The Annals of Mathematical Statistics
1972, Vol. 43, No. 6, 2045-2049

MINIMAL SUFFICIENT σ-FIELDS AND MINIMAL SUFFICIENT STATISTICS. TWO COUNTEREXAMPLES

BY DIETER LANDERS AND LOTHAR ROGGE

University of Cologne

In this paper there are solved two problems raised by Bahadur (1954) which concern the relation between the existence of a minimal sufficient σ-field and the existence of a minimal sufficient statistic. Two examples show that the existence of a minimal sufficient σ-field is neither necessary nor sufficient for the existence of a minimal sufficient statistic.

1. Introduction. Let $\mathfrak{P}$ be a family of probability measures (p-measures) on a σ-field $\mathscr{A}$ over a basic set X. If $A, B \in \mathscr{A}$ we write $A \sim B(\mathfrak{P})$ iff $P(A \triangle B) = 0$ for all $P \in \mathfrak{P}$. If $\mathscr{B}$ and $\mathscr{C}$ are subsystems of $\mathscr{A}$ we write $\mathscr{B} \subset \mathscr{C}(\mathfrak{P})$ iff for every $B \in \mathscr{B}$ there exists $C \in \mathscr{C}$ with $B \sim C(\mathfrak{P})$. We write $\mathscr{B} \sim \mathscr{C}(\mathfrak{P})$ iff $\mathscr{B} \subset \mathscr{C}(\mathfrak{P})$ and $\mathscr{C} \subset \mathscr{B}(\mathfrak{P})$. If $P \mid \mathscr{A}$ is a p-measure, $\mathscr{A}_0$ a sub-σ-field of $\mathscr{A}$ and $f: X \to \mathbb{R}$ a P-integrable $\mathscr{A}$-measurable function, $P^{\mathscr{A}_0} f$ denotes the $P \mid \mathscr{A}_0$-equivalence class of conditional expectations of f, relative $P \mid \mathscr{A}$, given $\mathscr{A}_0$. We write $g \in \mathfrak{P}^{\mathscr{A}_0} f$ iff $g \in P^{\mathscr{A}_0} f$ for every $P \in \mathfrak{P}$. The σ-field $\mathscr{A}_0 \subset \mathscr{A}$ is sufficient for $\mathfrak{P} \mid \mathscr{A}$ iff $\mathfrak{P}^{\mathscr{A}_0} 1_A \neq \emptyset$ for every $A \in \mathscr{A}$. The σ-field $\mathscr{A}_0 \subset \mathscr{A}$ is minimal sufficient for $\mathfrak{P} \mid \mathscr{A}$ iff $\mathscr{A}_0$ is sufficient for $\mathfrak{P} \mid \mathscr{A}$ and $\mathscr{A}_0 \subset \mathscr{A}_1(\mathfrak{P})$ for every σ-field $\mathscr{A}_1 \subset \mathscr{A}$ which is sufficient for $\mathfrak{P} \mid \mathscr{A}$.

A statistic $T: X \to Y$ is sufficient for $\mathfrak{P} \mid \mathscr{A}$ iff the σ-field $\mathscr{A}_T := T^{-1}(\mathscr{P}(Y)) \cap \mathscr{A}$ is sufficient for $\mathfrak{P} \mid \mathscr{A}$, where $\mathscr{P}(Y)$ denotes the power set of Y. A sufficient statistic $T: X \to Y$ is minimal sufficient for $\mathfrak{P} \mid \mathscr{A}$ iff for every sufficient statistic $U: X \to Z$ there exists a function $g: Z \to Y$ such that $T = g \circ U$ $\mathfrak{P}$-a.e.

The concepts of minimal sufficient σ-fields and minimal sufficient statistics were introduced by Halmos–Savage [4] and Bahadur [1] in 1949 and 1954, respectively. Both concepts have received considerable attention. The concept of minimal sufficient statistics has a great intuitive appeal but it is more difficult to handle than the concept of minimal sufficient σ-fields. In 1954 Bahadur [1] investigated both concepts and showed that minimal sufficient σ-fields exist for dominated families of p-measures and minimal sufficient statistics exist for separable families of p-measures. Then Bahadur inquired for the exact relation between the notion of minimal sufficient statistics and minimal sufficient σ-fields; whether, for example, the existence of a minimal sufficient σ-field always implies the existence of a minimal sufficient statistic. The paper of Bahadur and Lehmann [2] casts some light upon this question but the problem remained unsolved. Up to that time it was not known whether minimal sufficient σ-fields and minimal sufficient statistics exist in general. In 1958 Pitcher [5] constructed a family of p-measures

Received November 8, 1971.

AMS 1970 *subject classifications*. Primary 62B05.

Key words and phrases. Minimal sufficient σ-field; minimal sufficient statistic.

for which neither a minimal sufficient σ-field nor a minimal sufficient statistic did exist. His example, however, did not answer the question whether the existence of a minimal sufficient σ-field always implies the existence of a minimal sufficient statistic or the converse. The following two examples show that in general none of these implications is true.

2. Two counterexamples. The following example shows that even for a dominated family of p-measures a minimal sufficient statistic need not exist. Consequently the existence of a minimal sufficient σ-field does not necessarily imply the existence of a minimal sufficient statistic.

We remark that our example is a slight modification of an example which was communicated by Savage to Bahadur (see [1] page 441) for showing that there exist dominated families of p-measures which are not separable. But neither Savage, nor Bahadur and Lehmann, seem to have realized that this example can also be used to solve their problem mentioned above.

EXAMPLE 1. Let $X = \{0, 1\}^{\mathbb{R}}$ and $\mathscr{A} = \times_{r\in\mathbb{R}} \mathscr{A}_r$ where $\mathscr{A}_r$ is the power set of $\{0, 1\}$. Let P, Q be p-measures on $\mathscr{A}_r$, defined by $P(1) = 1$, $Q(0) = Q(1) = \frac{1}{2}$. For each $r \in \mathbb{R}$ let $P(r) \mid \mathscr{A} = \times_{s\in\mathbb{R}} Q_s \mid \mathscr{A}_s$ where $Q_s = Q$ for $s \neq r$ and $Q_r = P$. Let $Q^{\mathbb{R}} \mid \mathscr{A}$ be the $\mathbb{R}$-fold product measure of Q, and $\bar{\mathscr{A}}$ the completion of $\mathscr{A}$ with respect to $Q^{\mathbb{R}}$. As $P(r) \mid \mathscr{A} \ll Q^{\mathbb{R}} \mid \mathscr{A}$ we may extend $P(r) \mid \mathscr{A}$ to $\bar{\mathscr{A}}$, $r \in \mathbb{R}$. Let $\mathfrak{P} \mid \bar{\mathscr{A}} := \{P(r) \mid \bar{\mathscr{A}} : r \in \mathbb{R}\} \cup \{Q^{\mathbb{R}} \mid \bar{\mathscr{A}}\}$. Then $\mathfrak{P} \mid \bar{\mathscr{A}}$ is dominated by $Q^{\mathbb{R}} \mid \bar{\mathscr{A}} \in \mathfrak{P} \mid \bar{\mathscr{A}}$. For each $r \in \mathbb{R}$ the function $h_r : X \to \mathbb{R}$ defined by "$h_r(x) = 2$ if $x_r = 1$ and $= 0$ if $x_r = 0$" is a density of $P(r) \mid \bar{\mathscr{A}}$ with respect to $Q^{\mathbb{R}} \mid \bar{\mathscr{A}}$. As $\mathscr{A}$ is the σ-field induced by the family of densities h_r, $r \in \mathbb{R}$, $\mathscr{A}$ is minimal sufficient for $\mathfrak{P} \mid \bar{\mathscr{A}}$ according to [1] page 439. Therefore each σ-field which is sufficient for $\mathfrak{P} \mid \bar{\mathscr{A}}$ is $Q^{\mathbb{R}} \mid \bar{\mathscr{A}}$-equivalent to $\bar{\mathscr{A}}$.

Now we shall show that there exists no minimal sufficient statistic for $\mathfrak{P} \mid \bar{\mathscr{A}}$. Assume, conversely, that $T : X \to Z$ is a minimal sufficient statistic for $\mathfrak{P} \mid \bar{\mathscr{A}}$. Then $\bar{\mathscr{A}}_T$ is sufficient for $\mathfrak{P} \mid \bar{\mathscr{A}}$ and hence $Q^{\mathbb{R}} \mid \bar{\mathscr{A}}$-equivalent to $\bar{\mathscr{A}}$. Therefore for each $r \in \mathbb{R}$ there exists $Z_r \subset Z$ such that $T^{-1}(Z_r) \in \bar{\mathscr{A}}$ and

$$T^{-1}(Z_r) \sim \pi_r^{-1}(1)(Q^{\mathbb{R}}) , \tag{1}$$

where π_r denotes the projection of X onto the rth component. It is easy to see that the system of all $\mathscr{A}$-measurable sets of positive $Q^{\mathbb{R}}$-measure has the same cardinal number as $\mathbb{R}$. Let $r_\alpha \in \mathbb{R}$, $\alpha < \eta$, be a well ordering of the real numbers and $A_\alpha \in \mathscr{A}$, $\alpha < \eta$, be a well ordering of the sets with positive $Q^{\mathbb{R}}$-measure.

Now we shall construct by induction over $\alpha < \eta$ elements $x_\alpha \in A_\alpha$ with $\{x_\alpha, \tilde{x}_\alpha{}^\alpha\} \cap \{x_\beta, \tilde{x}_\beta{}^\beta\} = \emptyset$ $(\alpha \neq \beta)$ and $T(x_\alpha) \neq T(\tilde{x}_\alpha{}^\alpha)$, where $\tilde{x}^\alpha$ is the element of X, distinguishing from $x \in X$ only in the component r_α. Assume that $x_\alpha \in A_\alpha$ is defined for all $\alpha < \beta$. According to (1) there exists $B \in \mathscr{A}$, $Q^{\mathbb{R}}(B) = 1$, such that for all $x \in B$

$$T(x) \in Z_{r_\beta} \quad \text{if and only if} \quad \pi_{r_\beta}(x) = 1 . \tag{2}$$

Let $\tilde{B}^\beta := \{\tilde{x}^\beta : x \in B\}$. Then $\tilde{B}^\beta \in \mathscr{A}$ and $Q^{\mathbb{R}}(\tilde{B}^\beta) = 1$. Therefore $C := A_\beta \cap B \cap \tilde{B}^\beta \in \mathscr{A}$ and $Q^{\mathbb{R}}(C) > 0$ because $Q^{\mathbb{R}}(A_\beta) > 0$. Since C has a cardinal number greater than that of $\mathbb{R}$ there exists $x_\beta \in C$ such that $\{x_\beta, \tilde{x}_\beta{}^\beta\} \cap \{x_\alpha, \tilde{x}_\alpha{}^\alpha\} = \emptyset$ for all $\alpha < \beta$. Since $x_\beta \in B \cap \tilde{B}^\beta$ we have $\tilde{x}_\beta{}^\beta \in B$ whence (2) implies $T(x_\beta) \neq T(\tilde{x}_\beta{}^\beta)$. As $x_\beta \in A_\beta$ this concludes the induction.

Now we shall define a sufficient statistic $U: X \to X$. Let $U(x) := x$ if $x \neq x_\alpha$, $\alpha < \eta$, and $U(x_\alpha) := \tilde{x}_\alpha{}^\alpha$, $\alpha < \eta$. Then $U^{-1}(\pi_{r_\alpha}^{-1}(1)) \triangle \pi_{r_\alpha}^{-1}(1) = \{x_\alpha\}$ for each $\alpha < \eta$. As $\{\pi_{r_\alpha}^{-1}(1) : \alpha < \eta\}$ generates $\mathscr{A}$ and $Q^{\mathbb{R}}\{x_\alpha\} = 0$, we obtain that $U^{-1}\mathscr{A} \sim \bar{\mathscr{A}}(Q^{\mathbb{R}})$, whence U is a sufficient statistic for $\mathfrak{P} | \bar{\mathscr{A}}$. As T is a minimal sufficient statistic for $\mathfrak{P} | \bar{\mathscr{A}}$ there exists $A \in \mathscr{A}$, $Q^{\mathbb{R}}(A) = 1$, such that $x, y \in A$ and $U(x) = U(y)$ imply $T(x) = T(y)$. By definition of $\mathscr{A}$ there exists a countable $R_0 \subset \{r_\alpha : \alpha < \eta\}$ such that A belongs to the σ-field generated by $\pi_r{}^{-1}\mathscr{A}_r$, $r \in R_0$, say $\mathscr{A}_{R_0}$. Since each $\mathscr{A}$-measurable set of positive $Q^{\mathbb{R}}$-measure contains uncountably many x_α, $\alpha < \eta$, there exists $r_\beta \notin R_0$ such that $x_\beta \in A$. As $A \in \mathscr{A}_{R_0}$ and $r_\beta \notin R_0$ we obtain $\tilde{x}_\beta{}^\beta \in A$, too. Since x_β, $\tilde{x}_\beta{}^\beta \in A$ and $U(x_\beta) = U(\tilde{x}_\beta{}^\beta)$ by definition, but $T(x_\beta) \neq T(\tilde{x}_\beta{}^\beta)$ by construction of x_β, we obtain a contradiction.

The following example shows that even for a family of p-measures on a countably generated σ-field the existence of a minimal sufficient statistic not necessarily implies the existence of a minimal sufficient σ-field.

EXAMPLE 2. Let $X := [0, 1] \times [0, 1]$, $\mathscr{B}$ the σ-field of all Borel sets of $[0, 1]$ and $\mathscr{B}^*$ be the σ-field of all Lebesgue-measurable sets of $[0, 1]$. Let $S \subset [0, 1]$ be a set of inner Lebesgue-measure 0 and outer Lebesgue-measure 1. Let $T := S \times [0, 1]$ and $\mathscr{A}$ be the σ-field on X generated by $\mathscr{B} \times \mathscr{B}$ and T, i.e.,

$$\mathscr{A} := \{A \cap T + B \cap \bar{T} : A, B \in \mathscr{B} \times \mathscr{B}\}.$$

For each $r \in [0, 1]$ let $Q_r | \mathscr{A}$ be the p-measure defined by

$$Q_r(A \cap T + B \cap \bar{T}) = \tfrac{1}{2}\lambda(A_r) + \tfrac{1}{2}\lambda(B_r), \qquad A, B \in \mathscr{B} \times \mathscr{B}$$

where λ denotes the L-measure on $[0, 1]$ and $A_r := \{s \in [0, 1] : (s, r) \in A\}$. Let $P_r | \mathscr{A}$ be the p-measure with density $p_r(s, t) := 2s(s, t \in [0, 1])$ with respect to $Q_r | \mathscr{A}$, $r \in [0, 1]$. As T is a set of inner measure 0 and outer measure 1 with respect to $\lambda \times \lambda | \mathscr{B} \times \mathscr{B}$ the set function $P | \mathscr{A}$ defined by

$$P(A \cap T + B \cap \bar{T}) := (\lambda \times \lambda)(A), \qquad A, B \in \mathscr{B} \times \mathscr{B}$$

is uniquely determined and a p-measure. Let $\mathfrak{P} := \{P_r, Q_r : r \in [0, 1]\} \cup \{P\}$.

We remark that for each $C \in \mathscr{A}$

(1) $\quad Q_r(C) = 0$ for all $r \in [0, 1]$ implies $P(C) = 0$ for all $P \in \mathfrak{P}$.

(i) Now we shall show that $\mathfrak{P} | \mathscr{A}$ admits no minimal sufficient σ-field:

It is easy to see that for each $r \in [0, 1]$ the σ-field $\mathscr{A}_r := \{A \in \mathscr{A} : A_r \in \mathscr{B}^*\}$ is sufficient for $\mathfrak{P} | \mathscr{A}$ and contains all $\mathfrak{P} | \mathscr{A}$-null sets. Therefore, if a minimal sufficient σ-field for $\mathfrak{P} | \mathscr{A}$ would exist, it is contained in $\bigcap \{\mathscr{A}_r : r \in [0, 1]\}$.

(a) At first we prove that the σ-field $\bigcap \{\mathscr{A}_r : r \in [0, 1]\}$ is $\mathfrak{P} | \mathscr{A}$ equivalent to $\mathscr{B} \times \mathscr{B}$: Obviously $\mathscr{B} \times \mathscr{B} \subset \bigcap \{\mathscr{A}_r : r \in [0, 1]\}$. Hence it suffices to show that for each $C := A \cap T + B \cap \bar{T} \in \bigcap \{\mathscr{A}_r : r \in [0, 1]\}$ there exists a $\mathfrak{P}$-equivalent element of $\mathscr{B} \times \mathscr{B}$, namely $A \cap B$. By definition of $\mathscr{A}_r$ we have $C_r = A_r \cap S + B_r \cap \bar{S} \in \mathscr{B}^*$ for each $r \in [0, 1]$. Since S is a set of inner λ-measure 0 and outer λ-measure 1 we obtain $\lambda(C_r \triangle A_r) = \lambda(C_r \triangle B_r) = 0$ whence $C_r \sim (A \cap B)_r(\lambda)$ for all $r \in [0, 1]$. This implies $C \sim A \cap B(Q_r)$ for all $r \in [0, 1]$ and hence according to (1) $C \sim A \cap B(\mathfrak{P})$. This proves (a).

(b) Now we prove that $\bigcap \{\mathscr{A}_r : r \in [0, 1]\}$ contains no sufficient σ-field for $\mathfrak{P} \mid \mathscr{A}$:
Assume conversely that $\mathscr{C} \subset \bigcap \{\mathscr{A}_r : r \in [0, 1]\}$ is a sufficient σ-field for $\mathfrak{P} \mid \mathscr{A}$. Hence there exists $f \in \mathfrak{P}^{\mathscr{C}} 1_{\bar{T}}$. As by (a) $\bigcap \{\mathscr{A}_r : r \in [0, 1]\}$ is $\mathfrak{P} \mid \mathscr{A}$ equivalent to $\mathscr{B} \times \mathscr{B}$ there exists a $\mathscr{B} \times \mathscr{B}$ measurable function g such that $f = g$ $\mathfrak{P}$-a.e. As $f \in Q_r^{\mathscr{C}} 1_{\bar{T}}$ and g is $\mathscr{B} \times \mathscr{B}$ measurable, we have

$$\lambda(g_r) = Q_r(g) = Q_r(f) = Q_r(\bar{T}) = \tfrac{1}{2}$$

for all $r \in [0, 1]$. Hence $P(g) = (\lambda \times \lambda)(g) = \frac{1}{2}$ according to the theorem of Fubini. However $f \in P^{\mathscr{C}} 1_{\bar{T}}$ and $P(\bar{T}) = 0$ imply $g = 0$ P-a.e. contradicting $P(g) = \frac{1}{2} > 0$. As a minimal sufficient σ-field has to be contained in $\bigcap_{r \in [0,1]} \mathscr{A}_r$, (b) implies (i).

(ii) Now we shall show that the identity $I: X \to X$ is a minimal sufficient statistic for $\mathfrak{P} \mid \mathscr{A}$:

(a) At first we prove that $\mathscr{B} \times \mathscr{B} \subset \mathscr{A}_U(\mathfrak{P})$ for every sufficient statistic $U: X \to Z$.

It is easy to see that for each $r \in [0, 1]$ $\mathscr{B} \times \mathscr{B}$ is minimal sufficient for $\mathfrak{P}_r := \{P_r, Q_r\}$. As U is sufficient statistic for $\mathfrak{P}_r$, we obtain $\mathscr{B} \times \mathscr{B} \subset \mathscr{A}_U(\mathfrak{P}_r)$. As $\mathscr{A}_U$ is sufficient for $\mathfrak{P}_0 := \bigcup \{\mathfrak{P}_r : r \in [0, 1]\}$ Lemma 2 implies that $\mathscr{B} \times \mathscr{B} \subset \mathscr{A}_U(\mathfrak{P}_0)$ and hence by (1), $\mathscr{B} \times \mathscr{B} \subset \mathscr{A}_U(\mathfrak{P})$. This proves (a).

As X, endowed with the usual topology is a complete separable metric space with Borel field $\mathscr{B} \times \mathscr{B}$ and I is $\mathscr{A}$, $\mathscr{B} \times \mathscr{B}$ measurable Lemma 1 implies according to (a) that $I = g_U \circ U$ $\mathfrak{P}$-a.e. for each sufficient statistic U with some appropriate function g_U. This proves (ii).

3. Auxiliary lemmas. In this section we prove two lemmas which are auxiliary to Example 2.

LEMMA 1. *Let $\mathfrak{P}$ be a family of p-measures on a σ-field $\mathscr{A}$ on X. Let Z be a complete separable metric space with Borel field $\mathscr{C}$ and $f: X \to Z$ be $\mathscr{A}$, $\mathscr{C}$ measurable. Then for each $g: X \to Y$ with $f^{-1}\mathscr{C} \subset \mathscr{A}_g(\mathfrak{P})$ there exists a function $h: Y \to Z$ such that $f = h \circ g$ $\mathfrak{P}$-a.e.*

PROOF. Since f is $f^{-1}\mathscr{C}$, $\mathscr{C}$ measurable, $f^{-1}\mathscr{C} \subset \mathscr{A}_g(\mathfrak{P})$ and $\mathscr{C}$ is the Borel field of a complete separable metric space it is easy to see that there exists an $\mathscr{A}_g$, $\mathscr{C}$-measurable function $f^*: X \to Z$ with $f = f^*$ $\mathfrak{P}$-a.e. Hence according to [3] page 251, there exists a function $h: Y \to Z$ such that $f^* = h \circ g$ whence $f = h \circ g$ $\mathfrak{P}$-a.e.

LEMMA 2. *Let $\mathfrak{P}_i$, $i \in I$, be families of p-measures on a σ-field $\mathscr{A}$ and $\mathfrak{P} = \bigcup \{\mathfrak{P}_i : i \in I\}$. If $\mathscr{A}_0, \mathscr{A}_1 \subset \mathscr{A}$ are σ-fields such that $\mathscr{A}_1$ is sufficient for $\mathfrak{P} \mid \mathscr{A}$ and $\mathscr{A}_0 \subset \mathscr{A}_1(\mathfrak{P}_i)$ for each $i \in I$ then $\mathscr{A}_0 \subset \mathscr{A}_1(\mathfrak{P})$.*

PROOF. Let $A_0 \in \mathscr{A}_0$ be given. Since $\mathscr{A}_1$ is sufficient for $\mathfrak{P} \mid \mathscr{A}$ there exists $f \in \mathfrak{P}^{\mathscr{A}_1} 1_{A_0}$. Then $A_1 := \{x : f(x) = 1\} \in \mathscr{A}_1$. As by assumption $A_0 \sim A_i(\mathfrak{P}_i)$ for some $A_i \in \mathscr{A}_1$, we have $1_{A_0} = 1_{A_i} = f\ \mathfrak{P}_i$-a.e., whence $A_0 \sim A_1(\mathfrak{P}_i)$. As this holds for each $i \in I$ we obtain $A_0 \sim A_1(\mathfrak{P})$.

REFERENCES

[1] BAHADUR, R. R. (1954). Sufficiency and statistical decision functions. *Ann. Math. Statist.* **25** 423–462.

[2] BAHADUR, R. R. and LEHMANN, E. L. (1955). Two comments on "Sufficiency and statistical decision functions." *Ann. Math. Statist.* **26** 139–142.

[3] BAUER, H. (1968). *Wahrscheinlichkeitstheorie und Grundzüge der Maßtheorie*. De Gruyter, Berlin.

[4] HALMOS, P. R. and SAVAGE, J. L. (1949) Application of the Radon–Nikodym theorem to the theory of sufficient statistics. *Ann. Math. Statist.* **20** 225–241.

[5] PITCHER, T. S. (1957). Sets of measures not admitting necessary and sufficient statistics or subfields. *Ann. Math. Statist.* **26** 267–268.

MATHEMATISCHES INSTITUT
DER UNIVERSITÄT ZU KÖLN
5 KÖLN-LINDENTHAL
WEYERTAL 86-90
KÖLN, GERMANY