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HITTING TIME DISTRIBUTIONS FOR GENERAL STOCHASTIC PROCESSES

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We show that under mild conditions the hitting time distributions of a general stochastic process are the unique solutions of an abstract Dirichlet problem.

It is the purpose of this paper to suggest, by a simple example, that the methods which have been so successful in the study of temporally homogeneous Markov processes can be applied equally successfully to general stochastic processes.

Let λ be a probability measure in the space $\mathscr{M}(\Omega, \mathscr{F})$ of all measures on the measurable space $(\Omega, \mathscr{F})$ of all functions ω mapping $R_+ = (0, \infty)$ into a measurable space (S, Σ) where $\mathscr{F}$ is the σ-field generated by the events $X_t(\omega) = \omega(t) \in U \in \Sigma$ and where S is a separable compact space with Borel sets Σ, and suppose that the continuous functions in Ω have λ-outer measure one. Let T_t, $t \in R_+$ be the semigroup of linear operators on $\mathscr{M}(\Omega, \mathscr{F})$ defined by $T_t\mu(X_{t_1} \in U_1, \cdots, X_{t_n} \in U_n) = \mu(X_{t+t_1} \in U_1, \cdots, X_{t+t_n} \in U_n)$ and let E_U, $U \in \Sigma$ be the resolution of the identity

$$E_U\,\mu(\Lambda) = \mu(X_0 \in U, \Lambda) .$$

Let Φ be the weak $*$ closure over the continuous functions on the product topology of $(\Omega, \mathscr{F})$ of the linear subspace of $\mathscr{M}(\Omega, \mathscr{F})$ which is generated by measures of the form $E_{U_n} T_{t_n} \cdots E_{U_1} T_{t_1} \lambda$ and let Φ_+ be the set of all probability measures in Φ.

Suppose now that τ is the first exit time of X from the interior U of S and let g be a continuous function on the boundary U' of U. Let Φ^* be the set of all linear functionals ϕ^* on Φ which are continuous in the weak $*$ topology of Φ. Let

$$T_t{}^*\phi^*\phi = \phi^* T_t\,\phi , \qquad \phi^* \in \Phi^*,\ \phi \in \Phi_+$$

and

$$G^*\phi^*\phi = \lim_{h\to 0} h^{-1}[T_h{}^*\phi^*\phi - \phi^*\phi] , \qquad \phi^* \in \Phi^*,\ \phi \in \Phi_+$$

and let a^* and b^* be the linear functionals on Φ_+ defined by

$$a^*\phi = \int g(X_\tau)\, d\phi$$

and

$$b^*\phi = \int \frac{\tau}{1+\tau}\, d\phi .$$

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Let Φ_A be the set of all $\phi \in \Phi_+$ for which $E_A \phi = \phi$. If

I. a^* and b^* are in Φ^*,
II. $\phi(\tau \leqq t) = o(t)$, $\phi \in \Phi_K$, $K \subset U$ compact,
III. $\int \tau \, d\phi < \infty$, $\phi \in \Phi_+$

then we have the following

THEOREM 1. *Under conditions* I–III, a^* *is the unique solution in* Φ^* *of the Dirichlet problem*

$$G^* a^* \phi = 0 , \qquad \phi \in \Phi_K,\ K \subset U \text{ compact}$$

with boundary condition

$$a^* \phi = g(u) , \qquad \phi \in \Phi_{\{u\}},\ u \in U' .$$

PROOF. If $\phi \in \Phi_K$, then letting $\omega_h^+(t) = \omega(t+h)$,

$$\begin{aligned} G^* a^* \phi &= \lim_{h \to 0} h^{-1}[T_h^* a^* \phi - a^* \phi] \\ &= \lim_{h \to 0} h^{-1}[a^* T_h \phi - a^* \phi] \\ &= \lim_{h \to 0} h^{-1}[\textstyle\int g(X_\tau)\, dT_h \phi - \int g(X_\tau)\, d\phi] \\ &= \lim_{h \to 0} h^{-1} \textstyle\int [g(X_\tau(\omega_h^+)) - g(X_\tau(\omega))] \phi(d\omega) \\ &= \lim_{h \to 0} h^{-1} \textstyle\int_{[\tau > h]} [g(X_\tau(\omega_h^+)) - g(X_\tau(\omega))] \phi(d\omega) \\ &\qquad + \lim_{h \to 0} h^{-1} \textstyle\int_{[\tau \leqq h]} [g(X_\tau(\omega_h^+)) - g(X_\tau(\omega))] \phi(d\omega) \\ &= 0 \end{aligned}$$

since $g(X_\tau(\omega_h^+)) = g(X_\tau(\omega))$ on $[\tau > h]$ and

$$\left| \int_{[\tau \leqq h]} [g(X_\tau(\omega_h^+)) - g(X_\tau(\omega))] \phi(d\omega) \right| \leqq 2\|g\|_\infty \phi(\tau \leqq h) = 2\|g\|_\infty o(h).$$

To prove uniqueness, we note that for any $\phi \in \Phi_K$,

$$\begin{aligned} G^* b^* \phi &= \lim_{h \to 0} h^{-1} \int_{[\tau > h]} \left[\frac{\tau(\omega_h^+)}{1 + \tau(\omega_h^+)} - \frac{\tau(\omega)}{1 + \tau(\omega)} \right] \phi(d\omega) \\ &\qquad + \lim_{h \to 0} h^{-1} \int_{[\tau \leqq h]} \left[\frac{\tau(\omega_h^+)}{1 + \tau(\omega_h^+)} - \frac{\tau(\omega)}{1 + \tau(\omega)} \right] \phi(d\omega) \\ &= \lim_{h \to 0} h^{-1} \int_{[\tau > h]} \left[\frac{\tau - h}{1 + \tau - h} - \frac{\tau}{1 + \tau} \right] d\phi + \lim_{h \to 0} h^{-1} o(h) \\ &= - \int_{[\tau > 0]} \frac{1}{(1 + \tau)^2}\, d\phi < 0 . \end{aligned}$$

Thus $G^* b^* \phi < 0$, $\phi \in \Phi_K$ and $b^* \phi = 0$, $\phi \in \Phi_{U'}$.

Now suppose that c^* is another solution in Φ^* of our equation which satisfies the boundary condition. Then $\psi_\varepsilon^* = c^* - a^* + \varepsilon b^*$ is a solution of $G^* \psi_\varepsilon^* \phi < 0$ for all $\phi \in \Phi_K$ with boundary condition

$$\psi_\varepsilon^* \phi \geqq 0 , \qquad \phi \in \Phi_{U'} .$$

Since Φ_+ is weak * compact and since ψ_ε^* is continuous in this topology, ψ_ε^* must obtain its minimum value in Φ_+ at some point $\phi_0 \in \Phi_+$. It follows then from Choquet's theorem [2] that there exists a probability measure m on the extreme points Φ_{++} of Φ_+ such that

$$\psi_\varepsilon{}^*\phi_0 = \int_{\Phi_{++}} \psi_\varepsilon{}^*\phi m(d\phi) .$$

Since $\psi_\varepsilon{}^*$ obtains its minimum value in Φ_+ at ϕ_0, it must also obtain its minimal value at each $\phi \in \Phi_{++}$ which is in the support of m. As a result we can choose ϕ_0 to be in Φ_{++} and so there exists a point $u_0 \in S$ such that $\phi_0 \in \Phi_{\{u_0\}}$. Suppose that $\psi_\varepsilon{}^*\phi_0 < 0$. Then since $\psi_\varepsilon{}^*\phi \geqq 0$ when $\phi \in \Phi_{U'}$, it follows that $u_0 \in U$ and so there exists a compact set $K \subset U$ for which $\phi_0 \in \Phi_K$. By the minimum principle $G^*\psi_\varepsilon{}^*\phi_0 \geqq 0$. But $G^*\psi_\varepsilon{}^*\phi_0 < 0$ and so we have a contradiction. Thus $\psi_\varepsilon{}^*\phi_0 \geqq 0$ for all $\varepsilon > 0$ and so $c^*\phi \geqq a^*\phi$ for all $\phi \in \Phi_+$. Interchanging a^* and c^* yields, via the same reasoning, $a^*\phi \geqq c^*\phi$ for all $\phi \in \Phi_+$. Thus $a^* = c^*$ and the theorem is proved.

This theorem is well known when λ is a temporally homogeneous Markov process since in that case $T_t{}^*$ plays the role of the usual semigroup of operators associated with a Markov process.

The reader might also note that condition I will be satisfied if, for example, one lets τ_m be the minimum of m and the first exit time of $X_{k/2^n}$ from U, and then assume that

$$\phi(\tau_m > \tau + \varepsilon) \to 0 \quad \text{uniformly in} \quad \phi \in \Phi_+$$

and

$$\phi[|g(X_{\tau_m}) - g(X_\tau)| > \varepsilon] \to 0 \quad \text{uniformly in} \quad \phi \in \Phi_+ .$$

For example, a^* would then be continuous in the weak $*$ topology of Φ_+ since if $\phi_n \to \phi$ in the weak $*$ topology then

$$\begin{aligned} |a^*\phi_n - a^*\phi| &= |\textstyle\int g(X_\tau)\, d\phi_n - \int g(X_\tau)\, d\phi| \\ &\leqq \textstyle\int |g(X_\tau) - g(X_{\tau_m})|\, d\phi_n + |\int g(X_{\tau_m})\, d\phi_n - \int g(X_{\tau_m})\, d\phi| \\ &\qquad + \textstyle\int |g(X_{\tau_m}) - g(X_\tau)|\, d\phi \\ &\leqq 4\|g\|_\infty \sup_{\psi\in\Phi_+} \psi[|g(X_{\tau_m}) - g(X_\tau)| > \varepsilon/3] + \varepsilon/3 \\ &\qquad + |\textstyle\int g(X_{\tau_m})\, d\phi_n - \int g(X_{\tau_m})\, d\phi| . \end{aligned}$$

The first term can be made less than $\varepsilon/3$ by picking m sufficiently large. Then since $g(X_{\tau_m})$ is a continuous function in the product topology of Ω and since $\phi_n \to \phi$ in the weak $*$ topology of Φ it follows that the last term can be made less than $\varepsilon/3$ if n is picked sufficiently large.

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