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THE ASYMPTOTIC NONCENTRAL DISTRIBUTION OF HOTELLING'S GENERALIZED T_0^2 [1]

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In this paper a general method is proposed for deriving the asymptotic noncentral distribution of Hotelling's generalized T_0^2 statistic for the case of large error degrees of freedom n_2. The approximation is given here to order n_2^{-2} and further terms in the expansion can be readily obtained.

1. Introduction and summary. The T_0^2 statistic of Hotelling [5] and Lawley [9] is used as a test of significance in the multivariate analysis of variance and, in the noncentral case, is defined as

$$T = T_0^2/n_2 = \operatorname{tr} S_1 S_2^{-1} \tag{1.1}$$

where the $m \times m$ matrices S_1 and S_2 are independently distributed on n_1 and n_2 degrees of freedom respectively, estimating the same covariance matrix, with S_2 having the Wishart distribution and S_1 having the noncentral Wishart distribution with matrix of noncentrality parameters Ω. Constantine [1] has obtained the exact distribution of T_0^2 over the range $|T| < 1$ as a power series involving generalized Laguerre polynomials. This series is very difficult to work with and it appears necessary, in order to calculate powers of the test, to obtain asymptotic expansions for the distribution. Of particular interest is the case of large n_2, i.e., large error degrees of freedom. Here the distribution of T_0^2 tends to χ^2 on mn_1 degrees of freedom and noncentrality parameter $\operatorname{tr}\Omega$. An asymptotic expansion, up to order n_2^{-1}, has been obtained by Siotani [13] and Itô [7] and, more recently, the expansion has been extended up to order n_2^{-2} by Siotani [14] using perturbation techniques and by Hayakawa [3] using weighted sums of generalized Laguerre polynomials. It appears rather difficult to extend the expansion further using either of these two methods. In this paper a reasonably general derivation is outlined which can be readily utilized to give further terms in the expansion.

It is shown in Section 3 that the moment generating function (mgf) of T_0^2 can be expressed as an integral involving the generalized Bessel function of the second kind, B_δ, which was defined by Herz [4]. The integral concerned appears very difficult to evaluate but, using a system of partial differential equations satisfied by B_δ, an asymptotic expansion is derived for B_δ, which then allows immediate integration and expresses the expansion of the mgf, given here to order n_2^{-3}, in terms of generalized Laguerre polynomials. This expansion is inverted in Section 4 to yield the expansion, given here to order n_2^{-2}, of the cumulative distribution function (cdf) in terms of noncentral χ^2 density functions.

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2. Preliminary results. In the ensuing sections use will be made of the following definitions and results. Let R be an $m \times m$ symmetric matrix, $\Gamma_m(a) = \pi^{m(m-1)/4} \prod_{i=1}^m \Gamma(a - \frac{1}{2}(i-1))$ and, throughout the paper, $p = \frac{1}{2}(m+1)$. Then Muirhead [9] has shown that the confluent hypergeometric function $\Psi(a, c; R)$, defined by the integral

$$(2.1) \qquad \Psi(a, c; R) = \frac{1}{\Gamma_m(a)} \int_{S>0} \operatorname{etr}(-RS)(\det S)^{a-p} \det(I+S)^{c-a-p}\, dS\,,$$

satisfies each partial differential equation (pde) in the system

$$(2.2) \qquad R_i \frac{\partial^2 y}{\partial R_i^2} + \left\{c - \tfrac{1}{2}(m-1) - R_i + \tfrac{1}{2} \sum_{j=1, j\neq i}^m \frac{R_i}{R_i - R_j}\right\} \frac{\partial y}{\partial R_i} - \tfrac{1}{2} \sum_{j=1, j\neq i}^m \frac{R_j}{R_i - R_j} \frac{\partial y}{\partial R_j} = ay \qquad (i = 1, 2, \cdots, m)$$

where $R_1, R_2, \cdots, R_m$ are the distinct latent roots of R. This system yields a system of pde's for the generalized Bessel function of the second kind, $B_\delta(R)$, defined by Herz [4] as the integral

$$(2.3) \qquad B_\delta(R) = \int_{S>0} \operatorname{etr}(-RS) \operatorname{etr}(-S^{-1})(\det S)^{\delta-p}\, dS\,.$$

First we obtain B_δ as a limiting function from the confluent function Ψ.

LEMMA.

$$(2.4) \qquad \lim_{a\to\infty} \{\Gamma_m(a) a^{mp-mc}\, \Psi(a, c; a^{-1}R)\} = B_{c-p}(R)\,.$$

This can easily be proved using the integral representation (2.1) for Ψ. Now, from the system of pde's (2.2) satisfied by $\Psi(a, c; R)$ it is readily verified, using (2.4), that

THEOREM. *$B_{c-p}(R)$ satisfies each* pde *in the system*

$$(2.5) \qquad R_i \frac{\partial^2 y}{\partial R_i^2} + \left\{c - \tfrac{1}{2}(m-1) + \tfrac{1}{2} \sum_{j=1, j\neq i}^m \frac{R_i}{R_i - R_j}\right\} \frac{\partial y}{\partial R_i} - \tfrac{1}{2} \sum_{j=1, j\neq i}^m \frac{R_j}{R_i - R_j} \frac{\partial y}{\partial R_j} = y \qquad (i = 1, 2, \cdots, m)\,.$$

It is worth pointing out that this system of pde's is also satisfied by ${}_0F_1(c; R)$, the generalized Bessel function of the first kind (see Muirhead [8]).

Constantine [2] has defined the generalized Laguerre polynomial $L_\kappa^a(R)$ by the integral

$$(2.6) \qquad \operatorname{etr}(-R) L_\kappa^a(R) = \frac{1}{\Gamma_m(a+p)} \int_{S>0} \operatorname{etr}(-S)(\det S)^a C_\kappa(S)\, {}_0F_1(a+p; -SR)\, dS$$

where $C_\kappa(S)$ is the zonal polynomial of S corresponding to the partition $\kappa = (k_1, k_2, \cdots, k_m)$ of k into not more than m parts. The Laguerre polynomials have the expansion (see [2])

$$L_\kappa^a(R) = (a+p)_\kappa C_\kappa(I) \sum_{s=0}^k \sum_\sigma \frac{(-1)^s\binom{\kappa}{\sigma}}{(a+p)_\sigma} \frac{C_\sigma(R)}{C_\sigma(I)} \tag{2.7}$$

where $\binom{\kappa}{\sigma}$ is the coefficient of $C_\sigma(R)/C_\sigma(I)$ in the "binomial" expansion

$$C_\kappa(I+R)/C_\kappa(I) = \sum_{s=0}^k \sum_\sigma \binom{\kappa}{\sigma} C_\sigma(R)/C_\sigma(I) . \tag{2.8}$$

These coefficients have been tabulated up to $k = 8$ by Pillai and Jouris [11].

3. The asymptotic expansion of the mgf **of** T_0^2. The joint distribution of the matrices S_1 and S_2 in (1.1) may be written as

$$\frac{\operatorname{etr}(-\frac{1}{2}\Omega)}{\Gamma_m(\frac{1}{2}n_1)\Gamma_m(\frac{1}{2}n_2)} \operatorname{etr}(-S_1)\operatorname{etr}(-S_2)(\det S_1)^{\frac{1}{2}n_1-p}(\det S_2)^{\frac{1}{2}n_2-p}{}_0F_1(\tfrac{1}{2}n_1; \tfrac{1}{2}\Omega S_1) \tag{3.1}$$

where, without loss of generality, the population covariance matrix Σ is taken to be $\frac{1}{2}I_m$. The mgf of $T_0^2 = n_2 \operatorname{tr} S_1 S_2^{-1}$ is then

$$g(t, \Omega) = E[\operatorname{etr}(-tn_2 S_1 S_2^{-1})] .$$

Multiplying (3.1) by $\operatorname{etr}(-tn_2S_1S_2^{-1})$ and integrating over $S_2 > 0$ using (2.3) we obtain

$$g(t, \Omega) = \frac{\operatorname{etr}(-\frac{1}{2}\Omega)}{\Gamma_m(\frac{1}{2}n_1)\Gamma_m(\frac{1}{2}n_2)} \int_{S_1>0} \operatorname{etr}(-S_1)(\det S_1)^{\frac{1}{2}n_1-p}{}_0F_1(\tfrac{1}{2}n_1; \tfrac{1}{2}\Omega S_1) \times B_{-\frac{1}{2}n_2}(n_2tS_1)\, dS_1 . \tag{3.2}$$

It is possible to express $g(t, \Omega)$ as various other integrals (see Constantine [1]) which certainly appear more elementary than (3.2). Unfortunately these forms do not appear to lend themselves easily to asymptotic expansion.

It appears very difficult to carry out the integration with respect to S_1 in (3.2) to obtain $g(t, \Omega)$ explicitly, but an asymptotic expansion for $g(t, \Omega)$ can be obtained in the following way. Using the system of pde's derived in Section 2 for the Bessel function of the second kind we can expand the function $B_{-\frac{1}{2}n_2}(n_2tS_1)/\Gamma_m(\frac{1}{2}n_2)$ asymptotically for large n_2. Rearranging this expansion in terms of zonal polynomials $C_\kappa(S_1)$ we may then carry out the integration in (3.2), using (2.6) to obtain an asymptotic expansion for $g(t, \Omega)$ in terms of the generalized Laguerre polynomials.

Now, in the expression (3.2) for $g(t, \Omega)$ we have the function $B_{-\frac{1}{2}n_2}(\frac{1}{2}n_2R)$ with $R = 2tS_1$. Since $g(t, \Omega)$ is a mgf the boundary condition

$$\lim_{R\to 0+} B_{-\frac{1}{2}n_2}(\tfrac{1}{2}n_2R)/\Gamma_m(\tfrac{1}{2}n_2) = 1 \tag{3.3}$$

must clearly be imposed. From Section 2 we have that $y(R) = B_{-\frac{1}{2}n_2}(\frac{1}{2}n_2R)$ satisfies the system of pde's

$$R_i \frac{\partial^2 y}{\partial R_i^2} + \left\{1 - \tfrac{1}{2}n_2 + \tfrac{1}{2}\sum_{j=1, j\neq i}^m \frac{R_i}{R_i - R_j}\right\} \frac{\partial y}{\partial R_i} - \tfrac{1}{2}\sum_{j=1, j\neq i}^m \frac{R_j}{R_i - R_j}\frac{\partial y}{\partial R_j} = \tfrac{1}{2}n_2 y \qquad (i = 1, 2, \cdots, m) \tag{3.4}$$

where $R_1, R_2, \cdots, R_m$ are the latent roots of R. It is easily shown, using either (2.4) or equation (5.7′) of Herz [4], that

$$\lim_{n_2\to\infty} B_{-\frac{1}{2}n_2}(\tfrac{1}{2}n_2 R)/\Gamma_m(\tfrac{1}{2}n_2) = \operatorname{etr}(-R)\,, \tag{3.5}$$

so putting

$$B_{-\frac{1}{2}n_2}(\tfrac{1}{2}n_2 R)/\Gamma_m(\tfrac{1}{2}n_2) = \operatorname{etr}(-R)G(R) \tag{3.6}$$

we may easily obtain the system of pde's satisfied by the function $G(R)$. We could now look for a solution of this system of the form

$$G(R) \sim 1 + \textstyle\sum_{k=1}^{\infty} P_k(R)n_2^{-k} \tag{3.7}$$

where $P_k(0) = 0$ for all k so that the boundary condition (3.3) is satisfied. However, a significant reduction in work is obtained if instead of G we consider the function

$$H(R) = \ln G(R) \sim \textstyle\sum_{k=1}^{\infty} Q_k(R)n_2^{-k} \tag{3.8}$$

where $Q_k(0) = 0$ for all k. The system of pde's satisfied by $H(R)$ is readily found to be

$$\begin{aligned} R_i\left\{\frac{\partial^2 H}{\partial R_i^2} + \left(\frac{\partial H}{\partial R_i}\right)^2\right\} + \left\{1 - \tfrac{1}{2}n_2 - 2R_i + \tfrac{1}{2}\textstyle\sum_{j=1,j\neq i}^{m} \dfrac{R_i}{R_i - R_j}\right\}\frac{\partial H}{\partial R_i} \\ - \tfrac{1}{2}\textstyle\sum_{j=1,j\neq i}^{m} \dfrac{R_j}{R_i - R_j}\dfrac{\partial H}{\partial R_j} = \tfrac{1}{2}(m+1) - R_i \qquad (i = 1, 2, \cdots, m)\,. \end{aligned} \tag{3.9}$$

Since we are only concerned with symmetric solutions of (3.9) we need only work with the first pde ($i = 1$). We substitute the series (3.8) into this pde and equate coefficients of like powers of n_2^{-1} on both sides. Equating constant terms gives

$$\partial Q_1/\partial R_1 = 2R_1 - (m+1)$$

so that

$$Q_1 = s_2 - (m+1)s_1 \tag{3.10}$$

where $s_r = R_1^r + \cdots + R_m^r$, since the Q_k are symmetric functions of $R_1, \cdots, R_m$. Similarly, equating the coefficients of n_2^{-1}, and of n_2^{-2}, and integrating gives

$$Q_2 = -\tfrac{8}{3}s_3 + (3m+4)s_2 + s_1^2 - (m+1)^2 s_1 \tag{3.11}$$

and

$$\begin{aligned} Q_3 = 10s_4 + \tfrac{8}{3}(5m+8)s_3 - 8s_1s_2 + (6m^2 + 16m + 13)s_2 \\ + (4m+5)s_1^2 - (m+1)^3 s_1\,. \end{aligned} \tag{3.12}$$

The coefficients of higher powers of n_2^{-1} may be obtained in a similar manner if required.

We now exponentiate H to give the function

$$G(R) = e^{H(R)} \sim 1 + \textstyle\sum_{k=1}^{\infty} P_k(R)n_2^{-k} \tag{3.13}$$

and we express the P_k in terms of zonal polynomials using the tables in James

[7]. For example

$$P_1 = Q_1 = s_2 - (m+1)s_1 \tag{3.14}$$
$$= C_{(2)}(R) - \tfrac{1}{2}C_{(1^2)}(R) - (m+1)C_{(1)}(R)$$
$$= (2t)^2[C_{(2)}(S_1) - \tfrac{1}{2}C_{(1^2)}(S_1)] - 2t(m+1)C_{(1)}(S_1)$$

since $R = 2tS_1$. In a similar way we can express

$$P_2 = Q_2 + \tfrac{1}{2}Q_1^2 \quad \text{and} \quad P_3 = Q_3 + Q_1Q_2 + \tfrac{1}{6}Q_1^3$$

as polynomials in the latent roots of S_1 of degree 4 and 6 respectively, but the lengthy expressions are not given here. Thus we have

$$\frac{B_{-\frac{1}{2}n_2}(n_2tS_1)}{\Gamma_m(\frac{1}{2}n_2)} = \operatorname{etr}(-2tS_1)[1 + n_2^{-1}P_1 + n_2^{-2}P_2 + n_2^{-3}P_3 + O(n_2^{-4})]\,. \tag{3.15}$$

Now substitute (3.15) in (3.2) and integrate term by term. The first term is

$$\frac{\operatorname{etr}(-\frac{1}{2}\Omega)}{\Gamma_m(\frac{1}{2}n_1)} \int_{S_1>0} \operatorname{etr}(-(1+2t)S_1)(\det S_1)^{\frac{1}{2}n_1-p}\,{}_0F_1(\tfrac{1}{2}n_1; \tfrac{1}{2}\Omega S_1)\, dS_1$$
$$= \operatorname{etr}\left(-\frac{t}{1+2t}\Omega\right)(1+2t)^{-\frac{1}{2}mn_1}\,.$$

The other terms are integrated using (2.6). For example, the coefficient of n_2^{-1} is

$$\frac{\operatorname{etr}(-\frac{1}{2}\Omega)}{\Gamma_m(\frac{1}{2}n_1)} \int_{S_1>0} \operatorname{etr}(-(1+2t)S_1)(\det S_1)^{\frac{1}{2}n_1-p}\,{}_0F_1(\tfrac{1}{2}n_1; \tfrac{1}{2}\Omega S_1)P_1(S_1)\, dS_1$$
$$= \operatorname{etr}\left(-\frac{t}{1+2t}\Omega\right)(1+2t)^{-\frac{1}{2}mn_1}\left\{\left(\frac{2t}{1+2t}\right)^2\left[L_{(2)}^c\left(-\frac{1}{2(1+2t)}\Omega\right)\right.\right.$$
$$\left.\left. - \tfrac{1}{2}L_{(1^2)}^c\left(-\frac{1}{2(1+2t)}\Omega\right)\right] - \left(\frac{2t}{1+2t}\right)(m+1)L_{(1)}^c\left(-\frac{1}{2(1+2t)}\Omega\right)\right\}$$

where $c = \frac{1}{2}n_1 - p$. Hence it is readily seen that we may write

$$g(t,\Omega) = \operatorname{etr}\left(-\frac{t}{1+2t}\Omega\right)(1+2t)^{-\frac{1}{2}mn_1} \tag{3.16}$$
$$\times \{1 + n_2^{-1}T_1 + n_2^{-2}T_2 + n_2^{-3}T_3 + O(n_2^{-4})\}$$

where T_1, T_2 and T_3 are obtained from P_1, P_2 and P_3 respectively by replacing the zonal polynomial $C_\kappa(S_1)$ by

$$(1+2t)^{-k}L_\kappa^c\left(-\frac{1}{2(1+2t)}\Omega\right) \quad \text{with} \quad c = \tfrac{1}{2}n_1 - p\,.$$

4. The inversion of the mgf. T_1, T_2 and T_3 in the expansion (3.16) for $g(t,\Omega)$ contain terms of the form

$$\left[\frac{2t}{1+2t}\right]^k L_\kappa^c\left(-\frac{1}{2(1+2t)}\Omega\right).$$

In order to obtain (3.16) in a form which allows easy inversion, it is necessary to express $[2t/(1+2t)]^k$ in terms of powers of $(1+2t)^{-1}$, and also to expand L_κ^c in terms of zonal polynomials using (2.7), which may then be expressed in terms of the power sums $\sigma_r = \omega_1^r + \cdots + \omega_m^r$ where $\omega_1, \cdots, \omega_m$ are the latent roots of Ω. Doing this we obtain

$$\begin{aligned} T_1 = {} & \frac{1}{4(1+2t)^4}\sigma_2 + \frac{1}{2(1+2t)^3}[-\sigma_2 + (m+n_1+1)\sigma_1] \\ & + \frac{1}{4(1+2t)^2}[\sigma_2 - 2(m+2n_1+1)\sigma_1 + mn_1(m+n_1+1)] \\ & + \frac{n_1}{2(1+2t)}[\sigma_1 - mn_1] + \frac{mn_1}{4}(n_1 - m - 1) \end{aligned}$$

and other much more lengthy expressions for T_2 and T_3. Now, since $(1+2t)^{-r/2}\operatorname{etr}(-t\Omega/(1+2t))$ is the mgf of a noncentral χ^2 distribution on r degrees of freedom and noncentrality parameter $\operatorname{tr}\Omega = \sigma_1$, we may invert the expansion (3.16) for $g(t, \Omega)$ term by term to give $\Pr(T_0^2 > x)$ in terms of noncentral χ^2 distribution functions. This expansion can then be rearranged in terms of noncentral χ^2 density functions, which are probably more convenient from a computational point of view. The final expansion is given, up to order n_2^{-2}, in the

THEOREM. *The noncentral* cdf *of* $T_0^2 = n_2 \operatorname{tr} S_1 S_2^{-1}$ *can be approximated up to the order* n_2^{-2} *by*

$$\Pr(T_0^2 > x) = \Pr(\chi^2_{mn_1}(\sigma_1) > x) + \frac{1}{2n_2}\sum_{j=1}^4 a_j \phi_{mn_1+2j}(x) + \frac{1}{48n_2^2}\sum_{j=1}^8 b_j \phi_{mn_1+2j}(x) + O(n_2^{-3}) \tag{4.1}$$

where $\Pr(\chi_k^2(\sigma_1) > x)$ *denotes the probability that a noncentral* χ^2 *variate on k degrees of freedom and noncentrality parameter* σ_1 *exceeds x*; $\phi_k(x)$ *denotes the probability density function, evaluated at x, of a noncentral* χ^2 *variate on k degrees of freedom and noncentrality parameter* σ_1;

$$a_1 = -mn_1(n_1 - m - 1), \qquad a_2 = -2n_1\sigma_1 + mn_1(m+n_1+1),$$

$$a_3 = -\sigma_2 + 2(m+n_1+1)\sigma_1, \qquad a_4 = \sigma_2,$$

$$\begin{aligned} b_1 = {} & -mn_1[3m^3n_1 - 2m^2(3n_1^2 - 3n_1 + 4) + 3m(n_1^3 - 2n_1^2 + 5n_1 - 4) \\ & - 4(2n_1^2 - 3n_1 - 1)], \end{aligned}$$

$$\begin{aligned} b_2 = {} & 12mn_1^2(m - n_1 + 1)\sigma_1 - mn_1[3m^3n_1 + 2m^2(3n_1^2 + 3n_1 - 4) \\ & - 3m(3n_1^3 - 2n_1^2 - 5n_1 + 4) - 4(2n_1^2 - 3n_1 - 1)], \end{aligned}$$

$$\begin{aligned} b_3 = {} & -12n_1^2\sigma_1^3 + 6n_1[m^2 - m(n_1 - 1) - 4]\sigma_2 \\ & - 12n_1[m^3 - 2m^2 - 3m(n_1^2 + 1) - 4(2n_1 + 1)]\sigma_1 \\ & + mn_1[3m^3n_1 - 2m^2(3n_1^2 - 3n_1 - 4) - 3m(3n_1^3 + 2n_1^2 + 11n_1 - 4) \\ & - 4(10n_1^2 + 9n_1 + 1)], \end{aligned}$$

$$\begin{aligned}
b_4 = {} & -12n_1\sigma_1\sigma_2 - 16\sigma_3 + 12[2mn_1 + (3n_1^2 + 2n_1 + 4)]\sigma_1^2 \\
& - 6[m^2n_1 - m(3n_1^2 - n_1 + 8) - 4(7n_1 + 4)]\sigma_2 \\
& - 12[m^2(3n_1^2 + 4) + 3m(n_1^3 + n_1^2 + 8n_1 + 4) + 8(2n_1^2 + 3n_1 + 2)]\sigma_1 \\
& + 3mn_1[m^3n_1 + 2m^2(n_1^2 + n_1 + 4) + m(n_1^3 + 2n_1^2 + 21n_1 + 20) \\
& + 4(2n_1^2 + 5n_1 + 5)] , \\
b_5 = {} & -3\sigma_2^2 + 12(m + 3n_1 + 1)\sigma_1\sigma_2 + 80\sigma_3 \\
& - 12[m^2 + 2m(2n_1 + 1) + (3n_1^2 + 4n_1 + 11)]\sigma_1^2 \\
& - 6[m^2n_1 + m(3n_1^2 + n_1 + 28) + 4(11n_1 + 12)]\sigma_2 \\
& + 12[m^3n_1 + 2m^2(n_1^2 + n_1 + 4) + m(n_1^3 + 2n_1^2 + 21n_1 + 20) \\
& + 4(2n_1^2 + 5n_1 + 5)]\sigma_1 , \\
b_6 = {} & 9\sigma_2^2 - 12(2m + 3n_1 + 2)\sigma_1\sigma_2 - 112\sigma_3 \\
& + 12[m^2 + 2m(n_1 + 1) + (n_1^2 + 2n_1 + 7)]\sigma_1^2 \\
& + 6[m^2n_1 + m(n_1^2 + n_1 + 20) + 4(5n_1 + 8)]\sigma_2 , \\
b_7 = {} & -9\sigma_2^2 + 12(m + n_1 + 1)\sigma_1\sigma_2 + 48\sigma_3 , \qquad b_8 = 3\sigma_2^2 .
\end{aligned}$$

The terms of order n_2^{-3} may also be obtained from previous results. When $\Omega = 0$ (4.1) agrees with the expansion of the central cdf obtained by Itô [6], Davis [2] and Muirhead [11].

REFERENCES

[1] CONSTANTINE, A. G. (1966). The distribution of Hotelling's generalized T_0^2. *Ann. Math. Statist.* **37** 215–225.

[2] DAVIS, A. W. (1968). A system of linear differential equations for the distribution of Hotelling's generalized T_0^2. *Ann. Math. Statist.* **39** 815–832.

[3] HAYAKAWA, T. (1970). On the derivation of the asymptotic distribution of the generalized Hotelling's T_0^2 (abstract). *Ann. Math. Statist.* **41** 1799.

[4] HERZ, C. S. (1955). Bessel functions of matrix argument. *Ann. of Math.* **61** 474–523.

[5] HOTELLING, H. (1947). Multivariate quality control, illustrated by the air testing of sample bomb-sights. *Techniques of Statistical Analysis*, 111–184. McGraw-Hill, New York.

[6] ITÔ, K. (1956). Asymptotic formulae for the distribution of Hotelling's generalized T_0^2 statistic. *Ann. Math. Statist.* **27** 1091–1105.

[7] ITÔ, K. (1960). Asymptotic formulae for the distribution of Hotelling's generalized T_0^2 statistic II. *Ann. Math. Statist.* **31** 1148–1153.

[8] JAMES, A. T. (1964). Distribution of matrix variates and latent roots derived from normal samples. *Ann. Math. Statist.* **35** 475–501.

[9] LAWLEY, D. N. (1938). A generalization of Fisher's *Z*-test. *Biometrika* **30** 180–187.

[10] MUIRHEAD, R. J. (1970a). Systems of partial differential equations for hypergeometric functions of matrix argument. *Ann. Math. Statist.* **41** 991–1001.

[11] MUIRHEAD, R. J. (1970b). Asymptotic distributions of some multivariate tests. *Ann. Math. Statist.* **41** 1002–1010.

[12] PILLAI, K. C. S. and JOURIS, G. M. (1969). On the moments of elementary symmetric functions of the roots of two matrices. *Ann. Inst. Statist. Math.* **21** 309–320.

[13] SIOTANI, M. (1957). Note on the utilization of the generalized student ratio in the analysis of variance or dispersion. *Ann. Inst. Statist. Math.* **9** 157–171.

[14] SIOTANI, M. (1971). An asymptotic expansion of the non-null distribution of Hotelling's generalized T_0^2-statistic. *Ann. Math. Statist.* **42** 560–571.

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