarrow \infty} (x_1 - b_n)/a_n = x_0$$

and

$$(2.3) \quad \lim_{n \rightarrow \infty} n f(a_n[x - x_0] + x_1) = g_k(x; \delta)$$

for every x , where the real number

$$\begin{aligned} x_1 &= \inf\{x: f(x) > 0\} && \text{whenever } k = 1, \delta > 0 \\ &= \sup\{x: f(x) > 0\} && \text{whenever } k = 2, \delta > 0 \\ &= 0 && \text{whenever } \delta < 0. \end{aligned}$$

From (2.2) and (2.3) necessary and sufficient conditions for the sequence $\{(X_{N_m} - b_m)/a_m\}$ to converge in distribution to a nonconstant random variable can be derived provided one assumes that

$$(2.4) \quad P[X = x_0] = 0 \text{ and the characteristic function of } \log |X - x_0| \text{ is not identically zero in some nondegenerate real interval.}$$

However, the proof of sufficiency, which is the main interest anyway, can be carried through under weaker hypotheses, so the results are stated in two theorems. For completeness, the requirement that the limiting distributions be nondegenerate is deleted throughout.

THEOREM 2.1. Suppose that $\{a_n\}$ and $\{b_n\}$ are of type k ($k = 1, 2$) with exponent δ and that $\{G_m(\cdot)\}$ converges weakly to a df $G(\cdot)$ with

$$(2.5) \quad G(0) = 0 \text{ if } \delta > 0 \quad \text{or} \quad G(+\infty) = 1 \text{ if } \delta < 0.$$

Then, $\{(X_{N_m} - b_m)/a_m\}$ converges in distribution to a random variable Y with cdf

$$P[Y \leq y] = \int_0^\infty P[X \leq x_0 + t^{1/\delta}(y - x_0)] dG(t).$$

Y is nonconstant if, and only if, either X is nonconstant and $G(+\infty) > 0$ when $\delta > 0$ (or $G(0) < 1$ when $\delta < 0$) or $G(\cdot)$ is nondegenerate and X is not identically zero.

THEOREM 2.2. Assume that the random variable X satisfies (2.4) and that the normalizing constants are of type k ($k = 1, 2$) with exponent δ . If $\{(X_{N_m} - b_m)/a_m\}$ converges in distribution to a random variable Y , then $\{G_m(\cdot)\}$ converges weakly to a df $G(\cdot)$ satisfying (2.5).

EXAMPLE 2.1. The asymptotic distribution of the maximum has been derived for various dependent sequences $\{X_n\}$; e.g., m -dependent (Newell, 1964), exchangeable (Berman, 1962) stationary (Berman, 1964), and independent-increment (Štatland, 1966) processes. If the normalizing constants used are of type 1 or 2 and if the limiting distribution satisfies (2.4), then Theorems 2.1 and 2.2 generalize Berman's Theorem 3.1 (1962) by considering the maximum, with random index, from such dependent sequences.

For example, suppose a sequence $\{Z_n, n \geq 1\}$ satisfies the invariance principle (see Billingsley, 1968) with scaling factors $n^{1/2}\sigma$. Set $S_n = Z_1 + \cdots + Z_n$ ($S_0 = 0$) and $X_n = \max_{0 \leq k \leq n} S_k$. Recall that $\{X_n/(n^{1/2}\sigma)\}$ converges in distribution to a "positive normal" random variable and note that, with

$$\begin{aligned} f(x) &= x^{-2} && \text{for } x \geq 1 \\ &= 1 && \text{for } x < 1, \end{aligned}$$

$nf(n^{\frac{1}{2}}x) \rightarrow g_1(x; -2)$ and $x_0 = 0$. Therefore, $\{X_{N_m}/(m^{\frac{1}{2}}\sigma)\}$ has a nonconstant limiting distribution if, and only if,

$$(2.6) \quad \{N_m/m\} \text{ converges in distribution to a random variable } T \text{ with } P[T = 0] < 1.$$

In that case, the limit law is given by

$$P[Y_1 \leq y] = (2/\pi)^{\frac{1}{2}} \int_0^\infty \int_0^{t^{-\frac{1}{2}}y} e^{-u^2/2} du dG(t)$$

for $y \geq 0$ and $= 0$ for $y < 0$, where $G(\cdot)$ is the cdf of T .

However, the theorems can be applied, as well, to the sequence of sums so that, equivalently, the sequence $\{S_{N_m}/(m^{\frac{1}{2}}\sigma)\}$ converges in distribution to a random variable with cdf

$$P[Y_2 \leq y] = (2\pi)^{-\frac{1}{2}} \int_0^\infty \int_{-\infty}^{t^{-\frac{1}{2}}y} e^{-u^2/2} du dG(t).$$

Robbins' Theorem 2 (1948) is a special case of the sufficiency portion of the last result (his hypotheses are slightly different, but they imply the ones used here if $G(0) = 0$).

Finally, let W_n be the index of the first maximum in $(0, S_1, \dots, S_n)$; then, $\{W_n/n\}$ converges in law to a random variable having an arc sin distribution. In this case, choose

$$\begin{aligned} f(x) &= x^{-1} & \text{for } x \geq 1 \\ &= 1 & \text{for } x < 1, \end{aligned}$$

so that $nf(nx) \rightarrow g_1(x; -1)$ and $x_0 = 0$. Thus, the cdf of the limiting law of the sequence $\{W_{N_m}/m\}$ is given (iff (2.6) holds) by

$$P[Y_3 \leq y] = (2/\pi) \int_y^\infty \arcsin(t^{-1}y)^{\frac{1}{2}} dG(t)$$

for $y \geq 0$ and $= 0$ for $y < 0$.

From the previous example it is obvious that scaling constants which are a positive power of n can be handled easily; however, Theorem 2.1 is equivalent, then, to Dobrushin's case (5) with his $\delta = 1$ (1955). The next example illustrates that other kinds of scaling constants can be dealt with via our theorems.

EXAMPLE 2.2. Suppose $\{X_n/(n \log n)\}$ converges in law to a nonconstant random variable X satisfying (2.4). If

$$\begin{aligned} f(x) &= x^{-1} \log x & \text{for } x \geq e \\ &= e^{-1} & \text{for } x < e, \end{aligned}$$

then $nf(nx \log n) \rightarrow g_1(x; -1)$ and $x_0 = 0$. Therefore, $\{X_{N_m}/(m \log m)\}$ has a nonconstant limiting distribution if, and only if, (2.6) holds.

THEOREM 2.3. If $\{a_n\}$ and $\{b_n\}$ are of type 3 with exponent δ and if $\{G_m(\cdot)\}$ converges weakly to a cdf $G(\cdot)$ with $G(0) = 0$, then $\{(X_{N_m} - b_m)/a_m\}$ converges in distribution to a random variable Y with cdf

$$P[Y \leq y] = \int_0^\infty P[X \leq y + \log t^{1/\delta}] dG(t).$$

Y is nonconstant if, and only if, either X is nonconstant or $G(\cdot)$ is nondegenerate.

Unfortunately, the author could not prove the converse, except in some special cases. It is conjectured that a theorem analogous to Theorem 2.2 is true with (2.4) replaced by the assumption: the characteristic function of X is not identically zero in some nondegenerate real interval.

EXAMPLE 2.3. Cramér (1965) states that if $\{Z_n, n \geq 0\}$ is a (standardized) normal stationary process which satisfies a strong mixing condition, then $\{(\max_{0 \leq k \leq n} Z_k - b_n)/a_n\}$ can converge in law to a random variable with the same cdf as though the Z_n 's were independent and identically distributed normal random variables; i.e., with the exponential type of extreme value distribution (see Gumbel, 1958). It follows immediately that

$$\{(\max_{0 \leq k \leq N_m} Z_k - b_m)/a_m\}$$

can have a limiting distribution with cdf

$$P[Y \leq y] = \int_0^\infty \exp(-t^{-1/\delta} e^{-y}) dG(t).$$

3. Proofs and lemmas. In order to give a more general statement of Gnedenko's characterizations of the domains of attraction, a positive function $h(\cdot)$ defined on the positive real numbers will be said to vary regularly at infinity with exponent ρ in case $h(x) = x^\rho L(x)$ where $-\infty < \rho < \infty$ and $L(\cdot)$ varies slowly; i.e.,

$$\lim_{x \rightarrow +\infty} L(xy)/L(x) = 1$$

for every $y > 0$ (see Karamata, 1930 or Feller, 1966).

PROPOSITION 3.1. $f(\cdot)$ belongs to the domain of attraction of type k ($k = 1, 2$) with exponent δ if, and only if, $f(x_1 - (-1)^k x^{\text{sgn}(-\delta)})$ varies regularly at infinity with exponent $\rho = -|\delta|$.

This proposition is helpful in finding the appropriate function $f(\cdot)$ when given the sequences of normalizing constants. Its proof, as well as the proofs of Propositions 2.1 and 2.2, is essentially the same as the proof of the original version and ultimately depends on the following generalization of a well-known result:

PROPOSITION 3.2. Let $g(\cdot)$ and $h(\cdot)$ be nondegenerate monotone functions with $g(-\infty) = h(-\infty)$ and $g(+\infty) = h(+\infty)$. If $\{f_n(\cdot), n \geq 1\}$ is a sequence of monotone functions such that

$$(3.1) \quad \lim_{n \rightarrow \infty} f_n(a_n x + b_n) = g(x) \quad \text{for every } x \text{ in } \mathcal{E}_g$$

$$(3.2) \quad \lim_{n \rightarrow \infty} f_n(\alpha_n x + \beta_n) = h(x) \quad \text{for every } x \text{ in } \mathcal{E}_h,$$

where $a_n > 0$, b_n , $\alpha_n > 0$, β_n are real constants, then there exist real constants $c > 0$, d such that $h(x) = g(cx + d)$ for every x and

$$(3.3) \quad \lim_{n \rightarrow \infty} \alpha_n/a_n = c \quad \text{and} \quad \lim_{n \rightarrow \infty} (\beta_n - b_n)/a_n = d.$$

Conversely, (3.1) and (3.3) imply (3.2) with $h(x) = g(cx + d)$.

Now, we turn to the proofs of the theorems in the previous section. They are conceptually simple and rely on the easily proved

LEMMA 3.1. Suppose that $\{\eta_m(\cdot), m \geq 1\}$ and $\{\xi_n(\cdot), n \geq 1\}$ are sequences of Borel measurable functions such that $\{\eta_m(N_m)\}$ converges in distribution to a random variable U and $\{\xi_n(X_n)\}$ converges in distribution to a random variable V . Let W be a random variable independent of U and identically distributed with V . Then, the sequence of random vectors $\{\eta_m(N_m), \xi_{N_m}(X_{N_m})\}$ converges in distribution to the random vector (U, W) and $h(\eta_m(N_m), \xi_{N_m}(X_{N_m}))$ converges in distribution to $h(U, W)$, for any continuous function, $h(\cdot, \cdot)$, of two variables.

PROOF OF THEOREM 2.1. The details of the proof will be given for $k = 1$ and $\delta < 0$; the other cases are similar. Now, (2.2) implies that $\{X_m/a_m\}$ converges in law to $X - x_0$; hence, writing

$$(X_{N_m} - b_m)/a_m = [a_{N_m}/a_m][X_{N_m}/a_{N_m}] - [b_m/a_m],$$

the first result will follow from Lemma 3.1 if $\{a_{N_m}/a_m\}$ converges in law to $T^{-1/\delta}$, where T is a random variable with cdf $G(\cdot)$. We prove a generalization of the latter which will be useful in the proof of Theorem 2.2:

LEMMA 3.2. If $\{a_n\}$ and $\{b_n\}$ are of type k ($k = 1, 2$) with exponent δ and if $\{G_m(\cdot)\}$ converges weakly to a df $G(\cdot)$ for some subsequence $\{m'\}$ of $\{m\}$, then

$$\begin{aligned} \lim_{m' \rightarrow \infty} P[a_{N_{m'}} \leq aa_{m'}] &= G(a^{-\delta}), & \text{if } \delta < 0 \\ &= 1 - G(a^{-\delta}), & \text{if } \delta > 0, \end{aligned}$$

provided $a > 0$ and $a^{-\delta}$ is in \mathcal{C}_G .

PROOF. In the case under consideration $f(\cdot)$ is a non-increasing function such that, for $x > 0$, $\lim_{n \rightarrow \infty} nf(a_n x) = \alpha_1 x^\delta$. Thus,

$$P[a_{N_{m'}} \leq aa_{m'}] \leq P[N_{m'} f(a_{N_{m'}}) \geq m' f(aa_{m'})(N_{m'}/m')]$$

and, by an obvious generalization of Section 20.6 of Cramér (1946),

$$\limsup_{m' \rightarrow \infty} P[a_{N_{m'}} \leq aa_{m'}] \leq G(a^{-\delta}).$$

Applying the same type of argument to $P[a_{N_{m'}} > aa_{m'}]$, we obtain

$$\liminf_{m' \rightarrow \infty} P[a_{N_{m'}} \leq aa_{m'}] \geq G(a^{-\delta}).$$

The necessary and sufficient condition for Y to be nonconstant is a result of the facts that X and T are "independent" and that the only characteristic functions whose reciprocals are characteristic functions belong to degenerate cdf's (Lukacs, 1960).

PROOF OF THEOREM 2.2. As above, take $k = 1$ and $\delta < 0$. We have $\{X_{N_m}/a_m\}$ converges in law to $Y - x_0$. Let $H_1(\cdot)$ and $H_2(\cdot)$ denote the cdf's of $|X - x_0|$ and $|Y - x_0|$, respectively, and let $\{m'\}$ be a subsequence of $\{m\}$ for which $\{G_{m'}(\cdot)\}$ converges weakly to some df $G(\cdot)$. Then, sending $m' \rightarrow \infty$ in

$$P[|X_{N_{m'}}| \leq ya_{m'}] \leq P[a_{N_{m'}} \leq aa_{m'}] + P[|X_{N_{m'}}| \leq ya_{N_{m'}}/a],$$

one obtains from Lemma 3.2 that

$$H_2(y) \leq G(a^{-\delta}) + H_1(y/a),$$

provided $a, y > 0$, y is in \mathcal{C}_{H_2} , $a^{-\delta}$ is in \mathcal{C}_G , and y/a is in \mathcal{C}_{H_1} . Letting $a \rightarrow \infty$ and, then, $y \rightarrow \infty$, $G(+\infty)$ must be one and there exists a random variable T such that $\{N_{m'}/m'\}$ converges in distribution to T . By Theorem 2.1., $Y - x_0$ must have the same distribution as $(X - x_0)T^{-1/\delta}$ where, now, T is considered to be independent of X . Any other weakly convergent subsequence of $\{G_m(\cdot)\}$ will lead in exactly the same way to a random variable S , independent of X , such that $(X - x_0)S^{-1/\delta}$ has the same distribution as Y and, thus, as $(X - x_0)T^{-1/\delta}$. Since $P[X = x_0] = 0$, $P[S = 0] = P[T = 0] = P[Y = 0]$ and, without loss of generality, we assume that S and T are positive. Then, (2.4) implies S and T have the same distribution (this result is equivalent to stating that, under the hypotheses, a scale parameter mixture of $|X - x_0|$ is identifiable—see Teicher (1961)). Hence, every weakly convergent subsequence of $\{G_m(\cdot)\}$ has the same limit and the theorem follows (Feller, 1966, page 261).

PROOF OF THEOREM 2.3. Write

$$(X_{N_m} - b_m)/a_m = [a_{N_m}/a_m][(X_{N_m} - b_{N_m})/a_{N_m} + (b_{N_m} - b_m)/a_{N_m}];$$

then, the theorem follows easily from Lemma 3.1 if $\{a_{N_m}/a_m\}$ converges in probability to one and if $\{(b_{N_m} - b_m)/a_{N_m}\}$ converges in distribution to (say $\delta < 0$) $\log T^{-1/\delta}$, where T is a positive random variable with cdf $G(\cdot)$. We again prove a generalization of these last statements, but, unfortunately this time, the results are not strong enough to prove a converse to Theorem 2.3:

LEMMA 3.3. *If $\{a_n\}$ and $\{b_n\}$ are of type 3 with exponent δ and if $\{G_m(\cdot)\}$ converges weakly to a df $G(\cdot)$ for some subsequence $\{m'\}$ of $\{m\}$, then*

$$(3.4) \quad \lim_{m' \rightarrow \infty} P[b_{N_{m'}} - b_{m'} \leq ba_{N_{m'}}] = G(e^{-\delta b}), \quad \text{if } \delta < 0 \\ = 1 - G(e^{-\delta b}), \quad \text{if } \delta > 0,$$

provided $e^{-\delta b}$ is in \mathcal{C}_G . Moreover, for $a > 1$

$$(3.5) \quad \limsup_{m' \rightarrow \infty} P[a_{N_{m'}} > aa_{m'}] \leq G(0) + 1 - G(+\infty);$$

and for $a < 1$

$$(3.6) \quad \limsup_{m' \rightarrow \infty} P[a_{N_{m'}} \leq aa_{m'}] \leq G(0) + 1 - G(+\infty).$$

PROOF. The proof of (3.4) is similar to that of Lemma 3.2 and will be omitted. It also follows that in (3.4) $a_{N_{m'}}$ can be replaced by $a_{m'}$. We prove (3.5) for $\delta < 0$. Let c belong to the interval $(1, a)$ and $b > 0$; then, $P[a_{N_{m'}} > aa_{m'}]$ is less than, or equal to,

$$P[(a_{N_{m'}} - ca_{m'})b \geq (b_{N_{m'}} - b_{m'})(a - c)] \leq P[b_{N_{m'}} - b_{m'} > ba_{m'}].$$

Applying the “usual argument” to the first term on the right and using (3.4) for the second term, we obtain

$$\limsup_{m' \rightarrow \infty} P[a_{N_{m'}} > aa_{m'}] \leq G(\exp[b\delta(c - 1)/(a - c)]) + 1 - G(e^{-\delta b}),$$

provided $\exp[b\delta(c - 1)/(a - c)]$ and $e^{-\delta b}$ are in \mathcal{C}_G . Now, send $b \rightarrow +\infty$. The proofs for $\delta > 0$ and for (3.6) are similar.

The last statement of the theorem follows as in Theorem 2.1.

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